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Inertial instability of a liquid film inside a rotating horizontal cylinder

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We examine the dynamics of a thin film of viscous fluid on the inside surface of a cylinder with horizontal axis, rotating about this axis. The stability of the film has been previously explored using the leading-order lubrication approximation, under which it was found to be neutrally stable. In the present paper, we examine how the stability of the film is affected by higher-order corrections, such as inertia (described by the material derivatives in the Navier–Stokes equations), surface tension, and the hydrostatic pressure gradient. Assuming that these effects are weak, we derive an asymptotic equation which takes them into account as perturbations. The equation is used to examine the stability of the steady-state distribution of film around the cylinder (rimming flow) with respect to linear disturbances with harmonic dependence on time (normal modes). It is shown that hydrostatic pressure gradient does not affect those at all, and the effect of surface tension is weak—whereas inertia always causes instability. The inertial instability, however, can be inhibited by viscosity, which can make the characteristic time of growth so large that the film would be effectively stable. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1905964]

I. INTRODUCTION

Rimming flows, where a thin film of liquid is entrained on the inside of a rotating cylinder, are of considerable practical significance. In rotational moulding,[1] liquid thermosetting plastic is placed inside a mould, which is then rotated to distribute the liquid as uniformly as possible. In the coating of fluorescent light bulbs,[2] a suspension consisting of a coating solute and a solvent is placed inside a spinning glass tube. Then, the solvent is evaporated off to leave the coating on the tube.

Most of the theoretical work to date has made use of the so-called lubrication approximation, where the inertia and pressure terms of the Navier–Stokes equations are dominated by viscous effects and gravity. The seminal work has been carried out in Ref. 3, where a leading-order lubrication theory was derived. This was further developed in Ref. 4, where the lubrication approximation was used to calculate the steady-state distribution of film around the cylinder.

Note, however, that almost all experimenters[5] have commented on the occurrence of instability, which disrupts the film and prevents the solute from distributing evenly. Yet, it was demonstrated in BPT93 that the leading-order lubrication model is neutrally stable. As neutral stability is the most precarious stability possible, an obvious approach is to extend the lubrication model to a higher order—to see if weak higher-order effects sway the leading-order balance to either asymptotic stability or instability. Four such attempts have been made.[6]

In Ref. 7, several higher-order models, describing the effects of inertia, surface tension, and transverse (axial) variability were examined analytically. Reference 8, in turn, dealt with a model including *all* effects simultaneously—given its complexity, it was examined numerically. The main conclusion of both papers was that the instability is mostly caused by inertia. An alternative source of instability, resulting from the interaction of hydrostatic pressure and surface tension, has been suggested in Refs. 9 and 10. Finally, a mechanism of instability based solely on the hydrostatic pressure effect has been put forward in Refs. 11 and 12.

In the present paper, we shall reexamine the effect of inertia, with surface tension and hydrostatic pressure also included in the model. We shall confine ourselves to normal modes, i.e., linear disturbances with harmonic dependence on time, which will enable us to examine the problem analytically and in detail.

The paper has the following structure. In the following section, the problem is formulated mathematically. In Sec. III, we develop an asymptotic method for studying the stability of the film and test it on a simple particular case, where the inertia and hydrostatic pressure are weak and the dynamics of the film is determined by surface tension. In Sec. IV, we apply our method to the full problem, and thus obtain a stability criterion reflecting the balance of the (destabilizing) effect of inertia and the (stabilizing) effect of surface tension; it will also be demonstrated that the hydrostatic pressure gradient does not affect the stability of normal modes. In Sec. V, we shall consider examples based on the parameters of "real" fluids (water and glycerin). Finally, in Sec. VI, we shall explain the differences between our results and those obtained in Refs. 7–9.

II. FORMULATION

Consider a thin film of incompressible liquid on the inside surface of a cylinder of radius R, with a horizontal axis, which is rotating about this axis with constant angular velocity Ω (see Fig. 1). We shall use polar coordinates (r,θ), with the origin at the center of the cylinder, so the thickness h of

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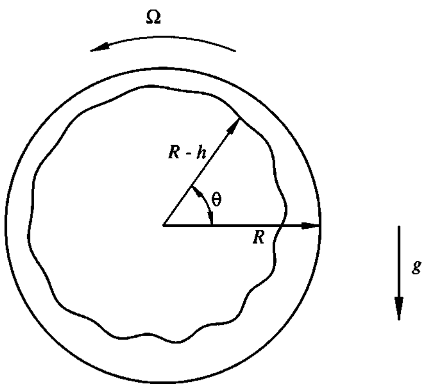

FIG. 1. Film in a rotating cylinder with horizontal axis.

the film depends on the polar angle θ and the time t. We shall also introduce the density ρ of the film, its kinematic viscosity ν, and surface tension γ, and also the acceleration due to gravity g.

From a physical viewpoint, there are five effects governing the film: viscosity, gravity, surface tension, hydrostatic pressure, and inertia (described by the material derivatives in the Navier–Stokes equations). The only approach to modeling these effects developed so far is based on the lubrication approximation, where the first two effects are assumed dominant and the last two are treated as perturbations (mathematically, surface tension can also be treated as a leading-order effect, but, in all applications, it is weak—see Sec. V). Thus, depending on the parameters involved, various asymptotic equations, taking into account some or all of the governing effects, can be derived (see, e.g., Refs. 7, 8, and 11).

In the present paper, we shall use a (most general) asymptotic equation, which takes into account all five effects (it is derived in Appendix A and compared to previous models in Sec. VI). This equations will be expressed in terms of the following nondimensional variables:

$$\hat{\theta} = \theta, \qquad \hat{t} = \Omega t, \tag{1}$$

$$\hat{\eta} = \sqrt{\frac{gR}{\nu\Omega}}\left(\frac{h}{R} - \frac{h^2}{2R^2}\right) \tag{2}$$

[note that, within the framework of the lubrication approximation, the second term in (2) is small—hence, $\hat{\eta}$ is, essentially, the nondimensional thickness of the film]. Then, omitting the hats, we can write the governing equation [Eq. (A45) of Appendix A] in the form

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta}\left\{\eta - \frac{1}{3}\eta^3\cos\theta + \alpha\left[\frac{2}{15}\eta^6\frac{\partial \eta}{\partial \theta}(\cos\theta)^2 - \frac{8}{315}\eta^7\sin\theta\cos\theta - \frac{2}{15}\eta^5\sin\theta\right] + \frac{1}{3}\beta\eta^3\left(\frac{\partial \eta}{\partial \theta} + \frac{\partial^3 \eta}{\partial \theta^3}\right) + \frac{1}{3}\varepsilon\eta^3\frac{\partial \eta}{\partial \theta}\sin\theta\right\} = 0, \tag{3}$$

where

$$\alpha = \frac{\Omega^2 R}{g}, \quad \beta = \frac{\gamma}{\rho g R^2}\sqrt{\frac{\nu\Omega}{gR}}, \quad \varepsilon = \sqrt{\frac{\nu\Omega}{gR}}. \tag{4}$$

The first two terms in figure brackets describe the leading-order effects (viscosity and gravity); and the terms involving ε, α, and β describe hydrostatic pressure, inertia, and surface tension, respectively. We also comment that equations similar to (3) have been previously used in Ref. 8 and Refs. 9 and 10 (for comparisons with these papers, see Secs. VI A and VI C, respectively).

For a steady-state solution,

$$\eta(\theta,t) = \overline{\eta}(\theta),$$

Eq. (3) yields

$$\overline{\eta} - \frac{1}{3}\overline{\eta}^3\cos\theta + \frac{1}{3}\varepsilon\overline{\eta}^3\frac{d\overline{\eta}}{d\theta}\sin\theta + \alpha\left[\frac{2}{15}\overline{\eta}^6\frac{d\overline{\eta}}{d\theta}(\cos\theta)^2 - \frac{8}{315}\overline{\eta}^7\sin\theta\cos\theta - \frac{2}{15}\overline{\eta}^5\sin\theta\right] + \frac{1}{3}\beta\overline{\eta}^3\left(\frac{d\overline{\eta}}{d\theta} + \frac{d^3\overline{\eta}}{d\theta^3}\right) = q, \tag{5}$$

where q is a constant of integration (physically, q is the nondimensional mass flux). We shall also impose the periodicity condition,

$$\overline{\eta}(\theta + 2\pi) = \overline{\eta}(\theta). \tag{6}$$

In what follows, the evolution equation (3) will be used to examine the stability of $\overline{\eta}$ with respect to small disturbances. Assuming that

$$\eta(\theta,t) = \overline{\eta}(\theta) + \eta'(\theta,t),$$

where η' describes a disturbance, we substitute $\eta(\theta,t)$ into (3) and omit the nonlinear terms,

$$\frac{\partial \eta'}{\partial t} + \frac{\partial}{\partial \theta}\left[C(\theta)\eta' + D(\theta)\frac{\partial \eta'}{\partial \theta} + B(\theta)\left(\frac{\partial \eta'}{\partial \theta} + \frac{\partial^3 \eta'}{\partial \theta^3}\right)\right] = 0, \tag{7}$$

where

$$C(\theta) = 1 - \overline{\eta}^2\cos\theta + \varepsilon\overline{\eta}^2\frac{d\overline{\eta}}{d\theta}\sin\theta + \alpha\left[\frac{4}{5}\overline{\eta}^5\frac{d\overline{\eta}}{d\theta}(\cos\theta)^2 - \frac{8}{45}\overline{\eta}^6\sin\theta\cos\theta - \frac{2}{3}\overline{\eta}^4\sin\theta\right] + \beta\overline{\eta}^2\left(\frac{d\overline{\eta}}{d\theta} + \frac{d^3\overline{\eta}}{d\theta^3}\right), \tag{8}$$

$$D(\theta) = \frac{1}{3}\varepsilon\bar{\eta}^3 \sin\theta + \frac{2}{15}\alpha\bar{\eta}^6(\cos\theta)^2, \quad B(\theta) = \frac{1}{3}\beta\bar{\eta}^3. \quad (9)$$

Physically, C is the propagation speed of disturbances, the coefficient of the second derivative, $-(D+B)$, plays the role of effective diffusivity, whereas the fourth derivative term in Eq. (7) describes dissipative effects due to surface tension. Observe that, for some θ, the diffusivity is negative—which often causes instability.

In this paper, we shall confine ourselves to harmonic disturbances, i.e., solutions of the form

$$\eta'(\theta,t) = \phi(\theta)e^{i\omega t},$$

for which (7) yields

$$\frac{d}{d\theta}\left[C(\theta)\phi + D(\theta)\frac{d\phi}{d\theta} + B(\theta)\left(\frac{d\phi}{d\theta} + \frac{d^3\phi}{d\theta^3}\right)\right] - i\omega\phi = 0. \quad (10)$$

This equation and the condition of periodicity,

$$\phi(\theta + 2\pi) = \phi(\theta), \quad (11)$$

form an eigenvalue problem for ϕ and ω. If $\mathrm{Im}\,\omega > 0$, the film is unstable.

III. THE SPECIAL CASE IN WHICH INERTIA AND PRESSURE ARE NEGLIGIBLE ($\varepsilon=\alpha=0$, $\beta\neq 0$)

In this section, an asymptotic technique for the stability problem formulated above will be developed and tested on a simple particular case, where inertia and the pressure gradient are negligible, and the stability of the film is determined by surface tension.[13] Later on, the same technique will be applied to the general case.

Substituting $\varepsilon=\alpha=0$ into Eq. (5) for the steady state and (10) for the disturbance, we obtain

$$\bar{\eta} - \frac{1}{3}\bar{\eta}^3\cos\theta + \frac{1}{3}\beta\bar{\eta}^3\left(\frac{d\bar{\eta}}{d\theta} + \frac{d^3\bar{\eta}}{d\theta^3}\right) = q \quad (12)$$

and

$$\frac{d}{d\theta}\left[C(\theta)\phi + B(\theta)\left(\frac{d\phi}{d\theta} + \frac{d^3\phi}{d\theta^3}\right)\right] - i\omega\phi = 0, \quad (13)$$

where

$$C(\theta) = 1 - \bar{\eta}^2\cos\theta + \beta\bar{\eta}^2\left(\frac{d\bar{\eta}}{d\theta} + \frac{d^3\bar{\eta}}{d\theta^3}\right), \quad B(\theta) = \frac{1}{3}\beta\bar{\eta}^3. \quad (14)$$

In this section, we shall examine Eqs. (12) and (13). Given the stabilizing nature of surface tension, it is expected that the film is stable.

A. Steady state

Steady rimming flows with surface tension have been thoroughly investigated in Ref. 16. Several regimes were examined, including those with a "pool" of fluid formed at the bottom of the cylinder or a shock formed on the cylinder's side (see also Refs. 17–19). Unlike these papers, we shall confine ourselves to the simplest case, where the fluid is distributed continuously and surface tension is a small perturbation.

In this case, Eq. (12) can be solved using a simple perturbation technique based on the smallness of β,

$$\bar{\eta} = \bar{\eta}^{(0)} + \beta\bar{\eta}^{(1)} + O(\beta^2), \quad (15)$$

where $\bar{\eta}^{(0)}$ satisfies the leading-order approximation of Eq. (12),

$$\bar{\eta}^{(0)} - \frac{1}{3}\bar{\eta}^{(0)3}\cos\theta = q. \quad (16)$$

This (cubic) equation was examined in Refs. 3 and 4, where it was demonstrated that, for $q<\frac{2}{3}$, it has a smooth unique solution for $\bar{\eta}^{(0)}(\theta)$. Next, substitution of (15) into (12) yields

$$\bar{\eta}^{(1)} = -\frac{\frac{1}{3}\bar{\eta}^{(0)3}\left(\frac{d\bar{\eta}^{(0)}}{d\theta} + \frac{d^3\bar{\eta}^{(0)}}{d\theta^3}\right)}{1 - \bar{\eta}^{(0)2}\cos\theta}. \quad (17)$$

Following this path, we can calculate as many terms of expansion (15) as necessary.

Note that, when q approaches $\frac{2}{3}$, $\bar{\eta}^{(0)}$ develops steep gradients near $\theta=0$ (see, for example, Ref. 19), and the surface-tension term in Eq. (12) may no longer be treated as a perturbation (because it involves high-order derivatives). As a result, expansion (15) fails in the limit $q\to\frac{2}{3}$, β=const. In practice, however, even the leading term $\bar{\eta}^{(0)}$ provides a good approximation for up to $q=0.65$—see Fig. 2, where $\bar{\eta}^{(0)}$ is compared with the numerical solution of the exact equation (12) (the numerical technique is described in Appendix B 1).

B. Disturbances

Assuming (15), one can rearrange (14) as follows:

$$C = C^{(0)} + \beta C^{(1)} + O(\beta^2), \quad B = \beta B^{(1)} + O(\beta^2),$$

where

$$C^{(0)} = 1 - \bar{\eta}^{(0)2}\cos\theta, \quad (18)$$

$$C^{(1)} = -2\bar{\eta}^{(0)}\bar{\eta}^{(1)}\cos\theta + \bar{\eta}^{(0)2}\left(\frac{d\bar{\eta}^{(0)}}{d\theta} + \frac{d^3\bar{\eta}^{(0)}}{d\theta^3}\right),$$
$$B^{(1)}(\theta) = \frac{1}{3}\bar{\eta}^{(0)3}. \quad (19)$$

Then, (13) becomes

$$\frac{d}{d\theta}\left[C^{(0)}\phi + \beta C^{(1)}\phi + \beta B^{(1)}\left(\frac{d\phi}{d\theta} + \frac{d^3\phi}{d\theta^3}\right)\right] - i\omega\phi = 0. \quad (20)$$

Seek a solution of (20) as a series in β,

$$\phi = \phi^{(0)} + \beta\phi^{(1)} + O(\beta^2), \quad \omega = \omega^{(0)} + \beta\omega^{(1)} + O(\beta^2).$$

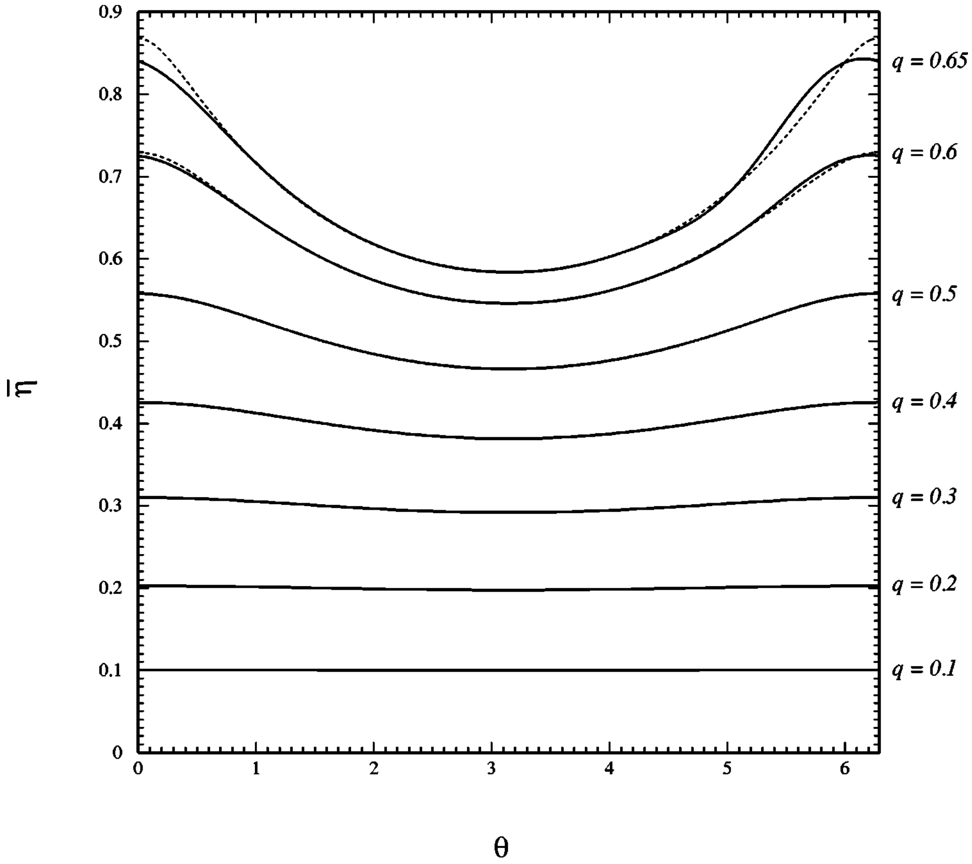

FIG. 2. Steady-state flow for $\varepsilon=\alpha=0$, $\beta=0.1$, and various values of the nondimensional flux q. Solid line shows the numerical solution of the exact equation (12), dotted line shows the leading-order asymptotic solution $\bar{\eta}^{(0)}$ [i.e., that of Eq. (16)].

The leading order yields a first-order ordinary differential equation (ODE)

$$\frac{d}{d\theta}(C^{(0)}\phi^{(0)}) - i\omega^{(0)}\phi^{(0)} = 0,$$

the solution to which is

$$\phi^{(0)} = \frac{1}{C^{(0)}(\theta)}\exp\left[i\omega^{(0)}\int_0^\theta \frac{d\theta'}{C^{(0)}(\theta')}\right]. \quad (21)$$

The periodicity condition requires

$$\omega^{(0)} = \frac{2\pi n}{\displaystyle\int_0^{2\pi}\frac{d\theta}{C^{(0)}(\theta)}}, \quad (22)$$

where n is an integer (the mode number). Observe that, to leading order, $\mathrm{Re}\,\omega=0$, i.e., the disturbance is neutrally stable, which agrees with the conclusions of BPT93 and Ref. 9 [the latter paper also presented (21) and (22)]. Finally, an example of the leading-order eigenfunction (21) is shown in Fig. 3.

The next order yields

$$\frac{d}{d\theta}(C^{(0)}\phi^{(1)}) - i\omega^{(0)}\phi^{(1)} = i\omega^{(1)}\phi^{(0)} - \frac{d}{d\theta}\left[B^{(1)}\left(\frac{d\phi^{(0)}}{d\theta} + \frac{d^3\phi^{(0)}}{d\theta^3}\right) + C^{(1)}\phi^{(0)}\right]. \quad (23)$$

This equation has a periodic solution for $\phi^{(1)}$ if and only if its right-hand side is orthogonal to the solution ϕ^+ of the adjoint problem. Given that we are currently considering a first-order ODE, the latter is easy to find,

$$\phi^+ = \exp\left[-i\omega^{(0)}\int_0^\theta \frac{d\theta'}{C^{(0)}(\theta')}\right].$$

In fact, one does not really need to know that ϕ^+ is the adjoint solution. One can simply multiply (23) with ϕ^+ and integrate from $\theta=0$ to $\theta=2\pi$. After straightforward algebra (involving integration by parts), $\phi^{(1)}$ can be eliminated, and we end up with

$$i\omega^{(1)}\int_0^{2\pi}\frac{d\theta}{C^{(0)}} - \int_0^{2\pi}\left\{B^{(1)}\left[\left(\frac{d^3}{d\theta^3}\frac{1}{C^{(0)}}\right) + \frac{4i\omega^{(0)}}{C^{(0)}}\left(\frac{d^2}{d\theta^2}\frac{1}{C^{(0)}}\right) + 3i\omega^{(0)}\left(\frac{d}{d\theta}\frac{1}{C^{(0)}}\right)^2 - 6\omega^{(0)2}\frac{1}{C^{(0)2}}\left(\frac{d}{d\theta}\frac{1}{C^{(0)}}\right) - i\omega^{(0)3}\left(\frac{1}{C^{(0)}}\right)^4 + \left(\frac{d}{d\theta}\frac{1}{C^{(0)}}\right) + \frac{i\omega^{(0)}}{C^{(0)2}}\right] + \frac{C^{(1)}}{C^{(0)}}\right\}\frac{i\omega^{(0)}}{C^{(0)}}d\theta = 0.$$

Expressing $\omega^{(1)}$ from this equation and taking the imaginary part, we obtain

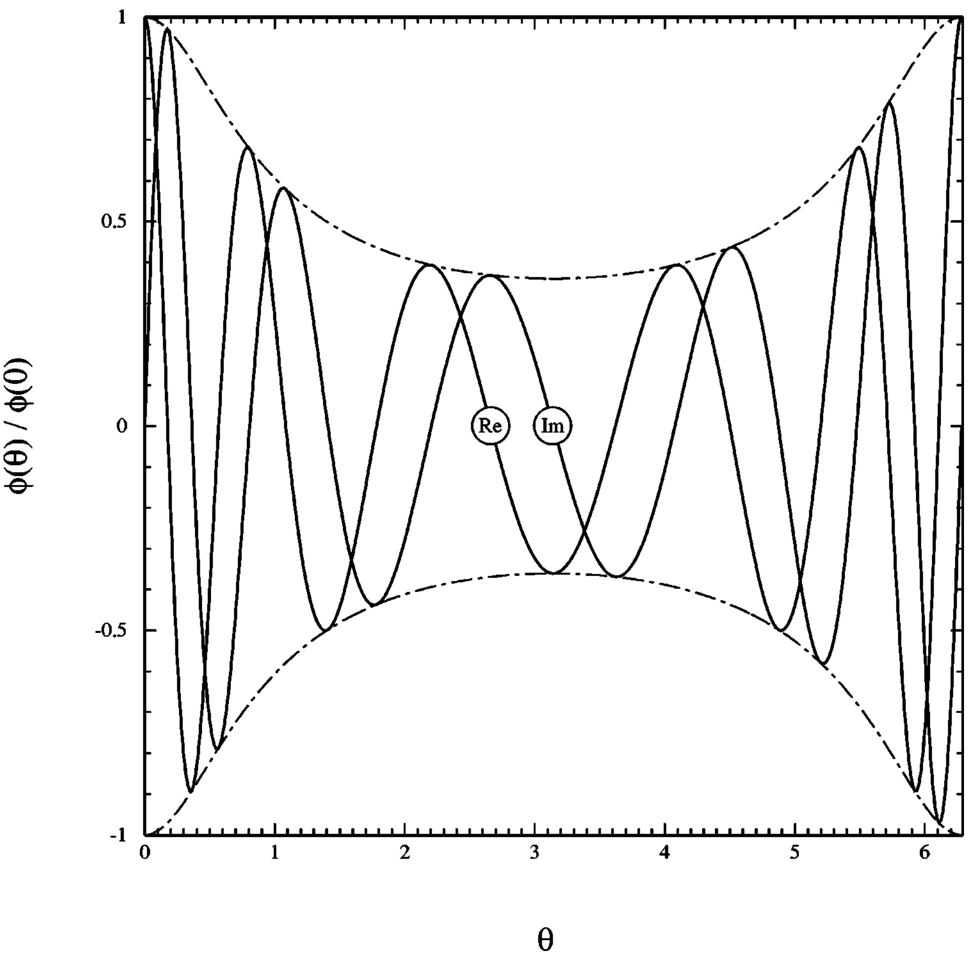

FIG. 3. The leading-order eigenfunction [given by (21)] for $q=0.6$, $n=5$. Solid line shows the real and imaginary parts, dashed-dotted line shows the absolute value.

$$\operatorname{Im}\omega^{(1)} = \frac{\omega^{(0)2}\int_0^{2\pi}\frac{B^{(1)}}{C^{(0)3}}\left\{1-\frac{1}{C^{(0)2}}\left[\omega^{(0)2}+4C^{(0)}\frac{d^2C^{(0)}}{d\theta^2}-11\left(\frac{dC^{(0)}}{d\theta}\right)^2\right]\right\}d\theta}{\int_0^{2\pi}\frac{d\theta}{C^{(0)}}}. \tag{24}$$

This expression [together with Eq. (16) for $\overline{\eta}^{(0)}$ and expressions (18), (19), and (22) for $C^{(0)}$, $B^{(1)}$, and $\omega^{(0)}$] is the main result of this section. The sign of $\operatorname{Im}\omega^{(1)}$ determines the stability—unfortunately, we were unable to determine it analytically. It has been computed numerically, and, expectedly, $\operatorname{Im}\omega^{(1)}$ turned out to be negative for all q, which means asymptotic stability of all steady-state flows.

C. Discussion

Formula (24) has been tested against the numerical solution of the exact eigenvalue problem (11) and (13) (the numerical technique is described in Appendix B). The results are shown in Fig. 4. One can see that the region of applicability of the asymptotic solution rapidly contracts with growing mode number.

To explain the deteriorating accuracy of formula (24) for higher modes, observe that the mode number is, essentially, the number of oscillations per period of the eigenfunction (see, for example, Fig. 3). Then, the capillary term in Eq. (13) is proportional to βn^4, whereas the leading-order terms are proportional to n. Accordingly, our expansion (based on the smallness of the former) fails for $\beta \gtrsim n^{-3}$. However, this condition is not very restrictive, as higher ($n \geqslant 2$) modes are less important than the first one (in the following section, this question will be discussed in detail).

Instead of the nondimensional flux q, it is sometimes more convenient to use the mean nondimensional thickness of the film, defined by

$$\langle\overline{\eta}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\overline{\eta}\,d\theta. \tag{25}$$

An approximate relationship between q and $\langle\overline{\eta}\rangle$ can be established by solving the leading-order equation (16) for a

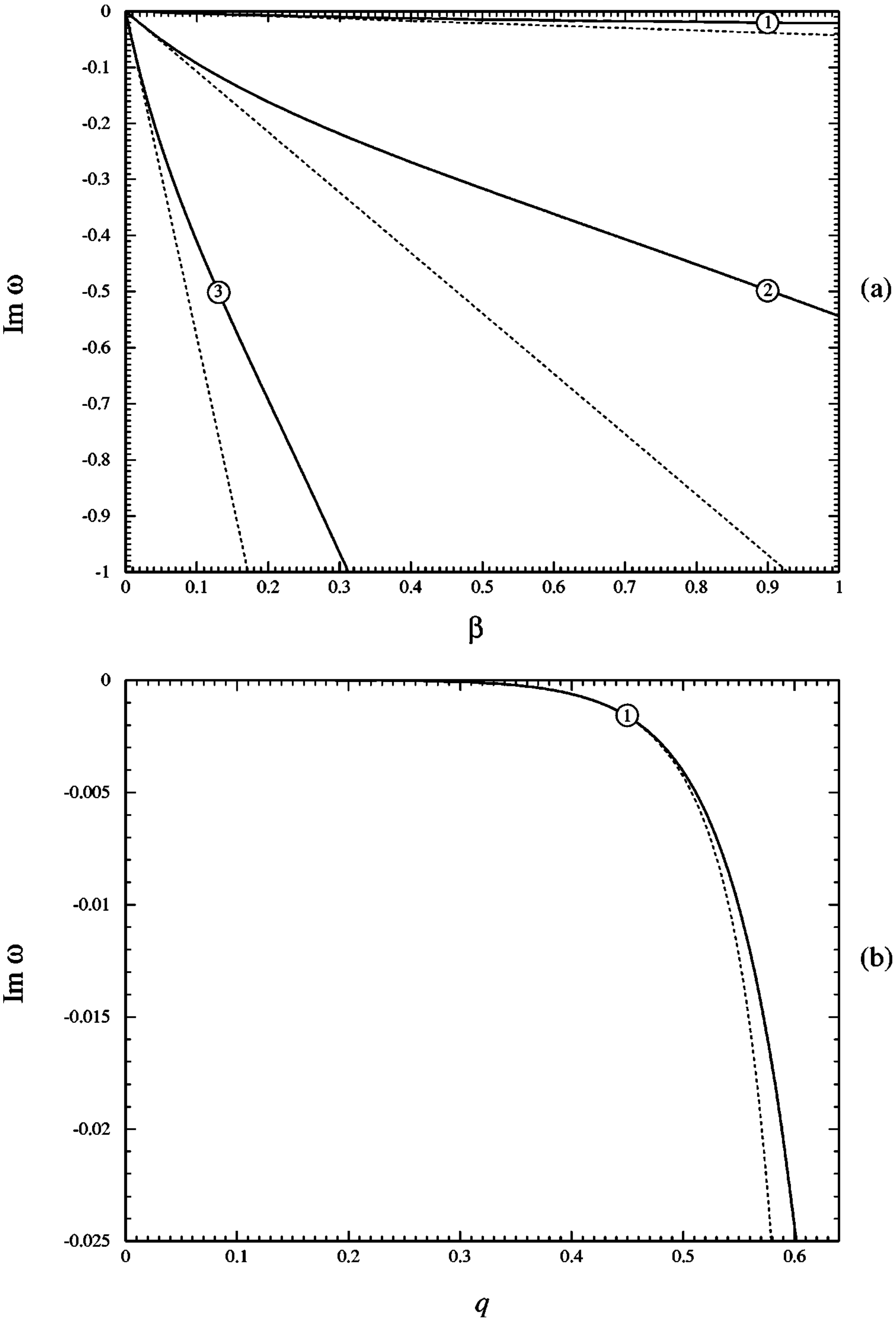

FIG. 4. The decay rate ($\operatorname{Im}\omega$) of eigenproblem (13) and (11). The curves are marked with the mode number. The solid line shows the numerical solution of (13) and (11), the dotted line shows the asymptotic solution, $\beta \operatorname{Im}\omega^{(1)}$, where $\omega^{(1)}$ is given by expression (24). (a) $\operatorname{Im}\omega$ vs β for $q=0.5$; (b) $\operatorname{Im}\omega$ vs q for $\beta=0.1$.

given q, then substituting the solution $\overline{\eta}^{(0)}$ into (25). It turns out that the two parameters are almost equal—see Fig. 5.

D. The small-q limit

It is instructive to examine what happens with our results as $q \to 0$. In this limit, the steady-state equation (16) can be solved using a series in powers of q^2,

$$\overline{\eta}^{(0)} = q + (1/3)q^3 \cos\theta + O(q^5) \quad \text{as } q \to 0. \tag{26}$$

This simple formula explains two important features of the results obtained. First, (26) shows that, for small q, $\overline{\eta}(\theta)$ becomes flat (see Fig. 2). Second, it explains the approximate equality of the nondimensional mean thickness $\langle\overline{\eta}\rangle$ and nondimensional flux q (see Fig. 5). Indeed, substituting (26) into (25), we obtain

$$\langle\overline{\eta}\rangle = q + O(q^5) \quad \text{as } q \to 0. \tag{27}$$

Strictly speaking, this conclusion applies only to small q—but Fig. 5 shows that $\langle\overline{\eta}\rangle \approx q$ for the whole range $0 < q < \frac{2}{3}$ [observe that, even for $q = \frac{2}{3}$, the error (27) can be estimated as $(\frac{2}{3})^5 \approx 0.0015$].

Equality (27) also allows one to relate q to the dimensional mean thickness $\langle\overline{h}\rangle$, and thus define the small-q limit in physical terms. It follows from (2) that

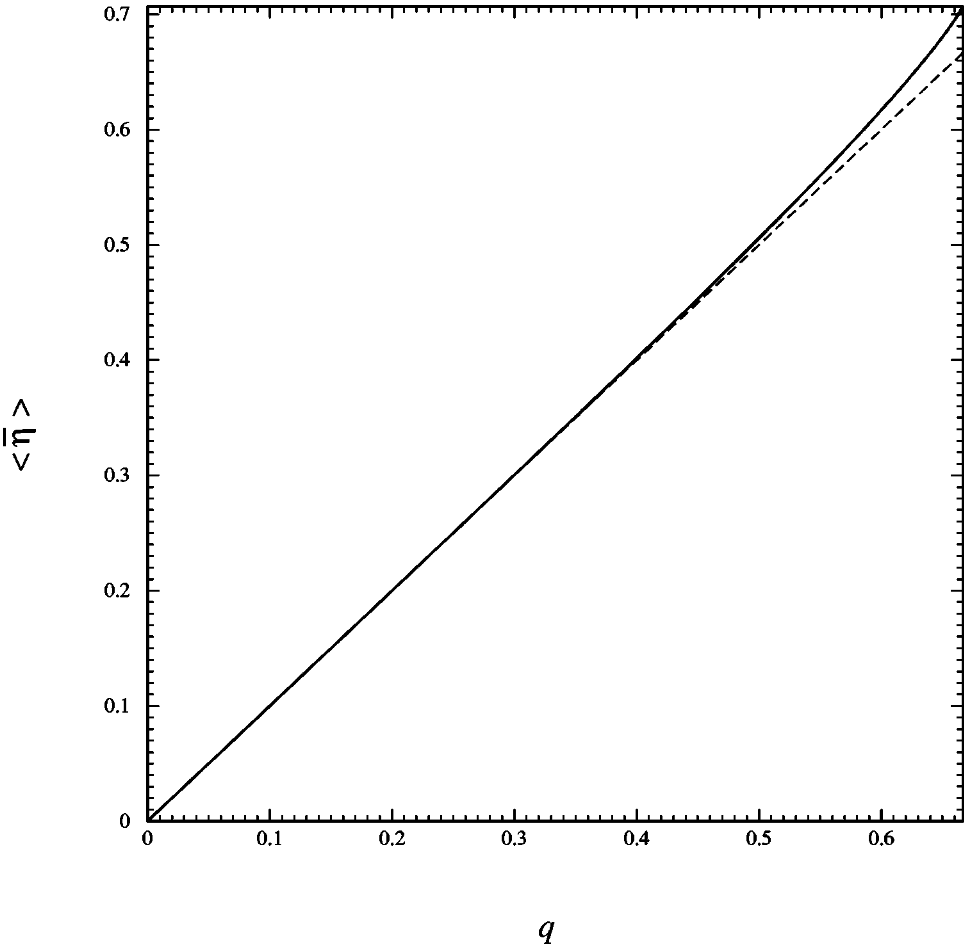

FIG. 5. The nondimensional mean thickness of the film [defined by (25)] vs nondimensional flux q, as determined by the leading-order steady-state equation (16) (i.e., for $\alpha=\beta=\varepsilon=0$). The dashed line corresponds to the small-q limit.

$$\langle\overline{\eta}\rangle \approx \sqrt{\frac{g}{\nu\Omega R}}\langle\overline{h}\rangle,$$

which we shall combine with (27) to obtain

$$q \approx \sqrt{\frac{g}{\nu\Omega R}}\langle\overline{h}\rangle \quad \text{if } q \ll 1. \tag{28}$$

Hence, the assumption of small q, or equivalently small $\langle\overline{\eta}\rangle$, implies

$$\langle\overline{h}\rangle \ll \sqrt{\frac{\nu\Omega R}{g}}, \tag{29}$$

which shows that the small-q limit is a kind of thin-film approximation.

Using the small-q form of the steady state, we can simplify formula for $\mathrm{Im}\,\omega^{(1)}$. Substituting (26) into expressions (18) and (19) for $C^{(0)}$, $B^{(1)}$, then substituting those into (24), we obtain

$$\mathrm{Im}\,\omega^{(1)} \approx \begin{cases} -3q^7 & \text{for } n=1 \\ -\dfrac{1}{3}n^2(n^2-1)q^3 & \text{for } n\geqslant 2, \end{cases} \quad \text{if } q \ll 1. \tag{30}$$

Observe that the decay rate of the first mode is $O(q^7)$, and those of higher modes are $O(q^3)$—hence, the former is much smaller than the latter. Moreover, even though this conclusion formally applies only to small q, Fig. 4 shows that it is valid for the whole range $0<q<\frac{2}{3}$. Thus, it is natural to assume that the first mode is always the first one to succumb to instability, should that appear in the system.

IV. THE GENERAL CASE

A. Steady state

As before, we shall need only the leading-order steady state, i.e., solution $\overline{\eta}^{(0)}$ determined by Eq. (16). Readers interested in further details are referred to the above-mentioned papers.[16–19]

B. Disturbances

Employing the same asymptotic method as in the case $\varepsilon=\alpha=0$ (preceding section), we obtain

$$\mathrm{Im}\,\omega = \varepsilon\,\mathrm{Im}\,\omega_\varepsilon^{(1)} + \alpha\,\mathrm{Im}\,\omega_\alpha^{(1)} + \beta\,\mathrm{Im}\,\omega_\beta^{(1)} + O(\varepsilon^2,\alpha^2,\beta^2,\varepsilon\alpha,\varepsilon\beta,\alpha\beta), \tag{31}$$

where

$$\mathrm{Im}\,\omega_\alpha^{(1)} = \frac{\omega^{(0)2}\displaystyle\int_0^{2\pi}\frac{D^{(1)}}{C^{(0)3}}d\theta}{\displaystyle\int_0^{2\pi}\frac{d\theta}{C^{(0)}}}, \tag{32}$$

$$\operatorname{Im}\omega_{\beta}^{(1)} = \frac{\omega^{(0)2}\displaystyle\int_0^{2\pi}\frac{B^{(1)}}{C^{(0)3}}\left\{1-\frac{1}{C^{(0)2}}\left[\omega^{(0)2}+4C^{(0)}\frac{d^2C^{(0)}}{d\theta^2}-11\left(\frac{dC^{(0)}}{d\theta}\right)^2\right]\right\}d\theta}{\displaystyle\int_0^{2\pi}\frac{d\theta}{C^{(0)}}}, \tag{33}$$

$$\operatorname{Im}\omega_{\varepsilon}^{(1)} = 0. \tag{34}$$

As before, $C^{(0)}$, $B^{(1)}$, and $\omega^{(0)}$ are given by (18), (19), and (22), respectively, and

$$D^{(1)} = (2/15)\,\overline{\eta}^{(0)6}(\cos\theta)^2.$$

Not surprisingly, $\omega_{\beta}^{(1)}$ coincides with $\omega^{(1)}$ calculated in the preceding section; observe also that the hydrostatic pressure gradient does not contribute to the growth rate of harmonic solutions [see (34)]. This does not, however, mean that it has no influence on the stability of the flow, as it may give rise to nonharmonic growing solutions (as shown in Refs. 11 and 12).

The other two effects, inertia and surface tension, do affect the stability of harmonic solutions. The former is always destabilizing [$\operatorname{Im}\omega_{\alpha}^{(1)}>0$, see (32)], whereas the latter, as shown in the preceding section, stabilizes the flow. Which of the two effects is stronger? The answer to this question depends on the parameters involved: for any α and β, there is a threshold value q_0, such that flows with $q<q_0$ are unstable. Indeed, substituting (32)–(34) into (31) and omitting small terms, one can derive the following condition of instability:

$$f(q) < \frac{\alpha}{\beta}, \tag{35}$$

where

$$f(q) = \frac{\displaystyle\int_0^{2\pi}\frac{B^{(1)}}{C^{(0)3}}\left\{\frac{1}{C^{(0)2}}\left[\omega^{(0)2}+4C^{(0)}\frac{d^2C^{(0)}}{d\theta^2}-11\left(\frac{dC^{(0)}}{d\theta}\right)^2\right]-1\right\}d\theta}{\displaystyle\int_0^{2\pi}\frac{D^{(1)}}{C^{(0)3}}d\theta}. \tag{36}$$

Observe that $f(q)$ depends on the mode number n only through $\omega^{(0)}$ in its numerator. Recalling that $\omega^{(0)}$ grows with n [see (22)], we conclude that higher ($n\geqslant 2$) modes can grow only if the first mode grows. Thus, the flow is unstable if and only if criterion (35) and (36) is satisfied for the *first* mode.

Outside the small-q limit (considered in the following section), the function $f(q)$ has to be computed numerically—or, physically, it would be more convenient to compute $f(\langle\overline{\eta}\rangle)$, where $\langle\overline{\eta}\rangle$ is the mean nondimensional thickness [recall that $\langle\overline{\eta}\rangle$ is related to q by formula (25)]. This has been done, and the instability region on the $(\alpha/\beta,\langle\overline{\eta}\rangle)$ plane, determined by criterion (35) and (36), is shown see Fig. 6. Observe that, counterintuitively, thinner films (with small $\langle\overline{\eta}\rangle$) are more unstable than thicker films—this question will be discussed in the following section.

C. The small-q limit

If q is small, criterion (35) and (36) can be simplified asymptotically. Putting $n=1$ (first mode) and assuming the small-q limit, we can reduce (31)–(34) to

$$\operatorname{Im}\omega^{(1)} \approx \frac{1}{15}\alpha q^6 - 3\beta q^7 \quad \text{if } q \ll 1. \tag{37}$$

Then, instability exists if and only if

$$45q \leqslant \frac{\alpha}{\beta} \quad \text{if } q \ll 1, \tag{38}$$

where the $45q$ is the small-q limit of $f(q)$.

It is instructive to rewrite the small-q criterion (38) in dimensional form. Making use of formula (28) and expressions (4) for α and β, we obtain

$$\langle\overline{h}\rangle < \frac{\rho\Omega^2R^4}{45\gamma} \quad \text{if } \langle\overline{h}\rangle \ll \sqrt{\frac{\nu\Omega R}{g}} \tag{39}$$

[keep in mind that this criterion holds subject to the (dimensional) thin-film condition (29)]. Thus, increasing the velocity of rotation or the cylinder's radius strengthens instability, whereas increasing surface tension or thickness of the film weakens it. On the other hand, (39) is independent of ν—hence, a change in viscosity cannot stabilize an otherwise unstable film or vice versa (note that this conclusion applies only to the small-q limit).

Condition (39) can also be written in the form

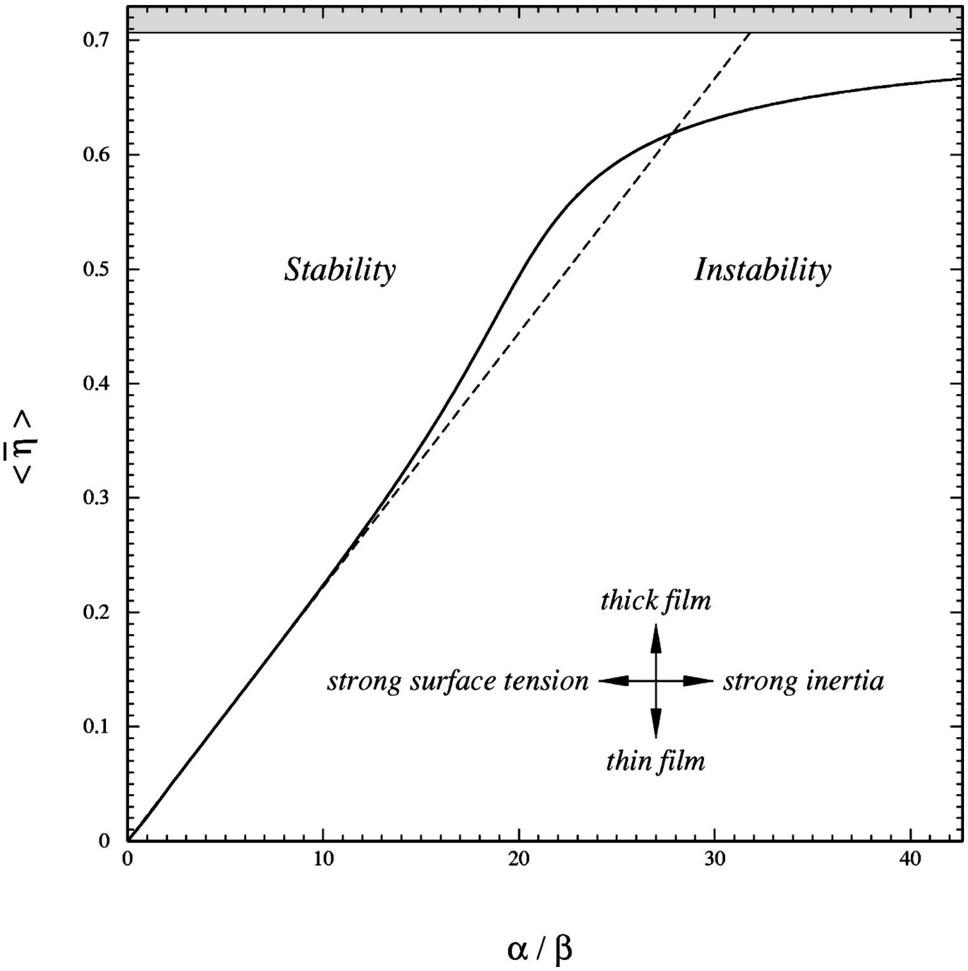

FIG. 6. The stability of the film on the $(\alpha/\beta, \langle\overline{\eta}\rangle)$ plane. α and β are nondimensional parameters characterizing inertia and surface tension $[\alpha/\beta = (gR/\nu\Omega)^{1/2}(\Omega^2R^3\rho/\gamma)]$, $\langle\overline{\eta}\rangle$ is the nondimensional thickness of the film $(\langle\overline{\eta}\rangle \approx (g/\nu\Omega R)^{1/2}\langle h\rangle)$. The dashed line corresponds to the small-q (thin film) limit; the shaded area corresponds to $q \geqslant \frac{2}{3}(\langle\overline{\eta}\rangle \gtrsim 0.707)$ where no continuous solutions exist for the leading-order steady-state equation (16).

$$B > 1,$$

where

$$B = \frac{\rho\Omega^2R^4}{45\gamma\langle\overline{h}\rangle} \tag{40}$$

is a nondimensional parameter characterizing the relative strengths of inertia and surface tension. Note that, even though (39) was derived for small q, B should retain its physical meaning for the general case as well.

The small-q limit also allows us to understand why thick films are more unstable than thin ones (see the end of the preceding section). Observe that the contribution of surface tension to the growth rate [formula (37)] is proportional to q^7, whereas that of inertia is proportional to q^6. Hence, we conclude that the (stabilizing) effect of surface tension depends on the film's thickness more strongly than the (destabilizing) effect of inertia—which explains the stabilization of the film with growing $\langle\overline{h}\rangle$.

V. EXAMPLES

In this section, we shall apply criterion (35) and (36) to "real" fluids, such as, for example, water, for which

$$\nu = 1.75 \times 10^{-6}\ \mathrm{m^2/s}, \quad \gamma = 0.0728\ \mathrm{N/m}, \qquad \rho = 1.0 \times 10^3\ \mathrm{kg/m^3} \tag{41}$$

(these parameters correspond to 20 °C). In order to increase the effect of surface tension and thus improve the chances of stability, we shall consider a relatively small cylinder,

$$R = 1.5\ \mathrm{cm}. \tag{42}$$

The so-called volume fraction will be fixed at 1%, i.e.,

$$\frac{2\langle\overline{h}\rangle}{R} = 0.01, \tag{43}$$

and the spin rate will be fixed at 1.5 revolutions per, i.e.,

$$\Omega = 2\pi \times 1.5\ \mathrm{s^{-1}}. \tag{44}$$

Using our asymptotic method to calculate the growth rate $\mathrm{Im}\,\omega^{(1)}$ for parameters (41)–(44), we obtain the following dimensional e-folding time:

$$\tau = \frac{1}{\Omega \, \mathrm{Im}\, \omega^{(1)}} \approx 11.5 \text{ min}$$

[it has been taken into account here that the time is scaled by Ω^{-1}, see (1)]. Note that, if we ignore surface tension, τ hardly changes ($\tau \approx 10.9$ min), which means that capillary effects are negligible. This conclusion can be verified by calculating B: substituting (41)–(44) into (40), we obtain $B \approx 18 \gg 1$. Thus, in order to stabilize inertial instability, we need to decrease R or Ω, or increase γ or $\langle \overline{h} \rangle$.

Unfortunately, R cannot be changed in most industrial applications (in coating of fluorescent bulbs, for example, R is the radius of the bulb). The thickness of the film $\langle \overline{h} \rangle$ cannot be controlled either, as it is determined by the amount of coating to be put on the bulb. Moreover, during the coating process, the solvent is being evaporated, i.e., $\langle \overline{h} \rangle$ is gradually decreasing to 0. Thus, to inhibit instability, Ω has to be continuously reduced (adjusted to the current value of $\langle \overline{h} \rangle$).

Note, however, that—instead of completely eliminating instability—we can reduce its growth rate by increasing viscosity. Replace, for example, water with glycerin, for which

$$\nu = 1.18 \times 10^{-3} \text{ m}^2/\text{s}, \quad \gamma = 0.0834 \text{ N/m},$$

$$\rho = 1.26 \times 10^{3} \text{ kg/m}^3.$$

Then, the corresponding e-folding time,

$$\tau \approx 7 \times 10^{12} \text{ min},$$

is so large that the instability can be simply ignored. Note that B, in this case, is still large ($B \approx 27$), i.e., surface tension is negligible.

In order to further illustrate the strong dependence of τ on ν, we shall calculate τ for the small-q limit. To do so, assume for simplicity that $\beta = 0$ (no surface tension), after which (37) yields

$$\mathrm{Im}\, \omega^{(1)} \approx (1/15)\alpha q^6 \quad \text{if } q \ll 1.$$

Then, make use of expression (28) for q and expression (4) for α,

$$\tau = \frac{15 R^2 \nu^3}{g^2 \langle \overline{h} \rangle^6} \quad \text{if } \langle \overline{h} \rangle \ll \sqrt{\frac{\nu \Omega R}{g}}. \tag{45}$$

Thus, if ν is increased by a factor of 5, the instability slows down by a factor of 125.

Given that the viscosity coefficient ν of any fluid can be dramatically increased by mixing it with another fluid (of a greater ν), this "method" of inhibiting inertial instability is much simpler than the one based on surface tension.

VI. COMPARISON TO EARLIER WORK

In this section, we shall compare our results on inertial instability to those obtained in Refs. 7–9.

A. The paper by Hosoi and Mahadevan (Ref. 8)

In this paper, the effect of inertia was examined using the same approximations as ours, but the inertia terms obtained in the two papers differ significantly. The discrepancy stems from the assumption made in Ref. 8 that "events equilibrate themselves quickly over the fast time scale," where the "fast time scale" means the period of rotation of the cylinder. On the basis of this assumption, the time variable was nondimensionalized by the slow time scale determined by viscosity [see formula (2.14) of Ref. 8].

Note, however, that the time scale of an evolution equation cannot be determined on the basis of an assumption. It should be determined by balancing the time derivative and the largest of the other terms in the equation. If this approach is applied to the problem at hand, it yields the *fast* time scale. Indeed, observe that t in Eq. (3) is nondimensionalized using the period of rotation Ω^{-1}—and any other choice would violate the balance between the time derivative and the other terms in the equation.

As a result of the alternative choice of the time scale, a term has been lost in the expression for mass flux due to inertia derived in Ref. 8.

To be specific, consider the mass flux due to inertia obtained in our paper,

$$\left(\int_0^h v \, dr \right)_{\text{inertia}} = \alpha \left[\frac{5}{24} h^4 \frac{\partial h}{\partial t} \cos\theta + \frac{5}{24} h^4 \frac{\partial h}{\partial \theta} \cos\theta - \frac{2}{15} h^5 \sin\theta - \frac{3}{40} h^6 \frac{\partial h}{\partial \theta} (\cos\theta)^2 + \frac{37}{840} h^7 \cos\theta \sin\theta \right] \tag{46}$$

[this expression has been extracted from formula (A39) of Appendix A and changed to the notation of the main body of the paper]. After straightforward algebra described in Appendix A, (46) gives rise to the following diffusion term in Eq. (3):

$$\alpha \frac{\partial}{\partial \theta} \left[\frac{2}{15} \eta^6 (\cos\theta)^2 \frac{\partial \eta}{\partial \theta} \right],$$

where the factor

$$D_{\text{BO}} = -(2/15)\alpha \eta^6 (\cos\theta)^2$$

plays the role of diffusivity.

If, however, we compare (46) to the corresponding expression derived in Ref. 8 [see the formula without a number following their Eq. (2.20)], the latter misses the time-derivative term [the first term in (46)]. All other terms of the two expressions coincide, but the omission results in a different expression for the diffusivity,

$$D_{\text{HM}} = \alpha \left[-(5/24) h^4 \cos\theta + (3/40) h^6 (\cos\theta)^2 \right].$$

Observe that "our" expression, D_{BO}, is always *negative*, whereas D_{HM} can be *either negative or positive*, depending on h and θ.

Given that negative diffusivity is the main mechanism of inertial instability, this omission can have a significant impact on the dynamics of the flow.

B. The paper by Johnson (Ref. 7)

This paper examined three limiting cases. Inertia was taken into account in the one considered in Sec. IV.

Unlike the present paper and Ref. 8, Ref. 7 examined the effect of inertia using a short-wave approximation. Accordingly, the steady-state flow was assumed "slowly varying in the θ direction" and, hence, "locally independent of θ." Note, however, that the steady-state flow is defined on an order-one interval, $0 \leqslant \theta < 2\pi$, and therefore cannot be assumed slowly varying. What the author meant was, in fact, that the θ scale of the steady state is much larger than the wavelength of the disturbance—which effectively implies that the wavelength of the latter is short. Most importantly, this assumption requires introduction of a short-scale spatial variable. In other words, since the spatial scale of the steady state cannot be scaled up, the spatial scale of the disturbance should be scaled down.

Unfortunately, this has not been done, and the problem was analyzed in terms of the "natural" variable θ. As a result, Eqs. (15a), (17), (21), and (22) derived in Ref. 7 miss a number of important terms involving high-order derivatives with respect to θ.

C. The papers by O'Brien (Ref. 9) and Benilov, Kopteva, and O'Brien (Ref. 10)

These papers examined the stability of the film affected by hydrostatic pressure and surface tension (inertia was not taken into account), using an equation equivalent to our Eq. (3) with $\alpha = 0$. It was shown numerically that unstable modes exist in this case, which appears to contradict our results (recall that we concluded that pressure gradient is a neutrally stable effect, while surface tension is even a stabilizing one).

To reconcile the results obtained in Refs. 9 and 10 with those of the present work, observe that, in the former, instability was found for *higher* eigenmodes (i.e., those that make many oscillations per period). The present work, on the other hand, is concerned mainly with the first eigenmode, as the asymptotic method that we use is inapplicable to the higher modes.

Note also that, for the higher modes, the hydrostatic pressure term in the original equation (3) is comparable to the leading-order terms, *which violates the lubrication approximation under which* (3) *was derived*. In other words, the unstable modes found in Refs. 9 and 10 should be reexamined using a model that is not based on the lubrication approximation.

VII. CONCLUDING REMARKS

We have examined the stability of a thin viscous film in a rotating cylinder with respect to harmonic disturbances. Three effects were taken into account: inertia was demonstrated to cause instability, surface tension turned out to be a stabilizing influence, whereas the hydrostatic pressure gradient does not affect the stability at all. An instability criterion [condition (35) and (36)] has been derived, reflecting the balance of inertia and surface tension—using which it was shown that the effect of the latter is weak. The inertial instability, however, can still be inhibited by viscosity, which can make the characteristic time of growth so large (literally, millions of years) that the film would appear effectively stable.

Finally, we note that at least some of the experiments with rimming flows show that unstable disturbances depend on the axial variable (i.e., vary along the cylinder's axis). This suggests the need to extend the results of the present study to the full three-dimensional problem.

APPENDIX A: DERIVATION OF EQ. (3)

1. The nondimensional parameters

Consider a thin film of incompressible liquid on the inside surface of a cylinder of radius R, with a horizontal axis, which is rotating about this axis with constant angular velocity Ω (see Fig. 1). We shall use polar coordinates (r,θ), with the origin at the center of the cylinder, so the thickness h of the film depends on the polar angle θ and the time t. We shall also introduce the density ρ of the film, its kinematic viscosity ν and surface tension γ, and also the acceleration due to gravity g.

In addition to R, the problem includes another characteristic length. To introduce it, observe that the volume of fluid per unit length of the cylinder,

$$\int_0^{2\pi}\int_{R-h}^{R} r\,dr\,d\theta = \int_0^{2\pi}\left(Rh - \frac{1}{2}h^2\right)d\theta,$$

is conserved, and we can define

$$\langle h\rangle = \frac{1}{2\pi R}\int_0^{2\pi}\left(Rh - \frac{1}{2}h^2\right)d\theta.$$

If $h \ll R$ (which is the limit we are interested in), $\langle h\rangle$ represents the mean thickness of the film.

The evolution of the film is governed by four nondimensional parameters. First, we shall introduce the nondimensional mean thickness,

$$\delta = \frac{\langle h\rangle}{R}. \tag{A1}$$

Second, we define the ratio of the gravitational and centrifugal forces,

$$G = \frac{g}{\Omega^2 R}, \tag{A2}$$

where g is the acceleration due to gravity. Third, we introduce a parameter characterizing viscosity (again, relative to rotation),

$$N = \frac{\nu}{\Omega\langle h\rangle^2}, \tag{A3}$$

where ν is the kinematic viscosity. Finally, the most convenient parameter to characterize capillary effects is

$$\Gamma = \frac{\gamma\langle h\rangle}{\rho\Omega^2 R^4}, \tag{A4}$$

where γ is the coefficient of surface tension and ρ is the density of the film.

The lubrication approximation corresponds to the assumptions

$$\delta \ll 1, \quad N \gg 1, \quad G \lesssim N, \tag{A5}$$

in which case viscosity is on par or stronger than gravity, and in any case stronger than the hydrostatic pressure gradient and inertia. The importance of surface tension, in turn, depends on Γ. We shall consider the simplest case,

$$\Gamma \ll G \sim N, \tag{A6}$$

where capillary effects are weaker than gravity and viscosity.

In addition to surface tension, there are another two groups of small terms in the governing equations. First, there are terms associated with the hydrostatic pressure gradient and the curvature of the flow—the ratio of these to the leading-order terms is δ. Second, there are inertia terms, i.e., the material derivatives in the Navier–Stokes equations—the ratio of these to the leading-order terms is G^{-1}. In order to derive the most general equation, we assume that surface tension, hydrostatic pressure, and inertia are all of the same order

$$\frac{\Gamma}{G} \sim \delta \sim \frac{1}{G}. \tag{A7}$$

2. The governing and asymptotic equations

A two-dimensional flow of viscous fluid inside a cylinder with horizontal axis is convenient to describe by the radial and angular velocities $u(r,\theta,t)$ and $v(r,\theta,t)$, and the pressure $p(r,\theta,t)$ (r and θ are the polar coordinates, t is the time). In terms of u, v, and p, the governing equations are (see, e.g., Ref. 20)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \frac{1}{\rho}\frac{\partial p}{\partial r} = -g \sin\theta + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} - u - 2\frac{\partial v}{\partial \theta}\right)\right], \tag{A8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\left(\frac{\partial v}{\partial \theta} + u\right) + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = -g\cos\theta + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 v}{\partial \theta^2} - v + 2\frac{\partial u}{\partial \theta}\right)\right], \tag{A9}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta} = 0, \tag{A10}$$

where g is the acceleration due to gravity, ρ is the density, and ν is the kinematic viscosity. We assume that the cylinder is rotating with constant angular velocity Ω, which corresponds to the following boundary conditions:

$$u = 0, \quad v = \Omega R \quad \text{at } r = R, \tag{A11}$$

where R is the radius of the cylinder. We shall also need boundary conditions on the free surface of the film, i.e., at $r=R-h(\theta,t)$, where h is the thickness of the film. We shall require that

$$\mathbf{n}\cdot\sigma\mathbf{n} = \gamma\kappa, \quad \mathbf{t}\cdot\sigma\mathbf{n} = 0 \quad \text{at } r = R - h, \tag{A12}$$

where

$$\mathbf{n} = \frac{1}{\sqrt{1 + \frac{1}{(R-h)^2}\left(\frac{\partial h}{\partial \theta}\right)^2}}\begin{bmatrix} 1 \\ \frac{1}{R-h}\frac{\partial h}{\partial \theta}\end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} \frac{1}{R-h}\frac{\partial h}{\partial \theta} \\ -1 \end{bmatrix} \tag{A13}$$

are the unit normal and a tangent (not necessarily unit),

$$\sigma = \begin{bmatrix} 2\rho\nu\frac{\partial u}{\partial r} - p & \frac{\rho\nu}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \rho\nu\frac{\partial v}{\partial r} \\ \frac{\rho\nu}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \rho\nu\frac{\partial v}{\partial r} & \frac{2\rho\nu}{r}\left(\frac{\partial v}{\partial \theta} + u\right) - p \end{bmatrix} \tag{A14}$$

is the stress tensor (see Ref. 20), γ is the surface tension, and κ is the curvature of the surface. An expression for the curvature can be extracted from Ref. 21,

$$\kappa = \frac{(R-h)^2 + (R-h)\frac{\partial^2 h}{\partial \theta^2} + 2\left(\frac{\partial h}{\partial \theta}\right)^2}{\left[(R-h)^2 + \left(\frac{\partial h}{\partial \theta}\right)^2\right]^{3/2}}. \tag{A15}$$

We shall also require that the normal velocity of particles at the surface matches the normal velocity of the surface itself. A straightforward calculation yields

$$\frac{\partial h}{\partial t} + \frac{v}{R-h}\frac{\partial h}{\partial \theta} + u = 0 \quad \text{at } r = R-h.$$

This condition can be rewritten in a more convenient form. Integrate the continuity equation (A10) with respect to r from $R-h$ to R and use the first one of the boundary conditions (A11),

$$-[(R-h)u]_{r=R-h} + \frac{\partial}{\partial \theta}\int_{R-h}^{R} v\,dr - \frac{\partial h}{\partial \theta}v = 0.$$

Combining the last two equalities, we obtain

$$(R-h)\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\int_{R-h}^{R} v\,dr = 0. \tag{A16}$$

Equations (A8)–(A16) form a closed set.

As mentioned in the preceding section, the problem at hand is governed by four nondimensional parameters δ, G, N, and Γ [defined by (A1)–(A4)]. Summarizing applicability conditions (A5)–(A7), we obtain

$$N \sim G \sim \frac{1}{\delta}, \quad \Gamma \sim 1,$$

or, equivalently,

$$N=\frac{N'}{\delta},\quad G=\frac{G'}{\delta},\quad \Gamma=\Gamma', \tag{A17}$$

where N', G', and Γ' are constants of order one. The corresponding nondimensional variables are

$$\tilde{r}=\frac{1}{\delta}\frac{R-r}{R},\quad \tilde{\theta}=\theta,\quad \tilde{t}=\Omega t, \tag{A18}$$

$$\tilde{u}=\frac{1}{\delta}\frac{u}{R\Omega},\quad \tilde{v}=\frac{v}{R\Omega},\quad \tilde{p}=\frac{\delta p}{\rho(R\Omega)^2},\quad \tilde{\sigma}=\frac{\delta\sigma}{(R\Omega)^2}, \tag{A19}$$

$$\tilde{h}=\frac{1}{\delta}\frac{h}{R},\quad \tilde{\kappa}=R\kappa. \tag{A20}$$

Substituting (A17)–(A20) into the governing equations (A8)–(A16), we obtain (hats omitted)

$$\delta\left(\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial r}+\frac{v}{1-\delta r}\frac{\partial u}{\partial\theta}\right)-\frac{v^2}{1-\delta r}-\frac{1}{\delta^2}\frac{\partial p}{\partial r}=-\frac{G'}{\delta}\sin\theta+N'\left[\frac{\partial^2 u}{\partial r^2}-\frac{\delta}{1-\delta r}\frac{\partial u}{\partial r}+\frac{\delta}{(1-\delta r)^2}\left(\delta\frac{\partial^2 u}{\partial\theta^2}-\delta u-2\frac{\partial v}{\partial\theta}\right)\right], \tag{A21}$$

$$\frac{\partial v}{\partial t}-u\frac{\partial v}{\partial r}+\frac{v}{1-\delta r}\frac{\partial v}{\partial\theta}+\frac{\partial uv}{1-\delta r}+\frac{1}{1-\delta r}\frac{1}{\delta}\frac{\partial p}{\partial\theta}=-\frac{G'}{\delta}\cos\theta+\frac{N'}{\delta}\left[\frac{\partial^2 v}{\partial r^2}-\frac{\delta}{1-\delta r}\frac{\partial v}{\partial r}+\frac{\delta^2}{(1-\delta r)^2}\left(\frac{\partial^2 v}{\partial\theta^2}-v+2\delta\frac{\partial u}{\partial\theta}\right)\right], \tag{A22}$$

$$\delta u-(1-\delta r)\frac{\partial u}{\partial r}+\frac{\partial v}{\partial\theta}=0, \tag{A23}$$

$$u=0,\quad v=1,\quad \text{at } r=0, \tag{A24}$$

$$\mathbf{n}\cdot\sigma\mathbf{n}=\Gamma'\kappa,\quad \mathbf{t}\cdot\sigma\mathbf{n}=0\quad \text{at } r=h, \tag{A25}$$

$$\mathbf{n}=\frac{1}{\sqrt{1+\dfrac{\delta^2}{(1-\delta h)^2}\left(\dfrac{\partial h}{\partial\theta}\right)^2}}\begin{bmatrix}1\\ \dfrac{\delta}{1-\delta h}\dfrac{\partial h}{\partial\theta}\end{bmatrix},\qquad \mathbf{t}=\begin{bmatrix}\dfrac{\delta}{1-\delta h}\dfrac{\partial h}{\partial\theta}\\ -1\end{bmatrix}, \tag{A26}$$

$$\sigma=\begin{bmatrix}-2N'\delta^2\dfrac{\partial u}{\partial r}-p & \dfrac{\delta N'}{1-\delta r}\left[\delta^2\dfrac{\partial u}{\partial\theta}-\delta v-(1-\delta r)\dfrac{\partial v}{\partial r}\right]\\ \dfrac{\delta N'}{1-\delta r}\left[\delta^2\dfrac{\partial u}{\partial\theta}-\delta v-(1-\delta r)\dfrac{\partial v}{\partial r}\right] & \dfrac{2N'\delta^2}{1-\delta r}\left(\dfrac{\partial v}{\partial\theta}+\delta u\right)-p\end{bmatrix}, \tag{A27}$$

$$\kappa=\frac{(1-\delta h)^2+\delta(1-\delta h)\dfrac{\partial^2 h}{\partial\theta^2}+2\delta^2\left(\dfrac{\partial h}{\partial\theta}\right)^2}{\left[(1-\delta h)^2+\delta^2\left(\dfrac{\partial h}{\partial\theta}\right)^2\right]^{3/2}}, \tag{A28}$$

$$(1-\delta h)\frac{\partial h}{\partial t}+\frac{\partial}{\partial\theta}\int_0^h v\,dr=0. \tag{A29}$$

Observe that the pressure term in Eq. (A21) appears to exceed all other terms—which, however, does not create a problem as the leading-order p is constant (see below).

Now, seek a solution of the form

$$u=u^{(0)}+\delta u^{(1)}+O(\delta^2),\quad v=v^{(0)}+\delta v^{(1)}+O(\delta^2),$$

$$p=p^{(0)}+\delta p^{(1)}+O(\delta^2).$$

We shall pursue the following plan: using Eqs. (A21)–(A28), we shall relate v to h and thus close Eq. (A29).

To the leading order, (A21)–(A28) yield

$$-\frac{\partial p^{(0)}}{\partial r}=0,\quad \frac{\partial p^{(0)}}{\partial\theta}=G'\cos\theta-N'\frac{\partial^2 v^{(0)}}{\partial r^2},\qquad \frac{\partial u^{(0)}}{\partial r}=\frac{\partial v^{(0)}}{\partial\theta}, \tag{A30}$$

$$u^{(0)}=0,\quad v^{(0)}=1\quad \text{at } r=0, \tag{A31}$$

$$p^{(0)}=\Gamma',\quad \frac{\partial v^{(0)}}{\partial r}=0\quad \text{at } r=h. \tag{A32}$$

Solving (A30)–(A32), we obtain

$$p^{(0)} = \Gamma',$$

$$v^{(0)} = 1 + \frac{G'}{N'}\left(\frac{1}{2}r^2 - rh\right)\cos\theta, \tag{A33}$$

$$u^{(0)} = \frac{G'}{N'}\left[\left(\frac{1}{2}r^2 h - \frac{1}{6}r^3\right)\sin\theta - \frac{1}{2}r^2 h_\theta \cos\theta\right].$$

In the next order, we need only $p^{(1)}$ and $v^{(1)}$,

$$-\frac{\partial p^{(1)}}{\partial r} = -G'\sin\theta, \tag{A34}$$

$$\frac{\partial v^{(0)}}{\partial t} - u^{(0)}\frac{\partial v^{(0)}}{\partial r} + v^{(0)}\frac{\partial v^{(0)}}{\partial \theta} + \frac{\partial p^{(1)}}{\partial \theta} = N'\left(\frac{\partial^2 v^{(1)}}{\partial r^2} - \frac{\partial v^{(0)}}{\partial t}\right), \tag{A35}$$

$$v^{(1)} = 0 \quad \text{at } r = 0, \tag{A36}$$

$$p^{(1)} = \Gamma'\left(h + \frac{\partial^2 h}{\partial \theta^2}\right), \quad \frac{\partial v^{(1)}}{\partial r} = -v^{(0)} \quad \text{at } r = h. \tag{A37}$$

The solution to (A34)–(A37) is

$$p^{(1)} = (r-h)\sin\theta + \Gamma'\left(h + \frac{\partial^2 h}{\partial \theta^2}\right),$$

$$v^{(1)} = \frac{G'}{N'^2}\left[\left(\frac{1}{2}rh^2 - \frac{1}{6}r^3\right)\left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta}\right)\cos\theta - \left(\frac{1}{3}rh^3 - \frac{1}{6}r^3 h + \frac{1}{24}r^4\right)\sin\theta\right] + \frac{G'^2}{N'^3}\left[\left(\frac{1}{24}r^4 h - \frac{1}{6}rh^4\right)\frac{\partial h}{\partial \theta}(\cos\theta)^2 - \left(\frac{1}{360}r^6 - \frac{1}{60}r^5 h + \frac{1}{24}r^4 h^2 - \frac{1}{10}rh^5\right)\cos\theta\sin\theta\right] + \frac{G'}{N'}\left[\left(\frac{1}{3}r^3 - r^2 h + \frac{3}{2}rh^2\right)\cos\theta + \left(rh - \frac{1}{2}r^2\right)\frac{\partial h}{\partial \theta}\sin\theta\right] - r - \frac{\Gamma'}{N'}\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\left(\frac{r^2}{2} - rh\right). \tag{A38}$$

Summarizing (A33) and (A38), we obtain

$$\int_0^b v\,dr = 1 - \frac{G'}{N'}\frac{1}{3}h^2\cos\theta + \delta\left\{\frac{G'}{N'^2}\left[\frac{5}{24}h^4\left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta}\right)\cos\theta - \frac{2}{15}h^5\sin\theta\right] - \frac{G'^2}{N'^3}\left[\frac{3}{40}h^6\frac{\partial h}{\partial \theta}(\cos\theta)^2 - \frac{37}{840}h^7\cos\theta\sin\theta\right] + \frac{G'}{N'}\left(\frac{1}{2}h^4\cos\theta + \frac{1}{3}h^3 h_\theta \sin\theta\right) - \frac{1}{2}h^2 + \frac{\Gamma'}{N'}\left[\frac{1}{3}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right]\right\} + O(\delta^2). \tag{A39}$$

Substitution of (A39) into (A29) yields

$$(1-\delta h)\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left(h - \frac{G'}{N'}\frac{1}{3}h^3\cos\theta\right) + \delta\frac{\partial}{\partial \theta}\left\{\frac{G'}{N'^2}\left[\frac{5}{24}h^4\left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta}\right)\cos\theta - \frac{2}{15}h^5\sin\theta\right] - \frac{G'^2}{N'^3}\left[\frac{3}{40}h^6\frac{\partial h}{\partial \theta}(\cos\theta)^2 - \frac{37}{840}h^7\cos\theta\sin\theta\right] + \frac{G'}{N'}\left(\frac{1}{2}h^4\cos\theta + \frac{1}{3}h^3\frac{\partial h}{\partial \theta}\sin\theta\right) - \frac{1}{2}h^2 + \frac{\Gamma'}{N'}\left[\frac{1}{3}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right]\right\} = O(\delta^2). \tag{A40}$$

Note that the first two expressions in square brackets are contributed by inertia, the third one is contributed by the hydrostatic pressure gradient, and the fourth one, by surface tension.

Rewriting (A40) in terms of

$$\hat{\theta} = \theta, \quad \hat{t} = t, \quad \hat{h} = \sqrt{\frac{G'}{N'}}h, \tag{A41}$$

omitting the hats, and introducing

$$\alpha = \frac{\delta}{G'}, \quad \varepsilon = \delta\sqrt{\frac{N'}{G'}}, \quad \beta = \delta\frac{\Gamma' N'^{1/2}}{G'^{3/2}}, \tag{A42}$$

we obtain

$$(1-\varepsilon h)\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left(h - \frac{1}{3}h^3\cos\theta\right) + \frac{\partial}{\partial \theta}\left\{\alpha\left[\frac{5}{24}\left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta}\right)h^4\cos\theta - \frac{2}{15}h^5\sin\theta - \frac{3}{40}h^6\frac{\partial h}{\partial \theta}(\cos\theta)^2 + \frac{37}{840}h^7\sin\theta\cos\theta\right] + \varepsilon\left[\left(\frac{1}{2}h^4\cos\theta + \frac{1}{3}h^3\frac{\partial h}{\partial \theta}\sin\theta\right) - \frac{1}{2}h^2\right] + \beta\left[\frac{1}{3}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right]\right\} = O(\delta^2). \tag{A43}$$

This equation can be simplified. First, observe that its zeroth order,

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{3}h^3\cos\theta\right) + O(\delta),$$

can be "resubstituted" into the first of the first-order terms, which yields, after straightforward algebra,

$$(1-\varepsilon h)\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\Bigg\{h - \frac{1}{3}h^3\cos\theta + \alpha\left[\frac{2}{15}h^6\frac{\partial h}{\partial \theta}(\cos\theta)^2 - \frac{8}{315}h^7\sin\theta\cos\theta - \frac{2}{15}h^5\sin\theta\right] + \varepsilon\left[\left(\frac{1}{2}h^4\cos\theta + \frac{1}{3}h^3\frac{\partial h}{\partial \theta}\sin\theta\right) - \frac{1}{2}h^2\right] + \beta\left[\frac{1}{3}h^3\left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right]\Bigg\} = O(\delta^2). \quad \text{(A44)}$$

[Note that this equation is not equivalent to (A43), but *asymptotically* equivalent, as we have omitted terms $O(\delta^2)$.] Second, we shall introduce

$$\eta = h - (1/2)\varepsilon h^2.$$

It terms of η, (A44) becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta}\Bigg\{\eta - \frac{1}{3}\eta^3\cos\theta + \alpha\left[\frac{2}{15}\eta^6\frac{\partial \eta}{\partial \theta}(\cos\theta)^2 - \frac{8}{315}\eta^7\sin\theta\cos\theta - \frac{2}{15}\eta^5\sin\theta\right] + \frac{1}{3}\varepsilon\eta^3\frac{\partial \eta}{\partial \theta}\sin\theta + \frac{1}{3}\beta\eta^3\left(\frac{\partial \eta}{\partial \theta} + \frac{\partial^3 \eta}{\partial \theta^3}\right)\Bigg\} = O(\delta^2) \quad \text{(A45)}$$

[again, this equation is *asymptotically* (not *exactly*) equivalent to its predecessor].

Finally, dropping the unspecified small terms on the right-hand side of (A45), we obtain Eq. (3), as required. Expressions (4) for the coefficients of this equation follow from (A17) and (A42), whereas expressions (1) and (2) for the nondimensional variables follow (A18), (A20), (A41), and (A17) and (A1)–(A4)

APPENDIX B: NUMERICAL METHODS

In this appendix, we shall describe the numerical techniques used for solving the boundary-value problem (12) and (6) (for the steady state), and the eigenvalue problem (13) and (11) (for disturbances).

1. Boundary-value problem (12) and (6)

Rewrite Eq. (12) as a set of three first-order equations

$$\frac{du_1}{d\theta} = u_2, \quad \frac{du_2}{d\theta} = u_3, \quad \frac{du_3}{d\theta} = \frac{q - u_1 + (1/3)u_1^3\cos\theta}{(1/3)\beta u_1^3} - u_2, \quad \text{(B1)}$$

where $u_1 = \overline{\eta}$. As the system is subject to periodic boundary conditions, the easiest way to deal with these is to introduce three extra unknowns, v_1, v_2, v_3, which satisfy

$$\frac{dv_1}{d\theta} = 0, \quad \frac{dv_2}{d\theta} = 0, \quad \frac{dv_3}{d\theta} = 0 \quad \text{(B2)}$$

(this method has been previously used in BPT93). Then, the boundary conditions are

$$u_1(0) = v_1(0), \quad u_2(0) = v_2(0), \quad u_3(0) = v_3(0), \quad \text{(B3)}$$

$$u_1(2\pi) = v_1(2\pi), \quad u_2(2\pi) = v_2(2\pi), \quad u_3(2\pi) = v_3(2\pi). \quad \text{(B4)}$$

The two point boundary-value problem (B1)–(B4) can then be solved using the an appropriate NAG routine, D02TKF.

2. Eigenvalue problem (13) and (11)

Observe that the coefficients of (13) involve four derivatives of $\overline{\eta}(\theta)$. The first two derivatives are available directly as u_2, u_3 (see the preceding section), and the third derivative can be computed as the right-hand side of the last equation of set (B1). Finally, the fourth derivative can be obtained using the formula

$$\frac{d^2u_3}{d\theta^2} = -\frac{9qu_2}{\beta u_1^4} + \frac{6u_2}{\beta u_1^3} - u_3,$$

which can be derived by differentiating the last equation of set (B1).

Next, rewrite the eigenvalue problem (13) and (11) in the form

$$a_0(\theta)\phi + a_1(\theta)\frac{d\phi}{d\theta} + a_2(\theta)\frac{d^2\phi}{d\theta^2} + a_3(\theta)\frac{d^3\phi}{d\theta^3} + a_4(\theta)\frac{d^4\phi}{d\theta^4} = s\phi, \quad \text{(B5)}$$

where $s = i\omega$ and

$$a_0 = \frac{dC}{d\theta}, \quad a_1 = C + \frac{dB}{d\theta}, \quad a_2 = B, \quad a_3 = \frac{dB}{d\theta}, \quad a_4 = B.$$

[Recall that B and C are related to $\overline{\eta}$ by (14).] We shall write $a_k(\theta)$ and $\phi(\theta)$ as complex Fourier series,

$$a_k(\theta) = \sum_{j=-\infty}^{\infty} a_{k,n}e^{in\theta}, \quad \phi(\theta) = \sum_{j=-\infty}^{\infty} \phi_n e^{in\theta},$$

where $a_{k,n}$ are given by

$$a_{k,n} = \frac{1}{2\pi}\int_0^{2\pi} a_k(\theta)e^{-in\theta}d\theta,$$

and the Fourier coefficients ϕ_n are unknown. Using the identity

$$\left(\sum_{n=-\infty}^{\infty}\alpha_n e^{in\theta}\right)\left(\sum_{n=-\infty}^{\infty}\beta_n e^{in\theta}\right) = \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\alpha_{m-n}\beta_n\right)e^{im\theta},$$

we can rewrite (B5) in the form

$$\sum_{k=0}^{4}\sum_{m=-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}a_{k,m-n}(in)^k\phi_n\right]e^{im\theta} = s\sum_{m=-\infty}^{\infty}\phi_m e^{im\theta},$$

which can be further rearranged into

$$\sum_{n=-\infty}^{\infty} A_{mn}\phi_n = s\phi_m, \tag{B6}$$

where

$$A_{mn} = \sum_{k=0}^{k=4} a_{k,m-n}(in)^k.$$

Equation (B6) is, essentially, an eigenvalue problem for a given matrix $\mathbf{A}=\{A_{mn}\}$ of infinite order, with $\phi=\{\phi_n\}$ and s being the eigenvector and eigenvalue. In practice, $\mathbf{A}$ and ϕ can be truncated at a large integer N, so that $-N \leq m,n \leq N$, in which case (B6) is approximated by an eigenvalue problem for a $(2N+1)\times(2N+1)$ matrix. This eigenvalue problem was solved using an appropriate NAG routine.

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