Thin three-dimensional drops on a slowly oscillating substrate

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(Received 20 September 2011; revised manuscript received 10 November 2011; published 1 December 2011)

We examine the evolution of a liquid drop on an inclined substrate oscillating vertically. The drop is assumed thin, and the oscillations are assumed weak and slow (the latter makes the liquid's inertia and viscosity negligible, so the drop's shape is determined by a balance of surface tension, gravity, and vibration-induced inertial force). On the basis of these approximations, asymptotic expressions are derived for the mean velocities of two- (2D) and three-dimensional (3D) drops. It is shown that, if the amplitude of the substrate's oscillations exceeds a certain threshold value, both 2D and 3D drops climb uphill. The two cases, however, exhibit different behaviors of the threshold amplitude of the oscillations on their frequency, in the low-frequency limit: to make 2D drops climb uphill, the oscillations must be much stronger than those in the 3D case. This difference is important, as the 2D behavior does not fit the existing experimental results.

DOI: 10.1103/PhysRevE.84.066301 PACS number(s): 47.55.D−, 47.55.nb

I. INTRODUCTION

One's intuition suggests that liquid drops on a vibrating solid object should slide down, yet experiments of Brunet, Eggers, and Deegan [1,2] demonstrate that drops on a vibrating inclined substrate can actually climb uphill. Their results were extended by the experiments of Ref. [3], where it was shown that drops on a horizontal substrate vibrating at an angle to itself can be driven in a given direction, at a given velocity by "tuning" the phase difference between the vertical and horizontal components of the vibration.

Two attempts to explain the observed effects have been made, and, in both cases, the drops were assumed thin and two-dimensional. Reference [4] examined the limit of low Reynolds numbers and modeled the drop's contact lines by precursor films due to van der Waals forces. It was shown that, if the substrate's acceleration is strong enough, the drop climbs uphill. A similar conclusion was reached in Ref. [5] for the opposite limit of small viscosity, with contact lines governed by a "contact-line law" (i.e., a functional dependence of the contact line's velocity on the contact angle). Reference [5] also demonstrated that the drop's uphill motion is due to an interaction of "swaying" and "spreading" oscillatory modes induced by the vibration.

All of the above experimental and theoretical results were obtained for harmonic oscillations of the substrate. Anharmonic oscillations were also examined, both experimentally [6,7] and theoretically [8]. Two important conclusion were drawn. Reference [7] explored how drops on a anharmonically vibrating substrate are affected by the so-called hysteresis interval, i.e., the range of contact angles for which velocity of the drop's contact line is zero. It was shown that, counterintuitively, drops with a wider hysteresis interval can sometimes climb faster than those with a narrower one. Reference [8], in turn, showed that, if the dependence of the substrate's acceleration on time involves narrow deep "troughs" and wide low "crests," the drop's uphill motion is much stronger than that caused by a harmonic vibration.

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Note, however, that all of the above theoretical models were two-dimensional (2D), and it is unclear how they relate to the experiments of Refs. [1–3,6,7] where the drops were, of course, three-dimensional (3D). Potentially, the behaviors of 2D and 3D drops can be quite different, as in the latter case, vibration-induced disturbances can travel *around* the drop's crest, whereas in the former, they have to climb *over* the crest (which requires more energy and may inhibit their development). One can furthermore conjecture that the 2D-3D asymmetry should be stronger for low-frequency vibrations, for which the disturbances have more time to travel. The quantitative comparison of the two cases, however, is hampered by the fact that 3D flows with free boundaries and contact lines are much more difficult to model than their 2D counterparts.

The present work is a first step toward a 3D model of climbing drops on oscillating substrates. Given the complexity of the problem, we use a number of approximations, outlined in Sec. II A. The problem is formulated mathematically in Secs. II B–II C, and it is analyzed in Sec. III. In Sec. IV we compare the theoretical results obtained through 3D and 2D models and, thus, clarify to which extent the latter grasps the essential physics of the problem. Finally, in Sec. V, we outline how a more general 3D model of drops on oscillating substrates can be developed.

II. FORMULATION

A. Qualitative aspects of the model

Consider a drop of liquid (of density ρ, kinematic viscosity ν, surface tension σ) on a vibrating substrate inclined at an angle α to the horizontal (see Fig. 1). We assume the substrate's vibration to be vertical, so the acceleration due to gravity g, whichever equation or boundary condition it normally appears in, can be replaced by an effective acceleration a due to both gravity and the vibration-induced inertial force. We shall assume that $a(t)$ is periodic with a characteristic amplitude a_0 and time scale t_0 [the latter being of the order of, but not necessarily equal to, the period of $a(t)$]. We also introduce the characteristic thickness h_0 of the drop and the mean radius R_0 of its base.

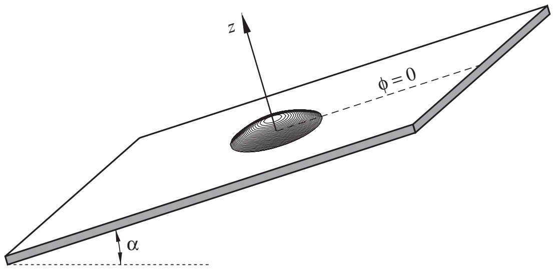

FIG. 1. The setting: a drop on an inclined substrate oscillating vertically.

All results obtained in this work rely on the assumption that the liquid's inertia and viscosity are weak, so the drop's shape is determined by a balance of surface tension, gravity, and vibration-induced inertial force. This assumption has been used for drops before, e.g., in Refs. [8,9]; in the latter paper, it was dubbed "the quasistatic approximation" (QSA) and was shown to hold if

$$\frac{\rho R_0^4}{\sigma h_0 t_0^2} \ll 1 \quad \frac{\rho \nu R_0^4}{\sigma h_0^3 t_0} \ll 1. \tag{1}$$

Observe that the QSA can be enforced by making the vibration sufficiently slow ($t_0 \to \infty$).

Next we introduce

$$\varepsilon = \frac{\rho R_0^3 a_0 \sin\alpha}{\sigma h_0}, \tag{2}$$

which characterizes the strength of gravity and vibration-induced inertial force relative to surface tension. We assume that

$$\varepsilon \ll 1.$$

This condition enables us to obtain a relatively simple asymptotic solution that will help to explore the essential physics of the problem.

We shall also assume that the drop is thin,

$$\frac{h_0}{R_0} \ll 1.$$

This condition is not crucial to our analysis, but it dramatically reduces the amount of algebra involved.

B. The governing equations

Let the z axis of the cylindrical coordinate system (r,ϕ,z) be perpendicular to the substrate and directed upward, with the angle ϕ counted from the direction of the steepest ascent (see Fig. 1). The shape of the drop's surface will be described by the equation $z = h(r,\phi,t)$, and we shall assume that the origin of our coordinate system is associated with the maximum of $h(r,\phi,t)$; hence,

$$\frac{\partial h}{\partial r} \to 0, \quad \frac{1}{r}\frac{\partial h}{\partial \phi} \to 0 \quad \text{as} \quad r \to 0. \tag{3}$$

In what follows, we shall briefly outline the derivation of the QSA (for more details, see Ref. [8]). First, conditions (1) are used to neglect the inertial and viscous terms in the Navier-Stokes equations. Second, the truncated equations (still including the pressure, gravity, and the vibration-induced inertial force) are integrated with respect to the spatial variables, which yields

$$\sigma\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2}\right] - \rho a\,(r\cos\phi\,\sin\alpha + h\cos\alpha) = P(t), \tag{4}$$

where $P(t)$ is a constant of integration (physically, it represents the nondimensional pressure at the drop's center).

Physically, the term involving σ in Eq. (4) describes the effect of surface tension (the expression in the square brackets is the curvature of the drop's surface under the thin-drop approximation). The term involving $\sin\alpha$ describes the down-the-slope acceleration of the liquid due to gravity and the inertial force, whereas the term involving $\cos\alpha$ describes the hydrostatic pressure due to the variability of the drop's thickness.

Let the equation $r = R(\phi,t)$ describe the contact line, i.e., the boundary of the drop's base, where

$$h = 0 \quad \text{if} \quad r = R. \tag{5}$$

As in the case of 2D drops [8], we shall explicitly require that the drop's mass be conserved,

$$\rho \int_0^{2\pi}\int_0^{R} h\, r\, dr\, d\phi = M, \tag{6}$$

where M should be treated as a known constant.

Following numerous other researchers (e.g., Refs. [10,11] and references therein), we shall assume that the normal velocity of the contact line is a known function, $v(\theta)$, of the contact angle θ. Expressing the former in terms of $R(\phi,t)$ and introducing the Cartesian velocity $[V_x(t),V_y(t)]$ of the origin of our coordinate system (i.e., the drop's center), we obtain

$$\frac{1}{\sqrt{1+\left(\frac{1}{R}\frac{\partial R}{\partial \phi}\right)^2}}\left[\frac{\partial R}{\partial t} + V_x\left(\cos\phi + \frac{1}{R}\frac{\partial R}{\partial \phi}\sin\phi\right) + V_y\left(\sin\phi - \frac{1}{R}\frac{\partial R}{\partial \phi}\cos\phi\right)\right] = v(\theta), \tag{7}$$

where the contact angle is, under the thin-drop approximation, given by

$$\theta = -\frac{1}{\sqrt{1+\left(\frac{1}{R}\frac{\partial R}{\partial \phi}\right)^2}}\left(\frac{\partial h}{\partial r} - \frac{1}{r^2}\frac{\partial h}{\partial \phi}\frac{\partial R}{\partial \phi}\right)_{r=R}. \tag{8}$$

Subject to a suitable initial condition, Eqs. (3)–(8) determine the evolution of $h(r,\phi,t)$, $R(\phi,t)$, $V(t)$, and $P(t)$. The functions $a(t)$ and $v(\theta)$ appear in (3)–(8) as given parameters.

Typically, the contact-line law (the dependence of the contact-line velocity v on the contact angle θ) is such that

$$\begin{aligned} v &< 0 \quad \text{if} \quad \theta < \theta_r, \\ v &= 0 \quad \text{if} \quad \theta_r \leqslant \theta \leqslant \theta_a, \\ v &> 0 \quad \text{if} \quad \theta > \theta_a, \end{aligned}$$

where the receding and advancing contact angles, θ_r and θ_a, are the boundaries of the hysteresis interval. In this paper we assume that the hysteresis interval is of zero length, i.e., $\theta_r = \theta_a = \theta_0$, where θ_0 is the equilibrium contact angle. This simple model (used previously in Refs. [5,8,9]) works well for substrates with narrow hysteresis intervals (e.g., microfibrillated polydimethylsiloxane (PDMS) (Dow Corning Sylgard 184) surfaces [see Ref. [12]]) and also can help us to explore qualitative aspects of the general case.

C. Nondimensionalization

We shall use the following nondimensional variables:

$$\begin{gathered}\hat{r}=\frac{r}{R_0},\quad \hat{\phi}=\phi,\quad \hat{t}=\frac{t}{t_0},\\ \hat{h}=\frac{h}{h_0},\quad \hat{P}=\frac{R_0^2}{\sigma h_0}P,\\ \hat{R}=\frac{R}{R_0},\quad \hat{V}_{x,y}=\frac{t_0}{R_0}V_{x,y},\\ \hat{a}=\frac{a}{a_0},\quad \hat{v}=\frac{v}{v_0},\quad \hat{\theta}=\frac{\theta}{\theta_0},\end{gathered}\tag{9}$$

where v_0 is the characteristic velocity of the contact line (i.e., the boundary of the drop's base), and the rest of the dimensional scales have been introduced in Sec. II A.

Let the scales h_0, R_0, and t_0 satisfy

$$\frac{M}{\rho h_0 (R_0)^2}=2\pi,\quad \frac{\theta_0 R_0}{h_0}=1,\quad \frac{v_0 t_0}{R_0}=1.\tag{10}$$

Note that these conditions will not be used to simplify the governing equations; thus, they are not new physical assumptions, but rather means of choosing the most convenient set of nondimensional variables.

Substituting (9)–(10) into (3)–(8) and omitting the hats, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2}-\varepsilon a\,(r\cos\phi+\gamma h)=P,\tag{11}$$

$$\int_0^{2\pi}\int_0^R h\,r\,dr\,d\phi=2\pi,\tag{12}$$

$$\frac{\partial h}{\partial r}\to 0,\quad \frac{1}{r}\frac{\partial h}{\partial \phi}\to 0\quad \text{as}\quad r\to 0,\tag{13}$$

$$h=0\quad \text{if}\quad r=R,\tag{14}$$

$$\frac{1}{\sqrt{1+\left(\frac{1}{R}\frac{\partial R}{\partial \phi}\right)^2}}\left[\frac{\partial R}{\partial t}+V_x\left(\cos\phi+\frac{1}{R}\frac{\partial R}{\partial \phi}\sin\phi\right)+V_y\left(\sin\phi-\frac{1}{R}\frac{\partial R}{\partial \phi}\cos\phi\right)\right]=v(\theta),\tag{15}$$

$$\theta=-\frac{1}{\sqrt{1+\left(\frac{1}{R}\frac{\partial R}{\partial \phi}\right)^2}}\left(\frac{\partial h}{\partial r}-\frac{1}{r^2}\frac{\partial h}{\partial \phi}\frac{\partial R}{\partial \phi}\right)_{r=R},\tag{16}$$

where ε is determined by (2) and

$$\gamma=\frac{h_0}{R_0\tan\alpha}\tag{17}$$

is the ratio of the slope of the drop's surface to the substrate's slope. Note that, since the former is small, so should be the latter; otherwise, the corresponding term in Eq. (11) is small.

In the next section, we shall construct an asymptotic solution of the boundary-value problem (11)–(16) under the assumption $\varepsilon \ll 1$.

III. THE ANALYSIS

Observe that, if the substrate's oscillations are strictly vertical, the drop can neither move nor tilt in the y direction, which implies

$$V_y=0,\tag{18}$$

$$\begin{gathered}h(t,r,-\phi)=h(t,r,\phi),\\ R(t,-\phi)=R(t,\phi),\\ R(t,-\phi)=R(t,\phi).\end{gathered}\tag{19}$$

Note also that a sustained uphill motion of the drop can be described only by a periodic solution; thus, in what follows, nonperiodic solutions will be discarded.

It is convenient to subdivide $a(t)$ into the oscillating and constant parts, representing the substrate's acceleration and gravity respectively, i.e.,

$$a=a^{(0)}(t)+a^{(1)},\tag{20}$$

where $a^{(0)}(t)$ is a periodic function with a zero mean and $a^{(1)}>0$ is a constant. Following the experiments [1,2], we assume

$$|a^{(0)}|\gg a^{(1)}.\tag{21}$$

This condition and the fact that a was nondimensionalized by its own magnitude [see (9)] imply $a^{(0)}\sim 1$ and $\alpha^{(1)}\ll 1$. The latter condition can be conveniently quantified by assuming

$$a^{(1)}=O(\varepsilon),$$

which actually includes the limits $a^{(1)}\ll\varepsilon$ and $a^{(1)}\gg\varepsilon$ as well.

For the sake of generality, $a^{(0)}(t)$ will initially be kept unspecified, but in the end we shall set

$$a^{(0)}(t)=\sin\omega t,\tag{22}$$

which, again, corresponds to the experiments [1,2].

In what follows, it will be convenient to expand the contact-line law, $v(\theta)$, in powers of the deviation of the contact angle from its equilibrium value. Recalling that, nondimensionally, the latter equals unity, we obtain

$$v=(\theta-1)+\tfrac{1}{2}v''(\theta-1)^2+\cdots,\tag{23}$$

where v'' is the second derivative of $v(\theta)$ at $\theta=1$, whereas the first derivative is implied to have been scaled out by an appropriate choice of v_0 in nondimensionalization (9).

The solution of Eqs. (11)–(16) and (18)–(19) will be sought in the form of an expansion in ε,

$$\begin{gathered}h=h^{(0)}+\varepsilon h^{(1)}+\cdots,\\ R=R^{(0)}+\varepsilon R^{(1)}+\cdots,\quad P=P^{(0)}+\varepsilon P^{(1)}+\cdots,\\ V_x=\varepsilon V_x^{(1)}+\cdots,\quad \theta=1+\varepsilon\theta^{(1)}+\cdots.\end{gathered}$$

The asymptotic results presented below can be better understood if they are interpreted physically. To this end, observe that the acceleration a in Eqs. (11)–(16) appears only together with a small parameter; hence, the leading-order solution describes a stationary drop with a shape determined by surface tension, i.e., a spherical cap (or rather its thin-film approximation).

The first-order solution, in turn, describes small periodic perturbations of the drop's surface and small-amplitude oscillations of the drop's position. Two oscillatory modes can be distinguished: an axisymmetric mode describing periodic spreading and contraction of the drop and an asymmetric mode describing "swaying" of the drop up and down the slope. Following Ref. [5], we shall refer to the former and latter as the "spreading" and "swaying" modes, respectively.

Finally, the second-order solution describes the nonlinear interaction of the spreading and swaying modes, resulting in a mean drift of the drop up or down the slope.

A. The leading-order solution

Assuming, as mentioned above, that the leading-order solution is axisymmetric and stationary (time-independent), one can reduce Eqs. (11)–(16) and (18) to

$$\frac{\partial^2 h^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial h^{(0)}}{\partial r} = P^{(0)},$$

$$2\pi \int_0^{R^{(0)}} h^{(0)}\, r\, dr = 2\pi,$$

$$h^{(0)} = 0 \quad \text{if} \quad r = R^{(0)},$$

$$1 = -\left(\frac{\partial h^{(0)}}{\partial r}\right)_{r=R^{(0)}}, \qquad \left(\frac{\partial h}{\partial r}\right)_{r=0} = 0.$$

The solution of this set of equations can be readily found:

$$P^{(0)} = -1, \quad h^{(0)} = 1 - \tfrac{1}{4}r^2, \quad R^{(0)} = 2. \tag{24}$$

The above expression for $h^{(0)}$ is, essentially, a thin-drop approximation of a spherical cap.

B. The first-order solution

In the next order, Eqs. (11)–(16), (18), (20), and (23) yield

$$\frac{\partial^2 h^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial h^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h^{(1)}}{\partial \phi^2} - a^{(0)}(r\cos\phi + \gamma h^{(0)}) = P^{(1)}, \tag{25}$$

$$\int_0^{2\pi}\left[\int_0^{R^{(0)}} h^{(1)} r\, dr + R^{(1)}(h^{(0)}r)_{r=R^{(0)}}\right] d\phi = 0, \tag{26}$$

$$h^{(1)} + \frac{\partial h^{(0)}}{\partial r} R^{(1)} = 0 \quad \text{if} \quad r = R^{(0)}, \tag{27}$$

$$\frac{\partial R^{(1)}}{\partial t} + V_x^{(1)}\cos\phi = \theta^{(1)}, \tag{28}$$

$$\theta^{(1)} = -\left(\frac{\partial h^{(1)}}{\partial r} + \frac{\partial^2 h^{(0)}}{\partial r^2} R^{(1)}\right)_{r=R^{(0)}}, \tag{29}$$

$$\left(\frac{\partial h^{(1)}}{\partial r}\right)_{r=0} = \left(\frac{\partial h^{(1)}}{\partial \phi}\right)_{r=0} = 0. \tag{30}$$

The general solution of Eq. (25) subject to the symmetry condition (19) is

$$h^{(1)} = h_0^{(1)}(r) + h_1^{(1)}(r)\cos\phi + \sum_{n=0}^{\infty} c_n r^n \cos n\phi, \tag{31}$$

where the last term (involving the infinite sequence of constants c_n) is the solution of the homogeneous (Laplace) equation, and the first two terms represent a particular solution of the nonhomogeneous (Poisson) equation. Physically, the former terms describe the drop's response to oscillations, whereas the latter describes an initial change of the drop's shape and, thus, is of no interest to us. Omitting it from expression (31) and substituting (31) into Eqs. (25)–(30), we find (after a certain amount of routine algebra)

$$P^{(1)} = -2A^{(1)} - \frac{5\gamma a^{(0)}}{6},$$

$$h^{(1)} = A^{(1)} + \left(\frac{\gamma a^{(0)}}{24} - \frac{1}{2}A^{(1)}\right) r^2 - \frac{\gamma a^{(0)}}{64} r^4 + \frac{a^{(0)} r^3}{8}\cos\phi, \tag{32}$$

$$R^{(1)} = -A^{(1)} - \frac{\gamma a^{(0)}}{12} + a^{(0)}\cos\phi, \tag{33}$$

$$\theta^{(1)} = \frac{3}{2}A^{(1)} + \frac{7\gamma a^{(0)}}{24} - a^{(0)}\cos\phi, \tag{34}$$

$$V^{(1)} = -a^{(0)} - \frac{da^{(0)}}{dt}, \tag{35}$$

where the function $A^{(1)}(t)$ satisfies

$$\frac{dA^{(1)}}{dt} + \frac{3}{2}A^{(1)} = -\frac{\gamma}{12}\left(\frac{da^{(0)}}{dt} + \frac{7a^{(0)}}{2}\right). \tag{36}$$

For the sinusoidal $a^{(0)}(t)$ [given by (22)], the only periodic solution of Eq. (36) is

$$A^{(1)} = -\frac{\gamma}{12}\left(\frac{21 + 4\omega^2}{9 + 4\omega^2}\sin\omega t - \frac{8\omega}{9 + 4\omega^2}\cos\omega t\right). \tag{37}$$

Observe that expressions (32)–(34) include axisymmetric terms and those involving $\cos\phi$, describing the spreading and swaying mode, respectively. The modes, however, do not interact at this order; as a result, the velocity $V^{(1)}$ [given by (35)] has zero mean.

To illustrate the zeroth- and first-order solution, the shape of the drop's base [given by (24) and (33)] and the drop's cross section [given by (24) and (32) with $\phi = 0,\pi$] are plotted in Fig. 2 for the sinusoidal acceleration (22) and four values of the phase of the oscillations [13]. One can see that, when the substrate accelerates upward ($\omega t = \pi/2$, $a^{(0)} = 1$), the drop's left-hand slope becomes steeper than its right-hand counterpart, while its base becomes wider: These two effects correspond to the swaying and spreading modes, respectively. The velocity $V^{(1)}$, at the same, is negative [as determined by (35) and (22)]; i.e., the drop slides down. When, in turn, the substrate accelerates downward ($\omega t = 3\pi/2$, $a^{(0)} = -1$), the drop becomes "skewed" to the right with its base contracting, whereas $V^{(1)}$ is positive.

Qualitatively, such a behavior agrees with that observed experimentally [1,2].

C. The second-order solution

Unfortunately, the second-order approximation involves a lot of cumbersome algebra. In a similar problem examined in Ref. [5], the similar calculations were bypassed by a relatively simple argument based on momentum conservation. The same argument can be used in the present case; however, it involves even more algebra than the straightforward approach. In the end, to verify the results obtained, both approaches have been followed through, but only the direct expansion will be presented below.

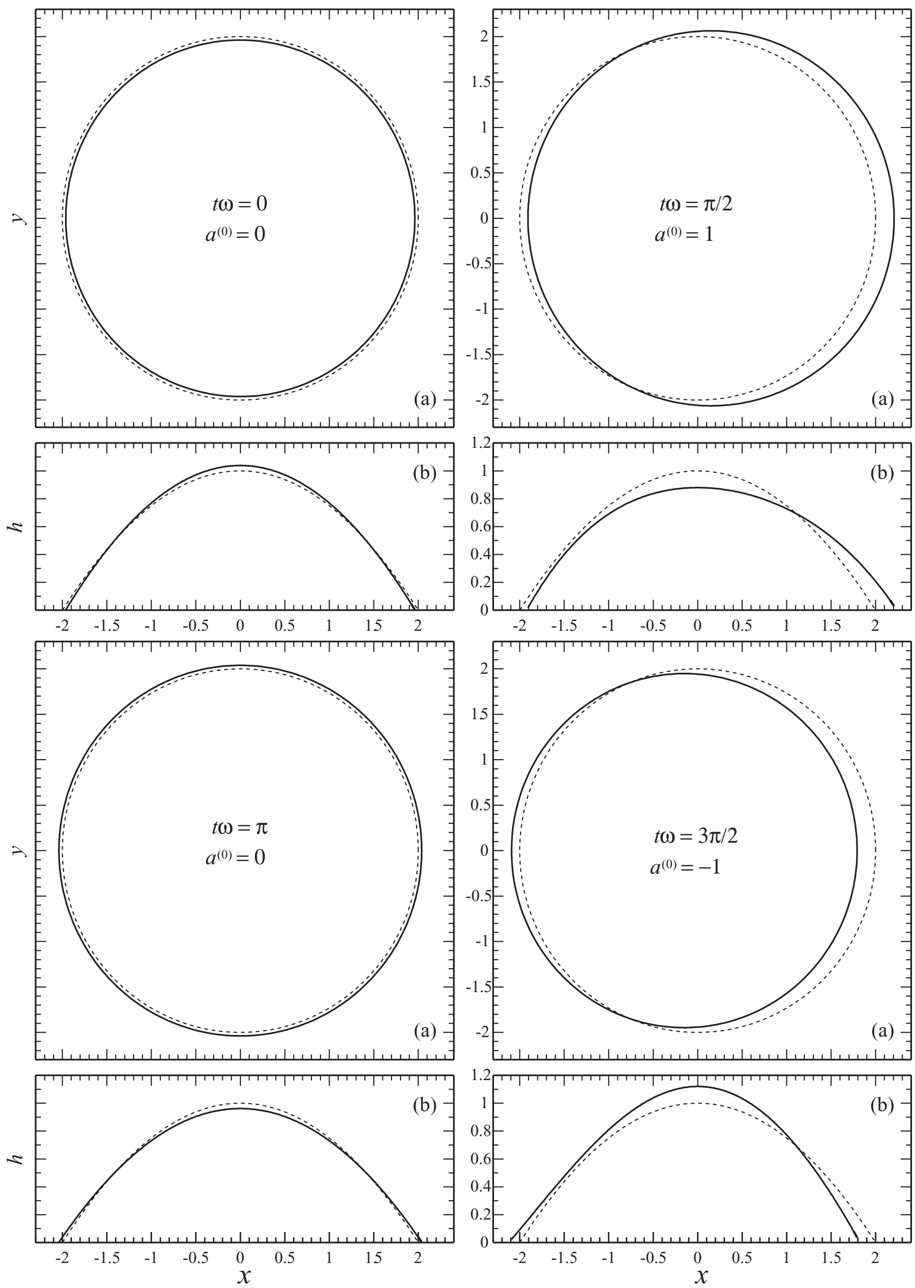

FIG. 2. The evolution of the drop over one period of oscillations, as described by the first-order expansion, for the sinusoidal acceleration (22) and $\varepsilon = 0.15$, $\gamma = 5$, $\omega = 1$. Panels marked (a) show the shape of the drop's base, panels (b) show the corresponding cross sections, $h(x)$, of the drop. The unperturbed curves are shown in dotted line. The corresponding phase of the oscillation, ωt, and the value of the inertial acceleration, $a^{(0)}$, are indicated in the panels (a) (when $a^{(0)} > 0$, the substrate accelerates upward, and vice versa).

In the second order, Eqs. (11), (13)–(16), (18), (20), and (23) yield

$$\frac{\partial^2 h^{(2)}}{\partial r^2} + \frac{1}{r}\frac{\partial h^{(2)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h^{(2)}}{\partial \phi^2} - \gamma\left[a^{(0)}h^{(1)} + \frac{a^{(1)}}{\varepsilon}(r\cos\phi + \gamma h^{(0)})\right] = P^{(2)}, \tag{38}$$

$$(h^{(2)})_{r=R^{(0)}} + \left(\frac{\partial h^{(0)}}{\partial r}\right)_{r=R^{(0)}} R^{(2)} + \left(\frac{\partial h^{(1)}}{\partial r}\right)_{r=R^{(0)}} R^{(1)} + \frac{1}{2}\left(\frac{\partial^2 h^{(0)}}{\partial r^2}\right)_{r=R^{(0)}} R^{(1)2} = 0, \tag{39}$$

$$\frac{\partial R^{(2)}}{\partial t} + V_x^{(2)}\cos\phi + V_x^{(1)}\frac{1}{R^{(0)}}\frac{\partial R^{(1)}}{\partial \phi}\sin\phi = \theta^{(2)} + \frac{1}{2}v''\theta^{(1)2}, \tag{40}$$

$$\theta^{(2)} + \frac{1}{2}\left(\frac{1}{R^{(0)}}\frac{\partial R^{(1)}}{\partial \phi}\right)^2 = -\left(\frac{\partial h^{(2)}}{\partial r} + \frac{\partial^2 h^{(0)}}{\partial r^2}R^{(2)} + \frac{\partial^2 h^{(1)}}{\partial r^2}R^{(1)} + \frac{1}{2}\frac{\partial^3 h^{(0)}}{\partial r^3}R^{(1)2} - \frac{1}{r^2}\frac{\partial h^{(1)}}{\partial \phi}\frac{\partial R^{(1)}}{\partial \phi}\right)_{r=R^{(0)}}, \tag{41}$$

$$\left(\frac{\partial h^{(2)}}{\partial r}\right)_{r=0} = \left(\frac{\partial h^{(2)}}{\partial \phi}\right)_{r=0} = 0. \tag{42}$$

Upon substitution of the zeroth- and first-order solutions, (24) and (32)–(35), into Eq. (38), one can see that its solution is of the form

$$h^{(2)}(t,r,\phi) = h_0^{(2)}(t,r) + h_1^{(2)}(t,r)\cos\phi. \tag{43}$$

It can be shown that, to calculate the rise velocity $V^{(2)}$, the function $h_0^{(2)}$ is not needed; hence, we present the solution only for

$$h_1^{(2)} = \frac{\gamma a^{(0)2}}{192} r^5 + \frac{a^{(1)}}{8\varepsilon} r^3. \tag{44}$$

Similarly, the solutions of Eqs. (39) and (41) have the form

$$\begin{aligned} R^{(2)} &= R_0^{(2)} + R_1^{(2)}\cos\phi + R_2^{(2)}\cos 2\phi, \\ \theta^{(2)} &= \theta_0^{(2)} + \theta_1^{(2)}\cos\phi + \theta_2^{(2)}\cos 2\phi, \end{aligned} \tag{45}$$

of which we only need

$$\begin{aligned} R_1^{(2)} &= -\frac{1}{4}\gamma a^{(0)2} + \frac{a^{(1)}}{\varepsilon} - 3a^{(0)}A^{(1)}, \\ \theta_1^{(2)} &= \frac{1}{4}\gamma a^{(0)2} - \frac{a^{(1)}}{\varepsilon} + a^{(0)}A^{(1)}. \end{aligned} \tag{46}$$

Substituting (43)–(46) into Eq. (40) and keeping track of terms involving $\cos\phi$, we obtain

$$\begin{aligned} V^{(2)} &= \frac{\gamma a^{(0)2}}{4} + a^{(0)}A^{(1)} - v''\left(\frac{3}{2}A^{(1)} + \frac{7\gamma a^{(0)}}{24}\right)a^{(0)} \\ &\quad - \frac{a^{(1)}}{\varepsilon} - \frac{d}{dt}\left(-\frac{\gamma a^{(0)2}}{4} + \frac{a^{(1)}}{\varepsilon} - 3a^{(0)}A^{(1)}\right). \end{aligned} \tag{47}$$

We are interested in the non-oscillating part of $V^{(2)}$, which can be obtained by averaging (47) over the period of the substrate's oscillations,

$$\langle V^{(2)}\rangle = \frac{\gamma\langle a^{(0)2}\rangle}{4} - v''\left(\frac{3\langle a^{(0)}A^{(1)}\rangle}{2} + \frac{7\gamma\langle a^{(0)2}\rangle}{24}\right) - \frac{a^{(1)}}{\varepsilon}, \tag{48}$$

where the angle brackets denote the averaging, and it was recalled that $a^{(1)}$ is constant (representing gravity), i.e.,

$$\langle a^{(1)}\rangle = a^{(1)}.$$

Finally, for sinusoidal oscillations described by (22) and (37), Eq. (48) yields

$$\langle V^{(2)}\rangle = \frac{\gamma(3 + 4\omega^2 - 2v''\omega^2)}{12(9 + 4\omega^2)} - \frac{a^{(1)}}{\varepsilon}. \tag{49}$$

This expression is the main result of the present work. If $\langle V^{(2)}\rangle$ is positive, the drop climbs uphill.

It is worth recalling here that γ is the slope of the drop's surface scaled by the substrate's slope [see (17)], v'' is the second derivative of the contact-line law $v(\theta)$ at the equilibrium value of the contact angle, $a^{(1)}$ is the nondimensional gravity (scaled by the substrate's acceleration), and ω is the nondimensional frequency.

IV. COMPARISON OF 3D AND 2D MODELS

We shall now compare expression (49) for the drop's rise velocity with its 2D counterpart derived under the same set of approximations (QSA, thin drop, $\varepsilon \ll 1$) in the Appendix.

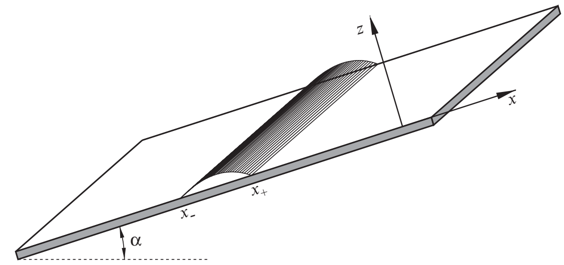

FIG. 3. A 2D drop (liquid ridge) on an inclined substrate.

Note that 2D drops are not really drops, but liquid ridges (see Fig. 3) and, thus, are different geometrically from the real, 3D drops. This leaves us with a question: which ridge should be regarded analogous to, and thus be compared with, a given 3D drop? It seems natural to choose the ridge whose width matches the drop's diameter; i.e., $D^{(0)}$ introduced in the Appendix should equal $R^{(0)}$ of Sec. III A.

The most important result to be deduced from formula (49) is the condition of zero rise velocity, which can be viewed as a curve in the (ω,ε_*) plane where

$$\varepsilon_* = \frac{\varepsilon}{a^{(1)}}. \tag{50}$$

Note that, even though ε is assumed small, ε_* can be large (provided $a^{(1)}$ is sufficiently small).

Now, equating $\langle V^{(2)}\rangle$ to zero, we obtain

$$\varepsilon_* = \frac{12\left(9 + 4\omega^2\right)}{\gamma[3 + (4 - 2v'')\omega^2]}. \tag{51}$$

Thus, if ε_* exceeds the threshold value (51), the drop climbs uphill.

The 2D equivalent of (51) follows from Eq. (A19),

$$\varepsilon_* = \frac{15\left(1 + \omega^2\right)a^{(1)}}{2\gamma\,(1 - v'')\,\omega^2}. \tag{52}$$

The curves of zero rise velocity determined by (51) and (52) are shown in Fig. 4, one can see that they differ mainly in the low-frequency limit. Indeed, as $\omega \to 0$, 2D drops can climb uphill only for an *increasingly large*acceleration of the substrate's oscillations, whereas, for 3D drops, the acceleration can be *finite*. This is an important difference, as the experimental results exhibit the latter pattern (see Fig. 2 of Ref. [1] and Fig. 1 of Ref. [2]).

We conclude that, for low frequencies, the 2D QSA-based model is qualitatively incorrect. The same conclusion applies to the 2D non-QSA model examined in Ref. [5]. The 2D model of Ref. [4], in turn, exhibits the correct behavior of $\varepsilon_*(\omega)$ as $\omega \to 0$. The difference between Ref. [4] and other 2D results is probably due to the fact that the former modeled the drop's contact lines by precursor films due to van der Waals forces, whereas the latter employed contact-line laws. This difference between the two models of contact lines in vibrating liquids is important and, thus, deserves further investigation.

In the opposite (high-frequency) limit, neither of the models presented in this paper is adequate. Indeed, the experimental

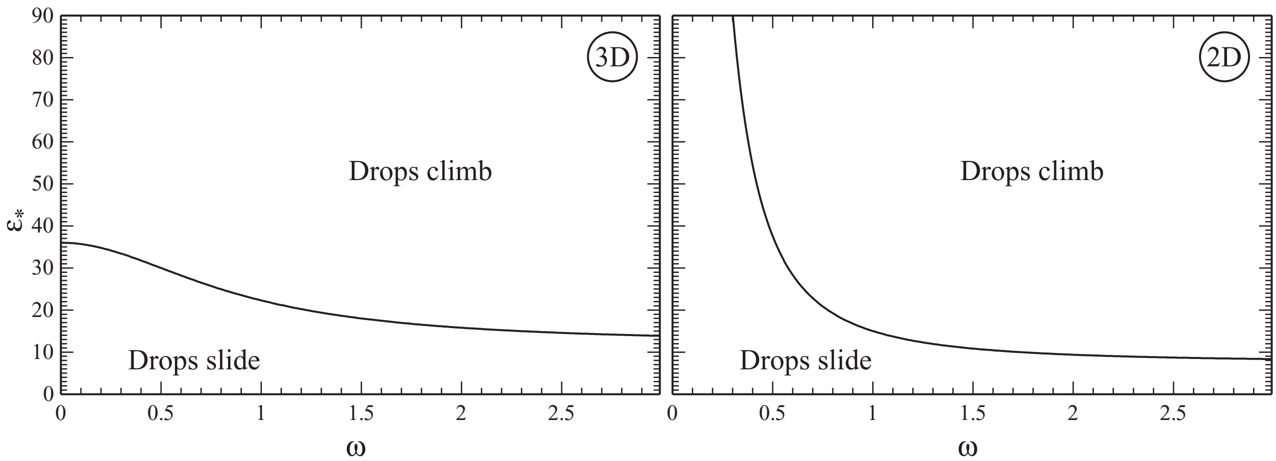

FIG. 4. The curves of zero rise velocity for 3D and 2D drops [described by (51) and (52), respectively], for $\gamma = 1$, $v'' = 0$. The parameter ε_* characterizes the substrate's acceleration [see (53)], and ω is the nondimensional frequency of its oscillations [given by (54)].

results suggest that ε_* *grows* as $\omega \to \infty$, whereas both (51) and (52) yield *finite* limiting values. This discrepancy is due to the inapplicability of the QSA at high frequencies. In other words, an adequate description of large ω can be obtained only if the QSA is not used (as in Refs. [4,5]).

Note also that both criteria (51) and (52) do not involve the static zone separating climbing and sliding drops, which is always observed in the experiments. That can be recovered only using a contact-line law with a nonzero hysteresis interval (this work is currently in progress).

Finally, we shall link the nondimensional criterion (51) to physically measurable parameters. Treating (10) as a set of equations for the scales h_0, R_0, and t_0, then substituting these into (2) and (50), and taking into account that $a^{(1)} = g/a_0$, we obtain

$$\varepsilon_* = \left(\frac{\text{the drops volume}}{2\pi\theta_0}\right)^{2/3} \frac{\sin\alpha}{\theta_0} \frac{\rho\,(a_0)^2}{\sigma g}, \tag{53}$$

$$\omega = \left(\frac{\text{the drops volume}}{2\pi\theta_0}\right)^{1/3} \frac{\omega_{\text{dim}}}{v_0}, \tag{54}$$

where ω_{dim} is the dimensional frequency, θ_0 is the equilibrium contact angle, v_0 is the derivative of the contact-line law $v(\theta)$ at $\theta = \theta_0$, σ is the surface tension, and α is the angle between the substrate and the horizontal.

V. CONCLUDING REMARKS

Thus, we have examined the evolution of drops on an inclined substrate oscillating vertically. The analysis was based on four assumptions: the quasistatic and thin-drop approximations, the assumption that the contact-line law does not involve an hysteresis interval, and that the substrate's oscillations are weak [so that the parameter ε, given by (2), is small]. Our main result is expression (49) for the drop's mean velocity, which was compared to its two-dimensional counterpart, and the latter turned out to be qualitatively incorrect in the limit of small frequencies of the substrate's oscillations.

In principle, expression (49) can be verified by a specifically designed experiment, but our assumptions are too restrictive for a comparison with the existing experiments [1,2] (where the drops were thick, ε was order one, and the contact-line law did involve an hysteresis interval).

Thus, to quantitatively model the results of Refs. [1,2], we shall need to give up three of the four assumptions used in this work. The quasistatic approximation, however, can be retained, as it did hold in some of the experiments of Refs. [1,2]. This is quite fortunate, in fact, as the QSA dramatically simplifies the *governing equations* and, thus, is crucial, whereas the other three assumptions simplify the *asymptotic analysis*. Even if they are not used, the problem can still be solved numerically, and by relatively simple means (this work is now in progress).

ACKNOWLEDGMENTS

The author acknowledges the support of the Science Foundation Ireland (RFP Grant 11/RFP.1/MTH3281 and Mathematics Initiative Grant 06/MI/005).

APPENDIX: THE TWO-DIMENSIONAL CASE

Consider a 2D drop (or rather a liquid ridge) on an inclined substrate vibrating vertically (see Fig. 3). It is governed by the following 2D equivalent of the 3D governing equations (4)–(8):

$$\sigma\frac{\partial^2 h}{\partial x^2} - \rho a\,(x\sin\alpha + h\cos\alpha) = P, \tag{A1}$$

$$\int_{x_-}^{x_+} h\,dx = A, \tag{A2}$$

$$h \to 0 \quad \text{as} \quad x \to x_\pm, \tag{A3}$$

$$\frac{dx_\pm}{dt} = \pm v(\theta_\pm), \qquad \theta_\pm = \mp\left(\frac{\partial h}{\partial x}\right)_{x=x_\pm}, \tag{A4}$$

where $\theta_\pm$ and $x_\pm$ are the contact angles and the coordinates of the contact lines, respectively, and A is the drop's (ridge's) cross-sectional area. We shall use the following nondimensional variables

$$\hat{x} = \frac{x}{x_0}, \quad \hat{t} = \frac{t}{t_0}, \quad \hat{h} = \frac{h}{h_0},$$
$$\hat{a} = \frac{a}{a_0}, \quad \hat{v} = \frac{v}{v_0}, \quad \hat{\theta} = \frac{\theta}{\theta_0}, \tag{A5}$$

and assume that

$$\frac{Ax_0}{h_0} = \frac{8}{3}, \quad \frac{\theta_0 x_0}{h_0} = 1, \quad \frac{v_0 t_0}{x_0} = 1, \tag{A6}$$

where the choice of the factor 8/3 will be explained later.

In terms of variables (A5)–(A6), Eqs. (A1)–(A4) become (hats omitted)

$$\frac{\partial^2 h}{\partial x^2} - \varepsilon a\,(x + \gamma h) = P, \tag{A7}$$

$$\int_{x_-}^{x_+} h\,dx = \frac{8}{3}, \tag{A8}$$

$$h \to 0 \quad \text{as} \quad x \to x_\pm, \tag{A9}$$

$$\frac{dx_\pm}{dt} = \pm v(\theta_\pm), \quad \theta_\pm = \mp\left(\frac{\partial h}{\partial x}\right)_{x=x_\pm}, \tag{A10}$$

where ε and γ are determined by (2) and (17).

As in the 3D case, it is convenient to change to a comoving reference frame, which amounts to introducing the position of the drop's center and its half-width,

$$X(t) = \frac{x_- + x_+}{2}, \quad D(t) = \frac{x_+ - x_-}{2},$$

and replacing x and P with

$$x_{\text{new}} = x - X, \quad P_{\text{new}} = P + \varepsilon a X.$$

Omitting the subscript "new," one can see that the change of variables does not affect (A7), whereas (A8)–(A10) become

$$\int_{-D}^{D} h\,dx = \frac{8}{3}, \tag{A11}$$

$$h \to 0 \quad \text{as} \quad x \to \pm D, \tag{A12}$$

$$V \pm \frac{dD}{dt} = \pm v(\theta_\pm), \quad \theta_\pm = \mp\left(\frac{\partial h}{\partial x}\right)_{x=\pm D}, \tag{A13}$$

where $V = dX/dt$ is, as before, the drop's velocity. Finally, we shall assume that the acceleration a is given by (A11)–(A13) and $v(\theta)$ by (23).

The first two orders of the expansion of Eqs. (A7) and (A11)–(A13) in ε yield

$$h^{(0)} = 1 - \frac{1}{4}x^2, \quad D^{(0)} = 2,$$

and

$$\begin{aligned} h^{(1)} &= a^{(0)}\left(\frac{x^3}{6} - \frac{\gamma x^4}{48}\right) - \frac{\gamma a^{(0)}}{15} - \frac{1}{2}D^{(1)} \\ &\quad - \frac{2a^{(0)}}{3}x + \left(\frac{\gamma a^{(0)}}{10} + \frac{3}{8}D^{(1)}\right)x^2, \\ \theta_\pm^{(1)} &= \mp\frac{4a^{(0)}}{3} + \frac{4\gamma a^{(0)}}{15} - D^{(1)}, \\ V^{(1)} &= -\frac{4a^{(0)}}{3}, \end{aligned} \tag{A14}$$

$$\frac{dD^{(1)}}{dt} + D^{(1)} = \frac{4\gamma a^{(0)}}{15}. \tag{A15}$$

Observe that the 2D drop's half-width $D^{(0)}$ equals the 3D drop's radius $R^{(0)}$ [see expression (24)], which gives us a "reference point" for comparing the 2D and 3D cases. We also mention that the factor 8/3 in the first equality of (A6) was chosen to ensure that $D^{(0)}$ coincides with $R^{(0)}$.

In the 2D case (unlike its 3D counterpart), there is no need to calculate the second-order solution; instead, the rise velocity can be found from an identity reflecting the momentum balance (similar to that used in Ref. [5]). We have verified that the two approaches yield the same result, but only the latter will be presented here (as it involves significantly less algebra).

To derive the above-mentioned identity, multiply the exact equation (A7) by h and integrate it with respect to x over $(-D,D)$. Taking into account conditions (A11)–(A12), we obtain

$$-\frac{1}{2}\left[\left(\frac{\partial h}{\partial x}\right)^2\right]_{x=D} + \frac{1}{2}\left[\left(\frac{\partial h}{\partial x}\right)^2\right]_{x=-D} = \frac{8\varepsilon a}{3}. \tag{A16}$$

Next use (A13) and (23) to relate $\partial h/\partial x$ to V and D,

$$\begin{aligned} \mp\left(\frac{\partial h}{\partial x}\right)_{x=\pm D} &= 1 + \varepsilon\left(\frac{dD^{(1)}}{dt} \pm V^{(1)}\right) \\ &\quad + \varepsilon^2\left[\frac{dD^{(2)}}{dt} \pm V^{(2)} - \frac{v''}{2}\left(\frac{dD^{(1)}}{dt} \pm V^{(1)}\right)^2\right] + O(\varepsilon^3). \end{aligned} \tag{A17}$$

Substituting (A17) into (A16) and averaging over the period of the substrate's oscillations, we obtain, to leading order,

$$\langle V^{(2)}\rangle = -(1 - v'')\left\langle\frac{dD^{(1)}}{dt}V^{(1)}\right\rangle - \frac{4a^{(1)}}{3\varepsilon}, \tag{A18}$$

where it was taken into account that

$$\langle a^{(1)}\rangle = a^{(1)}, \quad \langle V^{(1)}\rangle = \left\langle\frac{dD^{(1)}}{dt}\right\rangle = 0$$

[the former equality holds since $a^{(1)}$ is a constant, and the latter follows from (A14)–(A15) and (20)]. Finally, substituting (20) into (A14)–(A15), we find

$$V^{(1)} = -\frac{4\sin\omega t}{3}, \quad D^{(1)} = \frac{4\gamma\,(\sin\omega t - \omega\cos\omega t)}{15(1+\omega^2)}.$$

Finally, substitution of these expressions into (A18) yields

$$\langle V^{(2)}\rangle = \frac{8(1 - v'')\gamma\omega^2}{45(1+\omega^2)} - \frac{4a^{(1)}}{3\varepsilon}. \tag{A19}$$

This is the 2D equivalent of the 3D expression (49).

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[13] Note that $h(x)$ does not exactly vanish at the contact line (which is particularly visible at $\omega t = \pi/2$ and $\omega t = 3\pi/2$). This is because the solution presented is of the first order; i.e., it involves an error of $O(\varepsilon^2)$.