A note on the diffusion in fluids with wave-induced random motion. Applications to the barotropic Rossby waves

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An asymptotic theory, describing turbulent diffusion due to wave-induced random motion in incompressible or compressible fluids, is constructed. It is shown that even *weakly* nonlinear waves cause irreversible stretching of material lines. The results obtained are applied to the Rossby-wave-induced motion in the atmosphere or ocean. An expression for the effective coefficient of diffusivity is calculated, which indicates that the diffusion due to Rossby waves can be strongly anisotropic even for an isotropic wave spectrum.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

The transport of a passive tracer in a fluid with negligible molecular diffusion is governed by the following equation:

$$\frac{\partial c}{\partial t} + \nabla_j c u_j = 0, \tag{1}$$

where t and $\mathbf{x} = (x,y,z)$ are the time and the space coordinates, $c(\mathbf{x},t)$ is the concentration of the tracer and $u_j(\mathbf{x},t)$ is the velocity of the fluid $(j = x,y,z)$. The case of an incompressible fluid $(\nabla_j u_j = 0)$ will be considered first.

Let $u_j(\mathbf{x},t)$ be a random function with nonzero mean:

$$c = \langle c \rangle + c', \tag{2a}$$

where $\langle\ \rangle$ denotes averaging over the ensemble of realization, $\langle u_j' \rangle = 0$. Similarly,

$$u_j = \langle u_j \rangle + u_j'. \tag{2b}$$

Substitution of (2) into (1) and averaging yield

$$\frac{\partial \langle c \rangle}{\partial t} + \nabla_j(\langle c \rangle \langle u_j \rangle) + \langle c' u_j' \rangle) = 0. \tag{3}$$

Evidently, this equation alone cannot determine all the unknown functions $(\langle c \rangle, \langle c' u_j' \rangle)$; and to close it, we must use some hypothesis, relating the turbulent flux of the tracer to its mean concentration. The simplest example of such relation is

$$\langle c' u_j' \rangle = K_{ji} \nabla_i \langle c \rangle, \tag{4}$$

where K_{ji} is the so-called turbulent diffusivity tensor. The relationship (4) was originally suggested by Taylor[1] for the case of spatially homogeneous stationary turbulence in an incompressible fluid. Later Batchelor,[2] assuming the Gaussian distribution of the displacement of the tracer particles, obtained the following expression for the diffusivity tensor:

$$K_{ji} = \frac{1}{2} \int_0^\infty [B_{ji}^{(L)}(\tau) + B_{ij}^{(L)}(\tau)]\, d\tau, \tag{5}$$

where $B_{ji}^{(L)}$ is the so-called Lagrangian correlation function:

$$B_{ji}^{(L)}(\tau) = \langle u_j(\mathbf{X}(t),t) u_i(\mathbf{X}(t+\tau),t+\tau) \rangle;$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X},t), \quad \mathbf{X}(0) = \mathbf{x}.$$

Unfortunately, theoretical or experimental determination of this quantity is a very complicated problem; but if the fluid motion is induced by *small-amplitude waves*, $B_{ji}^{(L)}$ can be approximated by the Eulerian (conventional) correlation function:

$$B_{ij}^{(L)}(\tau) \approx B_{ij}^{(E)}(\tau) = \langle u_i'(\mathbf{x},t) u_j'(\mathbf{x},t+\tau) \rangle \tag{6}$$

(e.g., Phythian,[3] Lundgren and Pointin,[4] and Weinstock[5]). Formulas (5) and (6) have also been applied to a *deterministic* or even deterministic and *steady* motion (Andrews and McIntyre,[6] Plumb,[7] and Middleton and Loder[8]), while Mclaughlin *et al.*[9] modified it to describe the case of *short-scale* steady random velocity fields.

It is worth noting that all the papers mentioned above[1–9] considered *spatially homogeneous* turbulence in *incompressible* fluids. In addition to this, authors,[1–4] assumed the probability distribution of the fluid velocity (or the displacement of the tracer particles) to be Gaussian.

The present paper is devoted to the calculation of the effective diffusivity coefficient for the fluids with random wave-induced motion. Within the context of geophysical application, the correlation period of the wave-induced motion is likely to be much smaller than the time scale of the diffusion, while its spatial scale is usually of the same order as the diffusion spatial scale. These conditions enable us to calculate the diffusivity tensor using a direct perturbation technique, which is not based on any *a priori* assumption concerning probability distribution of any quantity and can be applied to the case of compressible fluids as well.

As a particular example, we consider fluid motion induced by a random spectrum of *Rossby waves*. It should be emphasized, that from the geophysical point of view these waves play an important role in the large- and mesoscale (100–1000 km) dynamics of the Earth's atmosphere and oceans (cf. Pedlosky[10]). We shall consider the simplest—*barotropic*—model of Rossby waves, where the fluid velocity does not depend on the vertical dimension and the "*effec-*

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tive" number of spatial variables is reduced to two. In addition, we shall assume the wave amplitude to be small and use the so-called *weakly nonlinear* approach to wave turbulence (e.g., Reznik and Soomere[11] and Zakharov[12]). With the help of this simplified model we hope (i) to obtain quantitative understanding of the horizontal structure of the atmospheric/oceanic diffusion and (ii) to develop a general approach to the calculation of diffusivity coefficient for fluids with weakly nonlinear wave motion.

II. TURBULENT DIFFUSION IN FLUIDS WITH WAVE-INDUCED RANDOM MOTION

A. The case of incompressible fluid

The nondimensional variables should be introduced as follows:

$$c=\tilde{c},\quad \mathbf{x}=\frac{\tilde{\mathbf{x}}}{L_{(d)}},\quad t=\frac{\tilde{t}}{T_{(t)}},\quad u_j=\frac{\tilde{u}_j}{U}, \tag{7}$$

where $L_{(d)}$ is the characteristic spatial scale of the diffusion, $T_{(t)}$ is the characteristic wave period, and U is the rms of the turbulent velocity. Substitution of (7) into (1) yields (tildes are omitted)

$$\frac{\partial c}{\partial t}+\epsilon\nabla_j c u_j=0,$$

where $\epsilon=UT_{(t)}/L_{(d)}$. We shall consider the case $\epsilon\ll 1$; i.e., $T_{(d)}\gg T_{(t)}$, where $T_{(d)}$ is the time scale of the diffusion.

Evidently, the fluctuations of the concentration are of the order of ϵ:

$$c=\langle c\rangle+\epsilon c',$$

while the mean component of the velocity of the fluid is assumed to be equal to zero: $\langle u_j\rangle=0$. The equations, governing the mean concentration and its fluctuations, are

$$\frac{\partial\langle c\rangle}{\partial t}+\epsilon^2\nabla_j\langle c'u_j'\rangle=0, \tag{8}$$

$$\frac{\partial c'}{\partial t}=-u_j'\nabla_j\langle c\rangle-\epsilon\nabla_j(c'u_j'-\langle c'u_j'\rangle). \tag{9a}$$

The latter equation should be supplemented with the initial condition

$$c'(\mathbf{x},0)=0. \tag{9b}$$

The solution of the Cauchy problem (9) can be found in the form of an asymptotic series

$$c'=c'^{(0)}+\epsilon c'^{(1)}+..., \tag{10}$$

where

$$c'^{(0)}(\mathbf{x},t)=-\int_0^t u_j'(\mathbf{x},\tau)\nabla_j\langle c(\mathbf{x},\tau)\rangle d\tau, \tag{11a}$$

$$c'^{(m+1)}(\mathbf{x},t)=-\int_0^t\nabla_j[u_j'(\mathbf{x},\tau)c'^{(m)}(\mathbf{x},\tau)-\langle u_j'(\mathbf{x},\tau)c'^{(m)}(\mathbf{x},\tau)\rangle]d\tau. \tag{11b}$$

Substitution of (11) and (10) into (8) yields a closed-form equation governing $\langle c\rangle$. Taking into account the lowest-order term only, we have

$$\frac{\partial\langle c(\mathbf{x},t)\rangle}{\partial t}-\epsilon^2\nabla_j\int_0^t\langle u_j'(\mathbf{x},t)u_i'(\mathbf{x},\tau)\rangle\times\nabla_i\langle c(\mathbf{x},\tau)\rangle d\tau=O(\epsilon^3). \tag{12}$$

Then, using the Eulerian correlation function (6) and the "slow" time variable $T=\epsilon^2 t$, we rewrite Eq. (12) as

$$\frac{\partial\langle c(\mathbf{x},T)\rangle}{\partial T}-\nabla_j\int_{-T/\epsilon^2}^0 B_{ji}^{(\mathrm{E})}(\mathbf{x},\tau)\times\nabla_i\langle c(\mathbf{x},T+\epsilon^2\tau)\rangle d\tau=O(\epsilon).$$

Finally, taking the limit $\epsilon\to 0$, we obtain the desired equation governing $\langle c(\mathbf{x},T)\rangle$:

$$\frac{\partial\langle c\rangle}{\partial T}=\nabla_j K_{ji}\nabla_i\langle c\rangle, \tag{13}$$

where

$$K_{ji}(\mathbf{x})=\int_0^\infty B_{ji}^{(\mathrm{E})}(\mathbf{x},\tau)d\tau \tag{14}$$

[we have taken into account here that $B_{ji}^{(\mathrm{E})}(\mathbf{x},\tau)$ is an even function of τ]. As could be expected beforehand, the diffusivity coefficient (14) coincides with the "exact" expression (5) with the only difference being in the particular expression for the correlation function. This difference is significant, since, in contrast with $B_{ji}^{(\mathrm{L})}(\mathbf{x},\tau)$, the Eulerian correlation function could be more easily determined in a laboratory or field experiment.

B. The case of compressible fluid

The case of a compressible fluid ($\nabla_j u_j\neq 0$) is more complicated. The compressibility-modified diffusion equation appears to be

$$\frac{\partial\langle c\rangle}{\partial T}=\nabla_j[(K_{ji}\nabla_i+P_j)]\langle c\rangle, \tag{15}$$

where

$$P_j(\mathbf{x})=\int_{-\infty}^0\langle u_j'(\mathbf{x},t)\nabla_i u_i'(\mathbf{x},t+\tau)\rangle d\tau.$$

If the wave spectrum is anisotropic, $P_j\neq 0$, and the effect of compressibility can strongly influence the process of turbulent diffusion [compare (13) and (15)].

III. AN EXAMPLE: LINEAR BAROTROPIC ROSSBY WAVES

The equation governing barotropic Rossby waves is (cf. Pedlosky[10])

$$\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial\Delta\Psi}{\partial x}+\beta\frac{\partial\Psi}{\partial x}=0, \tag{16}$$

where Ψ is the streamfunction:

$$u_x'=-\frac{\partial\Psi}{\partial y},\quad u_y'=\frac{\partial\Psi}{\partial x},\quad u_z'=0; \tag{17}$$

β is the gradient of Coriolis parameter and Δ is the two-dimensional Laplace operator. The general solution to the linearized equation (16) can be written in the form of a Fourier integral:

$$\Psi(\mathbf{x},t)=\int\left(\frac{1}{k}\right)a_{\mathbf{k}}e^{i\omega t-i\mathbf{k}\mathbf{x}}d\mathbf{k},$$

where $\mathbf{k} = (k_x, k_y)$ is the wave vector, $\omega(\mathbf{k}) = -\beta k_x/k^2$ is the frequency of Rossby waves, $k = |\mathbf{k}|$, and $a_\mathbf{k}$ is a symmetric function of $\mathbf{k}$:

$$a_{-\mathbf{k}} = a_\mathbf{k}^*, \tag{18}$$

where the asterisk denotes the complex conjugate. For simplicity, we assume that the wave field is spatially homogeneous, i.e., $\langle a_\mathbf{k} a_{\mathbf{k}'}\rangle = E_\mathbf{k}\delta(\mathbf{k}+\mathbf{k}')$, where $\delta(\mathbf{k})$ is the Dirac delta function and $E(\mathbf{k})$ can be interpreted as the energy spectrum of Rossby waves. The equality (18) yields

$$E_{-\mathbf{k}} = E_\mathbf{k}. \tag{19}$$

In terms of $E_\mathbf{k}$, the correlation matrix of the fluid velocity, derived from Eq. (14) and expressed in terms of $E_\mathbf{k}$, is

$$B_{xx} = \int \left(\frac{k_y}{k}\right)^2 E_\mathbf{k} \cos(\omega\tau)\, d\mathbf{k},$$

$$B_{yy} = \int \left(\frac{k_x}{k}\right)^2 E_\mathbf{k} \cos(\omega\tau)\, d\mathbf{k},$$

$$B_{xy} = -\int \left(\frac{k_x k_y}{k^2}\right) E_\mathbf{k} \cos(\omega\tau)\, d\mathbf{k}.$$

Then, substituting B_{ji} into (14) and making use of the formula

$$\int_0^\infty \cos(\omega\tau)\, d\tau = \pi\delta(\omega),$$

we obtain

$$K_{xx} = \frac{\pi}{2\beta}\int_{-\infty}^{\infty} (k_y)^2 E_\mathbf{k}\big|_{k_x=0}\, dk_y, \quad K_{xy} = K_{yy} = 0.$$

One can see that (i) linear Rossby waves do not transfer the tracer along the north–south (meridional) direction; and (ii) the main contribution into the diffusion process is given by waves of zero frequency. The physical meaning of these facts will be discussed in the next section.

IV. DISCUSSION

(1) Expressions for high-order corrections to the diffusion equation (13) can be classified into three groups:

$$\frac{\partial\langle c\rangle}{\partial T} = \nabla_j K_{ji}\nabla_i\langle c\rangle + G_1 + G_2 + G_3.$$

Here

$$G_1 = -\nabla_j \int_{-\infty}^{-T/\epsilon^2} B_{ji}(\tau)\, d\tau\, \nabla_i \langle c(T)\rangle + ..., \tag{20a}$$

$$G_2 = -\epsilon^2 \nabla_j \int_0^\infty B_{ji}(\tau)\tau\, d\tau\, \nabla_i \frac{\partial\langle c(T)\rangle}{\partial T} + ..., \tag{20b}$$

$$G_3 = \epsilon\nabla_j (Q_{ji}\nabla_i + M_{jin}\nabla_i\nabla_n)\langle c(T)\rangle + ..., \tag{20c}$$

where

$$Q_{ji} = \int_{-\infty}^0\int_{-\infty}^0 \langle u_j'(t) u_n'(t+\tau_1) \times \nabla_n u_i'(t+\tau_1+\tau_2)\rangle\, d\tau_1\, d\tau_2, \tag{21a}$$

$$M_{jin} = \int_{-\infty}^0\int_{-\infty}^0 \langle u_j'(t) u_n'(t+\tau_1) \times u_i'(t+\tau_1+\tau_2)\rangle\, d\tau_1\, d\tau_2. \tag{21b}$$

The corrections G_1 are associated with the fact that random fluctuations c' for a certain length of time "remember" the initial condition (9b). With time elapsing these corrections vanish (i.e., $G_1 \to 0$ with $T \to \infty$). The corrections G_2 describe weak transfer of the tracer by waves of *finite* frequency (as we saw in the previous section, the main contribution into the diffusion process is given by waves of *zero* frequency). Finally, G_3 takes into account high-order corrections $c'^{(m)}$, $m \geqslant 2$ and corresponds to the correlations in higher moments of fluid velocity field.

(2) Introducing the nondimensional parameter $\delta = L_{(t)}/L_{(d)}$ ($L_{(t)}$ is the correlation radius of the turbulence and $L_{(d)}$ is the spatial scale of the diffusion), we estimate the first term in the expression for G_3 as [cf. (20c) and (21a)]

$$Q_{ji} \sim 1/\delta, \quad G_3 \sim \epsilon/\delta.$$

Evidently, *if the spatial scale of the turbulent motion is sufficiently small* ($\delta \leqslant \epsilon$), *our expansion is invalid* [$G_3 = O(1)$]. This is reasonable, since for the case of spatially homogeneous "strong" turbulence in incompressible fluids we could obtain nothing but the equalities (4) and (5).

(3) As we have seen above, *the diffusivity coefficient is determined by waves with zero frequency*. Indeed, in the case of linear-wave-induced motion of high frequency, the orbits of fluid particles are closed and the distribution of the tracer is exposed to fast periodic distortion, resulting in no mixing (diffusion) at all. Thus, waves with periods smaller than the time scale of the diffusion cannot interact with slow changes in the tracer concentration field and produce nontrivial transfer of the tracer particles. Bearing in mind that $T_{(d)} \gg T_{(t)}$, we can see that only *infinitely* slow waves (i.e., waves with zero frequency) can contribute to the diffusivity coefficient within the framework of the *zeroth* order of the perturbation scheme. The corrections associated with the contribution of finite-frequency waves are taken into account in the higher orders.

(4) It is worth noting that the numerical simulation of the kinetic equation for weak Rossby wave turbulence (cf. Reznik and Soomere[11]) demonstrated that the zero-frequency component of the spectrum has a relatively high level and, consequently, can cause strong mixing.

(5) We can also provide a simple physical explanation of the zero values of diffusivity coefficient responsible for the meridional diffusion in the above example. Apparently, they vanish because *the meridional component of zero-frequency wave-induced velocity is equal to zero* (zero-frequency Rossby waves correspond to a *zonal* flow with random profile).

V. AN EXAMPLE: WEAKLY NONLINEAR BAROTROPIC ROSSBY WAVES

The difference between linear and (weakly) nonlinear wave spectra is that the latter comprises the so-called combinational harmonics. In contrast with the ordinary (resonant) waves, frequencies and wave vectors of these harmonics do not satisfy the dispersion relation, but are "composed" of frequencies and wave vectors of resonant waves: $\tilde{\mathbf{k}} = \Sigma\mathbf{k}_m$, $\tilde{\omega} = \Sigma\omega(\mathbf{k}_m)$. Accordingly, a weakly nonlinear

Rossby wave field is described by the following expression (cf. the Appendix):

$$\Psi(x,t) = \int \left(\frac{1}{k}\right)\left(a_{\mathbf{k}} \exp(i\omega t) + \iint iU^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \times \exp[i(\omega_1+\omega_2)t]\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2 + \iiint U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} \exp[i(\omega_1+\omega_2+\omega_3)t] \cdot\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3 + \cdots\right) \times \exp(-i\mathbf{k}\mathbf{x})\,d\mathbf{k}, \tag{22}$$

where $U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ are symmetric functions:

$$U^{(1)}_{\mathbf{k}\mathbf{k}_2\mathbf{k}_1} = U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}, \tag{23a}$$

$$U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_2} = U^{(2)}_{\mathbf{k}\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_2}. \tag{23b}$$

In addition, the condition of $\Psi(\mathbf{x},t)$ being real yields

$$U^{(1)}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} = -U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}, \qquad U^{(2)}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}} \tag{24}$$

(the explicit expressions for $U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ are given in the Appendix). Substituting (22) and (17) into the expression for the correlation function (14), we can split the fourth-order moments of $a_{\mathbf{k}}$ through the moments of the second order:

$$\langle a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}'_1} a_{\mathbf{k}'_2}\rangle = E_{\mathbf{k}_1} E_{\mathbf{k}_2}[\delta(\mathbf{k}_1+\mathbf{k}'_1)\delta(\mathbf{k}_2+\mathbf{k}'_2) + \delta(\mathbf{k}_1+\mathbf{k}'_2)\delta(\mathbf{k}_2+\mathbf{k}'_1)] + E_{\mathbf{k}_1} E_{\mathbf{k}'_2}\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\mathbf{k}'_1+\mathbf{k}'_2).$$

This procedure is justified because the distribution of any weakly nonlinear wave field is close to Gaussian distribution (e.g., Zakharov[12]). Using (19), (23), (24), and the equality corresponding to the zero value of the Rossby-wave Stokes drift,

$$U^{(1)}_{0,\mathbf{k},-\mathbf{k}} = 0,$$

we obtain

$$B_{yy}(\tau) = \iint \left[2\left(\frac{k_{x1}+k_{x2}}{|\mathbf{k}_1+\mathbf{k}_2|}\right)^2 (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \times \cos[(\omega_1+\omega_2)\tau] + [W_{\mathbf{k}_1\mathbf{k}_2}\cos(\omega_1\tau) + W_{\mathbf{k}_2\mathbf{k}_1}\cos(\omega_2\tau)]\right] E_{\mathbf{k}_1} E_{\mathbf{k}_2}\,d\mathbf{k}_1 d\mathbf{k}_2 + \dots,$$

where

$$W_{\mathbf{k}_1\mathbf{k}_2} = 3(k_{x1}/k_1)^2 U^{(2)}_{\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_2}.$$

Evidently, $W_{\mathbf{k}_1\mathbf{k}_2}\delta(\omega_1) = 0$; and after the substitution of B_{yy} into the expression for diffusivity coefficient (14), the second term in B_{yy} turns into zero:

$$K_{yy} = 2\pi \iint \left(\frac{k_{x1}+k_{x2}}{|\mathbf{k}_1+\mathbf{k}_2|}\right)^2 (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \cdot E_{\mathbf{k}_1} E_{\mathbf{k}_2}\delta(\omega_1+\omega_2)\,d\mathbf{k}_1\,d\mathbf{k}_2. \tag{25a}$$

Similarly,

$$K_{xy} = -2\pi \iint \left(\frac{(k_{x1}+k_{x2})(k_{y1}+k_{y2})}{|\mathbf{k}_1+\mathbf{k}_2|^2}\right) \times (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \cdot E_{\mathbf{k}_1} E_{\mathbf{k}_2}\delta(\omega_1+\omega_2)\,d\mathbf{k}_1\,d\mathbf{k}_2. \tag{25b}$$

Bearing in mind that Rossby waves, as a dynamic system, are isotropic in respect to the inversion of the y axis:

$$\omega(k_x,-k_y) = \omega(k_x,k_y),$$

$$U^{(1)}_{k_x,-k_y;k_{x1},-k_{y1};k_{x2},-k_{y2}} = -U^{(1)}_{k_x,k_y;k_{x1},k_{y1};k_{x2},k_{y2}}$$

[cf. (A4)]; we can see that $K_{xy} = 0$ for all symmetric $[E(k_x,-k_y) = E(k_x,k_y)]$ wave spectra.

VI. DISCUSSION

(1) The expressions (25) for the diffusivity coefficients have clear physical meaning. Indeed, the integrals in (25) sum up the contributions of all second-order combinational harmonics, while the corresponding delta functions "cut out" the harmonics with zero frequency only. Since the zero-frequency combinational harmonics can be formed by pairs of linear waves with *nonzero* frequencies, *the nonlinear corrections to the diffusivity tensor depend on the whole wave spectrum*.

(2) It must be emphasized that in the case of weakly nonlinear waves, the *meridional* mixing is much weaker than the *zonal* mixing.

(3) The nonzero values of the diffusivity coefficients indicate *the irreversible stretching of material lines*, caused by Rossby waves. This conclusion seems to be rather important for the understanding of the Rossby wave turbulence itself.

(4) It is also worth noting that the diffusion due to linear or nonlinear Rossby waves differs significantly from that caused by steady waves. In the latter case (studied by Andrews and McIntyre,[6] Plumb,[7] and Middleton and Loder[8]), the mixing is produced by the Stokes drift, which is, in the instance of Rossby waves, equal to zero.

VII. CONCLUDING REMARKS

Two problems have been discussed in this paper.

First, we have developed an asymptotic technique for the calculation of the coefficient of diffusion due to random-wave-induced motion in fluids. In contrast with the methods developed by the previous authors, this technique permits us to calculate the higher-order corrections to the diffusivity coefficient and can be applied to compressible fluids as well.

Second, we have calculated the diffusivity coefficient for the case, where the fluid motion is induced by linear or weakly nonlinear spectra of barotropic Rossby waves. It was shown that *the diffusion due to Rossby-wave-induced motion is strongly anisotropic*, the zonal diffusivity coefficient being much greater than the meridional one.

We emphasize that the results obtained can be easily generalized for other types of random wave motion (such as surface or internal gravity waves). In particular, the important conclusion about *the irreversible stretching of material lines by Rossby wave turbulence* can be applied to any nonlinear-wave-induced motion (even if the latter does not comprise a shear-flow component). The effect of weak nonlinearity seems to be sufficient for this phenomenon to occur.

APPENDIX: WEAKLY NONLINEAR SPECTRUM OF ROSSBY WAVES

In terms of the Fourier transform

$$\Psi_{\mathbf{k}}(t) = (2\pi)^{-2} k \int \Psi(\mathbf{x},t) \exp(i\mathbf{k}\mathbf{x}) d\mathbf{x},$$

the Rossby wave equation can be written as follows:

$$\frac{\partial}{\partial t} \Psi_{\mathbf{k}} - i\omega \Psi_{\mathbf{k}} = \iint V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \Psi_{\mathbf{k}_1} \Psi_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 \, d\mathbf{k}_2, \tag{A1}$$

where

$$V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{1}{2}\left(\frac{1}{k}\right)(k_{x1}k_{y2} - k_{y1}k_{x2})\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right).$$

Equation (A1) can be solved via iterations:

$$\Psi_{\mathbf{k}} = \Psi_{\mathbf{k}}^{(0)} + \Psi_{\mathbf{k}}^{(1)} + \dots, \tag{A2}$$

$$\Psi_{\mathbf{k}}^{(0)} = a_{\mathbf{k}} \exp(i\omega t). \tag{A3a}$$

The first iteration is

$$\Psi_{\mathbf{k}}^{(1)} = \iint i(\omega - \omega_1 - \omega_2)^{-1} V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \cdot \exp[i(\omega_1 + \omega_2)t] \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 \, d\mathbf{k}_2. \tag{A3b}$$

Comparing (A2) and (A3) to (22), we can see that

$$U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{(1)} = (\omega - \omega_1 - \omega_2)^{-1} V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}. \tag{A4}$$

Evidently, if $\omega - \omega_1 - \omega_2 \to 0$, $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{(1)} \to \infty$. This singularity corresponds to strong resonances between Rossby wave triplets and has nothing to do with combinational harmonics of zero frequency. Correspondingly, if $\omega_1 = -\omega_2$ and $\omega(\mathbf{k}_1 + \mathbf{k}_2) = 0$, the singularity disappears.

The coefficient $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{(2)}$ can be calculated in a similar way.

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A note on the diffusion in fluids with wave-induced random motion. Applications to the barotropic Rossby waves

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An asymptotic theory, describing turbulent diffusion due to wave-induced random motion in incompressible or compressible fluids, is constructed. It is shown that even *weakly* nonlinear waves cause irreversible stretching of material lines. The results obtained are applied to the Rossby-wave-induced motion in the atmosphere or ocean. An expression for the effective coefficient of diffusivity is calculated, which indicates that the diffusion due to Rossby waves can be strongly anisotropic even for an isotropic wave spectrum.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

The transport of a passive tracer in a fluid with negligible molecular diffusion is governed by the following equation:

$$\frac{\partial c}{\partial t} + \nabla_j c u_j = 0, \qquad (1)$$

where t and $\mathbf{x} = (x,y,z)$ are the time and the space coordinates, $c(\mathbf{x},t)$ is the concentration of the tracer and $u_j(\mathbf{x},t)$ is the velocity of the fluid ($j = x,y,z$). The case of an incompressible fluid ($\nabla_j u_j = 0$) will be considered first.

Let $u_j(\mathbf{x},t)$ be a random function with nonzero mean:

$$c = \langle c \rangle + c', \qquad (2a)$$

where $\langle\ \rangle$ denotes averaging over the ensemble of realization, $\langle u_j' \rangle = 0$. Similarly,

$$u_j = \langle u_j \rangle + u_j'. \qquad (2b)$$

Substitution of (2) into (1) and averaging yield

$$\frac{\partial \langle c \rangle}{\partial t} + \nabla_j(\langle c \rangle \langle u_j \rangle) + \langle c' u_j' \rangle) = 0. \qquad (3)$$

Evidently, this equation alone cannot determine all the unknown functions ($\langle c \rangle$, $\langle c' u_j' \rangle$); and to close it, we must use some hypothesis, relating the turbulent flux of the tracer to its mean concentration. The simplest example of such relation is

$$\langle c' u_j' \rangle = K_{ji} \nabla_i \langle c \rangle, \qquad (4)$$

where K_{ji} is the so-called turbulent diffusivity tensor. The relationship (4) was originally suggested by Taylor[1] for the case of spatially homogeneous stationary turbulence in an incompressible fluid. Later Batchelor,[2] assuming the Gaussian distribution of the displacement of the tracer particles, obtained the following expression for the diffusivity tensor:

$$K_{ji} = \frac{1}{2} \int_0^\infty [B_{ji}^{(L)}(\tau) + B_{ij}^{(L)}(\tau)]\, d\tau, \qquad (5)$$

where $B_{ji}^{(L)}$ is the so-called Lagrangian correlation function:

$$B_{ji}^{(L)}(\tau) = \langle u_j(\mathbf{X}(t),t) u_i(\mathbf{X}(t+\tau),t+\tau) \rangle;$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X},t), \quad \mathbf{X}(0) = \mathbf{x}.$$

Unfortunately, theoretical or experimental determination of this quantity is a very complicated problem; but if the fluid motion is induced by *small-amplitude waves*, $B_{ji}^{(L)}$ can be approximated by the Eulerian (conventional) correlation function:

$$B_{ij}^{(L)}(\tau) \approx B_{ij}^{(E)}(\tau) = \langle u_i'(\mathbf{x},t) u_j'(\mathbf{x},t+\tau) \rangle \qquad (6)$$

(e.g., Phythian,[3] Lundgren and Pointin,[4] and Weinstock[5]). Formulas (5) and (6) have also been applied to a *deterministic* or even deterministic and *steady* motion (Andrews and McIntyre,[6] Plumb,[7] and Middleton and Loder[8]), while Mclaughlin *et al.*[9] modified it to describe the case of *short-scale* steady random velocity fields.

It is worth noting that all the papers mentioned above[1–9] considered *spatially homogeneous* turbulence in *incompressible* fluids. In addition to this, authors,[1–4] assumed the probability distribution of the fluid velocity (or the displacement of the tracer particles) to be Gaussian.

The present paper is devoted to the calculation of the effective diffusivity coefficient for the fluids with random wave-induced motion. Within the context of geophysical application, the correlation period of the wave-induced motion is likely to be much smaller than the time scale of the diffusion, while its spatial scale is usually of the same order as the diffusion spatial scale. These conditions enable us to calculate the diffusivity tensor using a direct perturbation technique, which is not based on any *a priori* assumption concerning probability distribution of any quantity and can be applied to the case of compressible fluids as well.

As a particular example, we consider fluid motion induced by a random spectrum of *Rossby waves*. It should be emphasized, that from the geophysical point of view these waves play an important role in the large- and mesoscale (100–1000 km) dynamics of the Earth's atmosphere and oceans (cf. Pedlosky[10]). We shall consider the simplest—*barotropic*—model of Rossby waves, where the fluid velocity does not depend on the vertical dimension and *the "effec-*

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tive" number of spatial variables is reduced to two. In addition, we shall assume the wave amplitude to be small and use the so-called *weakly nonlinear* approach to wave turbulence (e.g., Reznik and Soomere[11] and Zakharov[12]). With the help of this simplified model we hope (i) to obtain quantitative understanding of the horizontal structure of the atmospheric/oceanic diffusion and (ii) to develop a general approach to the calculation of diffusivity coefficient for fluids with weakly nonlinear wave motion.

II. TURBULENT DIFFUSION IN FLUIDS WITH WAVE-INDUCED RANDOM MOTION

A. The case of incompressible fluid

The nondimensional variables should be introduced as follows:

$$c=\tilde{c},\quad \mathbf{x}=\frac{\tilde{\mathbf{x}}}{L_{(d)}},\quad t=\frac{\tilde{t}}{T_{(t)}},\quad u_j=\frac{\tilde{u}_j}{U},\tag{7}$$

where $L_{(d)}$ is the characteristic spatial scale of the diffusion, $T_{(t)}$ is the characteristic wave period, and U is the rms of the turbulent velocity. Substitution of (7) into (1) yields (tildes are omitted)

$$\frac{\partial c}{\partial t}+\epsilon\nabla_j c u_j=0,$$

where $\epsilon=UT_{(t)}/L_{(d)}$. We shall consider the case $\epsilon\ll 1$; i.e., $T_{(d)}\gg T_{(t)}$, where $T_{(d)}$ is the time scale of the diffusion.

Evidently, the fluctuations of the concentration are of the order of ϵ:

$$c=\langle c\rangle+\epsilon c',$$

while the mean component of the velocity of the fluid is assumed to be equal to zero: $\langle u_j\rangle=0$. The equations, governing the mean concentration and its fluctuations, are

$$\frac{\partial\langle c\rangle}{\partial t}+\epsilon^2\nabla_j\langle c'u_j'\rangle=0,\tag{8}$$

$$\frac{\partial c'}{\partial t}=-u_j'\nabla_j\langle c\rangle-\epsilon\nabla_j(c'u_j'-\langle c'u_j'\rangle).\tag{9a}$$

The latter equation should be supplemented with the initial condition

$$c'(\mathbf{x},0)=0.\tag{9b}$$

The solution of the Cauchy problem (9) can be found in the form of an asymptotic series

$$c'=c'^{(0)}+\epsilon c'^{(1)}+\ldots,\tag{10}$$

where

$$c'^{(0)}(\mathbf{x},t)=-\int_0^t u_j'(\mathbf{x},\tau)\nabla_j\langle c(\mathbf{x},\tau)\rangle d\tau,\tag{11a}$$

$$c'^{(m+1)}(\mathbf{x},t)=-\int_0^t\nabla_j\left[u_j'(\mathbf{x},\tau)c'^{(m)}(\mathbf{x},\tau)-\langle u_j'(\mathbf{x},\tau)c'^{(m)}(\mathbf{x},\tau)\rangle\right]d\tau.\tag{11b}$$

Substitution of (11) and (10) into (8) yields a closed-form equation governing $\langle c\rangle$. Taking into account the lowest-order term only, we have

$$\frac{\partial\langle c(\mathbf{x},t)\rangle}{\partial t}-\epsilon^2\nabla_j\int_0^t\langle u_j'(\mathbf{x},t)u_i'(\mathbf{x},\tau)\rangle\times\nabla_i\langle c(\mathbf{x},\tau)\rangle d\tau=O(\epsilon^3).\tag{12}$$

Then, using the Eulerian correlation function (6) and the "slow" time variable $T=\epsilon^2 t$, we rewrite Eq. (12) as

$$\frac{\partial\langle c(\mathbf{x},T)\rangle}{\partial T}-\nabla_j\int_{-T/\epsilon^2}^0 B_{ji}^{(E)}(\mathbf{x},\tau)\times\nabla_i\langle c(\mathbf{x},T+\epsilon^2\tau)\rangle d\tau=O(\epsilon).$$

Finally, taking the limit $\epsilon\to 0$, we obtain the desired equation governing $\langle c(\mathbf{x},T)\rangle$:

$$\frac{\partial\langle c\rangle}{\partial T}=\nabla_j K_{ji}\nabla_i\langle c\rangle,\tag{13}$$

where

$$K_{ji}(\mathbf{x})=\int_0^\infty B_{ji}^{(E)}(\mathbf{x},\tau)d\tau\tag{14}$$

[we have taken into account here that $B_{ji}^{(E)}(\mathbf{x},\tau)$ is an even function of τ]. As could be expected beforehand, the diffusivity coefficient (14) coincides with the "exact" expression (5) with the only difference being in the particular expression for the correlation function. This difference is significant, since, in contrast with $B_{ji}^{(L)}(\mathbf{x},\tau)$, the Eulerian correlation function could be more easily determined in a laboratory or field experiment.

B. The case of compressible fluid

The case of a compressible fluid ($\nabla_j u_j\neq 0$) is more complicated. The compressibility-modified diffusion equation appears to be

$$\frac{\partial\langle c\rangle}{\partial T}=\nabla_j[(K_{ji}\nabla_i+P_j)]\langle c\rangle,\tag{15}$$

where

$$P_j(\mathbf{x})=\int_{-\infty}^0\langle u_j'(\mathbf{x},t)\nabla_i u_i'(\mathbf{x},t+\tau)\rangle d\tau.$$

If the wave spectrum is anisotropic, $P_j\neq 0$, and the effect of compressibility can strongly influence the process of turbulent diffusion [compare (13) and (15)].

III. AN EXAMPLE: LINEAR BAROTROPIC ROSSBY WAVES

The equation governing barotropic Rossby waves is (cf. Pedlosky[10])

$$\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial\Delta\Psi}{\partial x}+\beta\frac{\partial\Psi}{\partial x}=0,\tag{16}$$

where Ψ is the streamfunction:

$$u_x'=-\frac{\partial\Psi}{\partial y},\quad u_y'=\frac{\partial\Psi}{\partial x},\quad u_z'=0;\tag{17}$$

β is the gradient of Coriolis parameter and Δ is the two-dimensional Laplace operator. The general solution to the linearized equation (16) can be written in the form of a Fourier integral:

$$\Psi(\mathbf{x},t)=\int\left(\frac{1}{k}\right)a_{\mathbf{k}}e^{i\omega t-i\mathbf{k}\mathbf{x}}d\mathbf{k},$$

where $\mathbf{k}=(k_x,k_y)$ is the wave vector, $\omega(\mathbf{k}) = -\beta k_x/k^2$ is the frequency of Rossby waves, $k=|\mathbf{k}|$, and $a_\mathbf{k}$ is a symmetric function of $\mathbf{k}$:

$$a_{-\mathbf{k}} = a_\mathbf{k}^*, \tag{18}$$

where the asterisk denotes the complex conjugate. For simplicity, we assume that the wave field is spatially homogeneous, i.e., $\langle a_\mathbf{k} a_{\mathbf{k}'}\rangle = E_\mathbf{k}\delta(\mathbf{k}+\mathbf{k}')$, where $\delta(\mathbf{k})$ is the Dirac delta function and $E(\mathbf{k})$ can be interpreted as the energy spectrum of Rossby waves. The equality (18) yields

$$E_{-\mathbf{k}} = E_\mathbf{k}. \tag{19}$$

In terms of $E_\mathbf{k}$, the correlation matrix of the fluid velocity, derived from Eq. (14) and expressed in terms of $E_\mathbf{k}$, is

$$B_{xx} = \int\left(\frac{k_y}{k}\right)^2 E_\mathbf{k}\cos(\omega\tau)d\mathbf{k},$$

$$B_{yy} = \int\left(\frac{k_x}{k}\right)^2 E_\mathbf{k}\cos(\omega\tau)d\mathbf{k},$$

$$B_{xy} = -\int\left(\frac{k_x k_y}{k^2}\right) E_\mathbf{k}\cos(\omega\tau)d\mathbf{k}.$$

Then, substituting B_{ji} into (14) and making use of the formula

$$\int_0^\infty \cos(\omega\tau)d\tau = \pi\delta(\omega),$$

we obtain

$$K_{xx} = \frac{\pi}{2\beta}\int_{-\infty}^{\infty}(k_y)^2 E_\mathbf{k}\,|_{k_x=0}\,dk_y, \quad K_{xy}=K_{yy}=0.$$

One can see that (i) linear Rossby waves do not transfer the tracer along the north–south (meridional) direction; and (ii) the main contribution into the diffusion process is given by waves of zero frequency. The physical meaning of these facts will be discussed in the next section.

IV. DISCUSSION

(1) Expressions for high-order corrections to the diffusion equation (13) can be classified into three groups:

$$\frac{\partial\langle c\rangle}{\partial T} = \nabla_j K_{ji}\nabla_i\langle c\rangle + G_1 + G_2 + G_3.$$

Here

$$G_1 = -\nabla_j\int_{-\infty}^{-T/\epsilon^2} B_{ji}(\tau)d\tau\nabla_i\langle c(T)\rangle + \ldots, \tag{20a}$$

$$G_2 = -\epsilon^2\nabla_j\int_0^\infty B_{ji}(\tau)\tau\,d\tau\,\nabla_i\frac{\partial\langle c(T)\rangle}{\partial T} + \ldots, \tag{20b}$$

$$G_3 = \epsilon\nabla_j(Q_{ji}\nabla_i + M_{jin}\nabla_i\nabla_n)\langle c(T)\rangle + \ldots, \tag{20c}$$

where

$$Q_{ji} = \int_{-\infty}^0\int_{-\infty}^0 \langle u'_j(t)u'_n(t+\tau_1) \times\nabla_n u'_i(t+\tau_1+\tau_2)\rangle d\tau_1\,d\tau_2, \tag{21a}$$

$$M_{jin} = \int_{-\infty}^0\int_{-\infty}^0 \langle u'_j(t)u'_n(t+\tau_1) \times u'_i(t+\tau_1+\tau_2)\rangle d\tau_1\,d\tau_2. \tag{21b}$$

The corrections G_1 are associated with the fact that random fluctuations c' for a certain length of time "remember" the initial condition (9b). With time elapsing these corrections vanish (i.e., $G_1\to 0$ with $T\to\infty$). The corrections G_2 describe weak transfer of the tracer by waves of *finite* frequency (as we saw in the previous section, the main contribution into the diffusion process is given by waves of *zero* frequency). Finally, G_3 takes into account high-order corrections $c'^{(m)}$, $m\geqslant 2$ and corresponds to the correlations in higher moments of fluid velocity field.

(2) Introducing the nondimensional parameter $\delta = L_{(t)}/L_{(d)}$ ($L_{(t)}$ is the correlation radius of the turbulence and $L_{(d)}$ is the spatial scale of the diffusion), we estimate the first term in the expression for G_3 as [cf. (20c) and (21a)]

$$Q_{ji}\sim 1/\delta, \quad G_3\sim\epsilon/\delta.$$

Evidently, *if the spatial scale of the turbulent motion is sufficiently small* ($\delta\leqslant\epsilon$), *our expansion is invalid* [$G_3 = O(1)$]. This is reasonable, since for the case of spatially homogeneous "strong" turbulence in incompressible fluids we could obtain nothing but the equalities (4) and (5).

(3) As we have seen above, *the diffusivity coefficient is determined by waves with zero frequency*. Indeed, in the case of linear-wave-induced motion of high frequency, the orbits of fluid particles are closed and the distribution of the tracer is exposed to fast periodic distortion, resulting in no mixing (diffusion) at all. Thus, waves with periods smaller than the time scale of the diffusion cannot interact with slow changes in the tracer concentration field and produce nontrivial transfer of the tracer particles. Bearing in mind that $T_{(d)}\gg T_{(t)}$, we can see that only *infinitely* slow waves (i.e., waves with zero frequency) can contribute to the diffusivity coefficient within the framework of the *zeroth* order of the perturbation scheme. The corrections associated with the contribution of finite-frequency waves are taken into account in the higher orders.

(4) It is worth noting that the numerical simulation of the kinetic equation for weak Rossby wave turbulence (cf. Reznik and Soomere[11]) demonstrated that the zero-frequency component of the spectrum has a relatively high level and, consequently, can cause strong mixing.

(5) We can also provide a simple physical explanation of the zero values of diffusivity coefficient responsible for the meridional diffusion in the above example. Apparently, they vanish because *the meridional component of zero-frequency wave-induced velocity is equal to zero* (zero-frequency Rossby waves correspond to a *zonal* flow with random profile).

V. AN EXAMPLE: WEAKLY NONLINEAR BAROTROPIC ROSSBY WAVES

The difference between linear and (weakly) nonlinear wave spectra is that the latter comprises the so-called combinational harmonics. In contrast with the ordinary (resonant) waves, frequencies and wave vectors of these harmonics do not satisfy the dispersion relation, but are "composed" of frequencies and wave vectors of resonant waves: $\tilde{\mathbf{k}} = \Sigma\mathbf{k}_m$, $\tilde{\omega} = \Sigma\omega(\mathbf{k}_m)$. Accordingly, a weakly nonlinear

Rossby wave field is described by the following expression (cf. the Appendix):

$$\Psi(x,t) = \int\left(\frac{1}{k}\right)\Big(a_{\mathbf{k}} \exp(i\omega t) + \iint iU^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \times\exp[i(\omega_1+\omega_2)t]\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2 + \iiint U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} \exp[i(\omega_1+\omega_2+\omega_3)t] \cdot\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3 + \cdots\Big) \times\exp(-i\mathbf{k}\mathbf{x})d\mathbf{k}, \tag{22}$$

where $U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ are symmetric functions:

$$U^{(1)}_{\mathbf{k}\mathbf{k}_2\mathbf{k}_1} = U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}, \tag{23a}$$

$$U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_1\mathbf{k}_2} = U^{(2)}_{\mathbf{k}\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_2}. \tag{23b}$$

In addition, the condition of $\Psi(\mathbf{x},t)$ being real yields

$$U^{(1)}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} = -U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2},\qquad U^{(2)}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} = U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}} \tag{24}$$

(the explicit expressions for $U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ are given in the Appendix). Substituting (22) and (17) into the expression for the correlation function (14), we can split the fourth-order moments of $a_{\mathbf{k}}$ through the moments of the second order:

$$\langle a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}'_1} a_{\mathbf{k}'_2}\rangle = E_{\mathbf{k}_1}E_{\mathbf{k}_2}[\delta(\mathbf{k}_1+\mathbf{k}'_1)\delta(\mathbf{k}_2+\mathbf{k}'_2) + \delta(\mathbf{k}_1+\mathbf{k}'_2)\delta(\mathbf{k}_2+\mathbf{k}'_1)] + E_{\mathbf{k}_1}E_{\mathbf{k}'_2}\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\mathbf{k}'_1+\mathbf{k}'_2).$$

This procedure is justified because the distribution of any weakly nonlinear wave field is close to Gaussian distribution (e.g., Zakharov[12]). Using (19), (23), (24), and the equality corresponding to the zero value of the Rossby-wave Stokes drift,

$$U^{(1)}_{0,\mathbf{k},-\mathbf{k}} = 0,$$

we obtain

$$B_{yy}(\tau) = \iint\Big[2\left(\frac{k_{x1}+k_{x2}}{|\mathbf{k}_1+\mathbf{k}_2|}\right)^2 (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \times\cos[(\omega_1+\omega_2)\tau] + [W_{\mathbf{k}_1\mathbf{k}_2}\cos(\omega_1\tau) + W_{\mathbf{k}_2\mathbf{k}_1}\cos(\omega_2\tau)]\Big]E_{\mathbf{k}_1}E_{\mathbf{k}_2}\,d\mathbf{k}_1 d\mathbf{k}_2 + \dots,$$

where

$$W_{\mathbf{k}_1\mathbf{k}_2} = 3(k_{x1}/k_1)^2 U^{(2)}_{\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_2}.$$

Evidently, $W_{\mathbf{k}_1\mathbf{k}_2}\delta(\omega_1) = 0$; and after the substitution of B_{yy} into the expression for diffusivity coefficient (14), the second term in B_{yy} turns into zero:

$$K_{yy} = 2\pi\iint\left(\frac{k_{x1}+k_{x2}}{|\mathbf{k}_1+\mathbf{k}_2|}\right)^2 (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \cdot E_{\mathbf{k}_1}E_{\mathbf{k}_2}\delta(\omega_1+\omega_2)d\mathbf{k}_1\,d\mathbf{k}_2. \tag{25a}$$

Similarly,

$$K_{xy} = -2\pi\iint\left(\frac{(k_{x1}+k_{x2})(k_{y1}+k_{y2})}{|\mathbf{k}_1+\mathbf{k}_2|^2}\right) \times (U^{(1)}_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2})^2 \cdot E_{\mathbf{k}_1}E_{\mathbf{k}_2}\delta(\omega_1+\omega_2)d\mathbf{k}_1\,d\mathbf{k}_2. \tag{25b}$$

Bearing in mind that Rossby waves, as a dynamic system, are isotropic in respect to the inversion of the y axis:

$$\omega(k_x,-k_y) = \omega(k_x,k_y),$$

$$U^{(1)}_{k_x,-k_y;k_{x1},-k_{y1};k_{x2},-k_{y2}} = -U^{(1)}_{k_x,k_y;k_{x1},k_{y1};k_{x2},k_{y2}}$$

[cf. (A4)]; we can see that $K_{xy} = 0$ for all symmetric $[E(k_x,-k_y) = E(k_x,k_y)]$ wave spectra.

VI. DISCUSSION

(1) The expressions (25) for the diffusivity coefficients have clear physical meaning. Indeed, the integrals in (25) sum up the contributions of all second-order combinational harmonics, while the corresponding delta functions "cut out" the harmonics with zero frequency only. Since the zero-frequency combinational harmonics can be formed by pairs of linear waves with *nonzero* frequencies, *the nonlinear corrections to the diffusivity tensor depend on the whole wave spectrum*.

(2) It must be emphasized that in the case of weakly nonlinear waves, the *meridional* mixing is much weaker than the *zonal* mixing.

(3) The nonzero values of the diffusivity coefficients indicate *the irreversible stretching of material lines*, caused by Rossby waves. This conclusion seems to be rather important for the understanding of the Rossby wave turbulence itself.

(4) It is also worth noting that the diffusion due to linear or nonlinear Rossby waves differs significantly from that caused by steady waves. In the latter case (studied by Andrews and McIntyre,[6] Plumb,[7] and Middleton and Loder[8]), the mixing is produced by the Stokes drift, which is, in the instance of Rossby waves, equal to zero.

VII. CONCLUDING REMARKS

Two problems have been discussed in this paper.

First, we have developed an asymptotic technique for the calculation of the coefficient of diffusion due to random-wave-induced motion in fluids. In contrast with the methods developed by the previous authors, this technique permits us to calculate the higher-order corrections to the diffusivity coefficient and can be applied to compressible fluids as well.

Second, we have calculated the diffusivity coefficient for the case, where the fluid motion is induced by linear or weakly nonlinear spectra of barotropic Rossby waves. It was shown that *the diffusion due to Rossby-wave-induced motion is strongly anisotropic*, the zonal diffusivity coefficient being much greater than the meridional one.

We emphasize that the results obtained can be easily generalized for other types of random wave motion (such as surface or internal gravity waves). In particular, the important conclusion about *the irreversible stretching of material lines by Rossby wave turbulence* can be applied to any nonlinear-wave-induced motion (even if the latter does not comprise a shear-flow component). The effect of weak nonlinearity seems to be sufficient for this phenomenon to occur.

APPENDIX: WEAKLY NONLINEAR SPECTRUM OF ROSSBY WAVES

In terms of the Fourier transform

$$\Psi_{\mathbf{k}}(t) = (2\pi)^{-2}k\int \Psi(\mathbf{x},t)\exp(i\mathbf{k}\mathbf{x})d\mathbf{x},$$

the Rossby wave equation can be written as follows:

$$\frac{\partial}{\partial t}\Psi_{\mathbf{k}} - i\omega\Psi_{\mathbf{k}} = \iint V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\Psi_{\mathbf{k}_1}\Psi_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2, \quad \text{(A1)}$$

where

$$V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{1}{2}\left(\frac{1}{k}\right)(k_{x1}k_{y2} - k_{y1}k_{x2})\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right).$$

Equation (A1) can be solved via iterations:

$$\Psi_{\mathbf{k}} = \Psi_{\mathbf{k}}^{(0)} + \Psi_{\mathbf{k}}^{(1)} + ..., \quad \text{(A2)}$$

$$\Psi_{\mathbf{k}}^{(0)} = a_{\mathbf{k}}\exp(i\omega t). \quad \text{(A3a)}$$

The first iteration is

$$\Psi_{\mathbf{k}}^{(1)} = \iint i(\omega-\omega_1-\omega_2)^{-1}V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1}a_{\mathbf{k}_2} \cdot\exp[i(\omega_1+\omega_2)t]\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2. \quad \text{(A3b)}$$

Comparing (A2) and (A3) to (22), we can see that

$$U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = (\omega-\omega_1-\omega_2)^{-1}V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}. \quad \text{(A4)}$$

Evidently, if $\omega-\omega_1-\omega_2\to 0$, $U^{(1)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\to\infty$. This singularity corresponds to strong resonances between Rossby wave triplets and has nothing to do with combinational harmonics of zero frequency. Correspondingly, if $\omega_1 = -\omega_2$ and $\omega(\mathbf{k}_1+\mathbf{k}_2) = 0$, the singularity disappears.

The coefficient $U^{(2)}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ can be calculated in a similar way.

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