TO THE INTEGRABILITY OF THE EQUATIONS DESCRIBING THE LANGMUIR-WAVE–ION-ACOUSTIC-WAVE INTERACTION

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It is shown that the system of equations describing the Langmuir-wave–ion-acoustic-wave interaction is not integrable via an inverse scattering transform.

1. Let us consider the following system of equations:

$$\mathrm{i}\Psi_t + \Psi_{xx} - u\Psi = 0\,, \quad u_t - u_{xxx} + (\alpha u^2 + |\Psi|^2)_x = 0\,. \tag{1}$$

The system describes the interaction between high-frequency electron plasma oscillations and associated low-frequency ion density perturbations under the assumptions of cold ions and weak nonlinearity of ion density perturbations (α is a nonzero constant). In fact the system is universal in regard to its applications in physics. For the special case $\alpha = 3$ system (1) was studied in ref. [1] by Hirota's technique. It is interesting to verify the integrability of the system via an inverse scattering transform or, in other words, the existence of the additional motion invariants with the quadratic major terms. To investigate the problem we make use of the method proposed by Zakharov and Schulman in ref. [2] and employed in refs. [3,4]. Theoretical ground for the method is presented in ref. [5]. Following ref. [4] we describe the general scheme of the method as follows.

Firstly, we rewrite the given system of equations in the canonical hamiltonian form. Then a process of the lowest ν-order is considered, such that the corresponding set of dispersion laws of the system is nondegenerative (for details see ref. [2]). Finally, if the ν-order amplitude is nonzero on the resonant manifold of the above-mentioned process, the initial system of equations is not integrable via an inverse scattering transform.

2. Let us rewrite system (1) in the canonical hamiltonian form. To do this we perform the Fourier transform of u, Ψ, obtaining new variables u_k, Ψ_k and introduce a new variable a_k by

$$a_k = u_k\,\theta(k)/\sqrt{k}\,,$$

where $\theta(k)$ is the step function, $\theta(k) = 1$, if $k \geqslant 0$, and $\theta(k) = 0$, if $k < 0$.

In new variables system (1) has the canonical hamiltonian form:

$$\dot{\Psi}_k + \mathrm{i}\delta H/\delta\Psi_k^* = 0\,, \quad \dot{a}_k + \mathrm{i}\delta H/\delta a_k^* = 0\,,$$

with hamiltonian

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$$H = \int \omega_k \Psi_k \Psi_k^* \mathrm{d}k + \int \Omega_k a_k a_k^* + \iiint U_{k_1 k_2 k_3}(a_{k_1}^* a_{k_2} a_{k_3} + a_{k_1} a_{k_2}^* a_{k_3}^*)\, \delta(k_1 - k_2 - k_3)\mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$$

$$+ \iiint V_{k_1 k_2 k_3}(\Psi_{k_1}^* a_{k_2} \Psi_{k_3} + \Psi_{k_1} a_{k_2}^* \Psi_{k_3}^*)\delta(k_1 - k_2 - k_3)\mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3\,, \qquad (2)$$

where

$$V_{k_1 k_2 k_3} = \theta(k_2)\sqrt{k_2/2\pi}\,, \quad U_{k_1 k_2 k_3} = \theta(k_1)\theta(k_2)\theta(k_3)\sqrt{k_1 k_2 k_3/2\pi}\;\alpha\,, \qquad \omega_k = k^2\,, \quad \Omega_k = k^3\,. \qquad (3,4)$$

The first nontrivial process in our case is the second-order process of scattering of two waves into two waves described by the following resonant conditions:

$$k_1 + k_2 = k_3 + k_4\,, \quad \omega(k_1) + \Omega(k_2) = \omega(k_3) + \Omega(k_4)\,. \qquad (5)$$

Eqs. (5) determine a two-dimensional manifold Γ in the four-dimensional space (k_1, k_2, k_3, k_4). One can obtain in the straightforward manner that in case (4) the manifold Γ can be parametrized in the following way:

$$k_1 = \tfrac{1}{2}(k_2^2 + k_2 k_4 + k_4^2 + k_4 - k_2)\,, \quad k_3 = \tfrac{1}{2}(k_2^2 + k_2 k_4 + k_4^2 + k_2 - k_4)\,. \qquad (6)$$

Let us now make sure that the set of dispersion laws (4) is nondegenerative with respect to the process (5).

To do this we should prove that every solution of the functional equation

$$f(k_1) + \varphi(k_2) = f(k_3) + \varphi(k_4) \qquad (7)$$

defined on the manifold Γ has the form:

$$f(k) = A\omega(k) + Bk + C\,, \quad \varphi(k) = A\Omega(k) + Bk + D\,.$$

A, B, C and D are arbitrary constants. Substituting (6) into (7), differentiating two times with respect to k_4, one time with respect to k_2 and assuming that $k_2 = k_4 = k$ and that $3k^2/2 = y$, we find: $f'''(y) = 0$, for all y, except possibly $y = 1/6$.

Similarily, if we substitute (6) into (7), differentiate four times with respect to k_4 and assume that $k_4 = k_2$ we find $\varphi'''(k) = 0$ for all k, except possibly, $k = \pm 1/3$.

Therefore

$$f(k) = Ak^2 + Bk + C\,, \quad \varphi(k) = Mk^3 + Nk^2 + Lk + D\,. \qquad (8)$$

Finally, substituting (8) into (7) on the manifold Γ we find after some simple calculations:

$$f(k) = Ak^2 + Bk + C\,, \quad \varphi(k) = Ak^3 + Bk + D\,, \qquad (9)$$

where A, B, C and D are arbitrary constants. Thus we have proved the nondegenerativeness of the set of dispersion laws (4).

3. Applying the perturbation theory [6] to the hamiltonian (2) and summarising the second-order terms, we obtain the amplitude of the process (5). Simple but rather extensive calculations result in

$$T_{k_1 k_2 k_3 k_4} = 2\left(\frac{V_{k_1+k_2,k_2,k_1} V_{k_3+k_4,k_4,k_3}}{\omega_{k_1} + \Omega_{k_2} - \omega_{k_1+k_2}} + \frac{V_{k_1,k_4,k_1-k_4} V_{k_3,k_2,k_3-k_2}}{\omega_{k_1} - \Omega_{k_4} - \omega_{k_1-k_4}}\right)$$

$$+ 4\left(\frac{V_{k_1,k_1-k_3,k_3} U_{k_4,k_2,k_4-k_2}}{\Omega_{k_4} - \Omega_{k_2} - \Omega_{k_4-k_2}} + \frac{V_{k_3,k_3-k_1,k_1} U_{k_2,k_4,k_2-k_4}}{\Omega_{k_2} - \Omega_{k_4} - \Omega_{k_2-k_4}}\right)\,. \qquad (10)$$

Substituting (3), (4), (6) into (10) one can see that the amplitude T is nonzero on the resonant manifold Γ. Hence, system (1) does not have any additional motion invariants quadratic at small amplitudes and in this case an inverse scattering transform is inapplicable. Therefore system (1) does not have an n-soliton solution. One can show by direct substitution that the n-soliton solution of system (1) ($\alpha = 3$) found in ref. [1] is not correct.

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References

[1] Y.C. Ma, Stud. Appl. Math. 60 (1979) 73.
[2] V.E. Zakharov and E.I. Schulman, Physica 1D (1980) 192.
[3] V.E. Zakharov and E.I. Schulman, Physica 4D (1982) 270.
[4] E.I. Schulman, Dokl. Akad. Nauk SSSR 259 (1982) 579.
[5] V.E. Zakharov, Proc. Congress on Mathematical physics (Springer, Berlin, 1982).
[6] V.E. Zakharov, Izv. Vuzov SSSR Radiofiz. 17 (1974) 431; 18 (1975) 1470.