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The generation of radiating waves in a singularly-perturbed Korteweg–de Vries equation

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We consider a fifth-order KdV equation, where the fifth-order derivative term is multiplied by a small parameter. It has been conjectured that this equation admits a non-local solitary wave solution which has a central core and an oscillatory tail either behind or in front of the core. We prove that this solution cannot be exactly steady, and instead the amplitude of the central core decays due to the energy flux generated in the oscillatory tail. The decay rate is calculated in the limit as the parameter tends to zero.

In order to verify the analytical results, we have developed a high-precision spectral method for numerical integration of this equation. The analytical and numerical result show good agreement.

1. Introduction

Recently there has been considerable interest in non-local solitary waves. These consist of a central part resembling classical solitary waves which, however, are accompanied by co-propagating oscillatory tails. They occur for solitary water waves in the presence of surface tension when the Bond number lies between 0 and $\frac{1}{3}$ (e.g. see Hunter and Vanden-Broeck [9] and Vanden-Broeck [14] for numerical studies, and Beale [3] and Sun [13] for theoretical work), and also for internal solitary waves of mode number greater than two (e.g. Vanden-Broeck and Turner [15] for a numerical investigation, and Akylas and Grimshaw [1] for an analytical study). A lot of attention has been focused on the fifth-order Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} + \varepsilon^2 u_{xxxxx} = 0 , \qquad (1)$$

which has been obtained by Kakutani and Ono [10] for magneto-acoustic waves and Hasimoto [7], and later by Hunter and Scheurle [8] as a model for capillary-gravity waves of small amplitude when the Bond number is close to but just less than $\frac{1}{3}$. See for instance the analytical studies by Pomeau et al. [12], Byatt-Smith [5], Amick and Toland [2], Karpman [11] and Grimshaw and Joshi [6], and the comprehensive numerical investigation by Boyd [4] who has called these non-local solitary waves "nanopterons" and drawn attention to their prevalence in a variety of physical contexts.

Most attention has focused on steadily propagating symmetric non-local solitary waves, and two types have been described in the literature. There are *symmetric* non-local solitary waves with identical oscillatory tails on both sides of the central core, and *asymmetric* non-local solitary waves with the oscillatory tail on one side only. Pomeau et al. [12] examined both types asymptotically in the limit $\varepsilon \to 0$ and found the

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relationship between the amplitudes of the solitary wave core and the tail oscillations which are exponentially small with respect to ε (or more strictly, with respect to $\delta = \varepsilon^2 A$ where A is the core amplitude).

However, symmetric non-local solitary waves will have an energy flux due to the oscillatory tails, and since this will be nonzero in general, it is clear that such symmetric waves require an energy source on one side of the wave and an energy sink on the other. Hence it seems very unlikely that they could be readily generated in practical situations, or realized as the outcome of an initial-value problem with a localized initial condition. The latter conclusion was made by Pomeau et al. [12], but they did not discuss the energetics of this localized initial-value problem. Our expectation is that a localized initial condition will produce one-sided non-local solitary waves, with the oscillatory waves appearing only on that side of the central core consistent with the energy flux of the oscillatory waves being directed outwards.

But now it is apparent that *steady* asymmetric non-local solitary waves cannot exist since the central core must lose energy to the outwardly-propagating oscillatory waves. Indeed, Boyd's [4] attempt to numerically calculate steady asymmetric solutions was unsuccessful, which led him to conclude that "... only symmetric nanopterons exist in a strict sense, that is, with an error smaller than $\mathcal{O}(a^2)$. It would be very interesting to have a rigorous proof of this hypothesis. It would be even better if analysis could explain *why* the nanopteron must be symmetric." (Here a is the amplitude of the oscillatory tail.) In section 2.1 we shall show that the existence of *steady* non-local asymmetric solitary waves contradicts the energy conservation law, and a rigorous proof of their non-existence will be presented, based on this contradiction.

In the remainder of this paper we consider the time evolution of non-local asymmetric solitary wave solutions of the fifth-order Korteweg–de Vries equation (1), emphasizing again that these are inherently unsteady and the solitary wave core decays as it loses energy to the outwardly-propagating waves in the oscillatory tail. However, as $\varepsilon \to 0$ this decay is exponentially slow, and the result of Pomeau et al. [12] for the amplitude of the oscillatory tail is asymptotically correct, even though it was obtained for the (non-existent) steady solution. This discussion is presented in sections 2.2 and 2.3. Then in section 3 we describe our numerical results which confirm this asymptotic result.

2. Analytical results

2.1. Steady asymmetric solutions

First we shall discuss the non-existence of steady, non-local, asymmetric solutions to eq. (1). We seek a solution of (1) in the form

$$u(x, t) = u(\theta)\,, \quad \theta = x - ct\,. \tag{2}$$

Substitution of (2) into (1) gives

$$-cu_\theta + 6uu_\theta + u_{\theta\theta\theta} + \varepsilon^2 u_{\theta\theta\theta\theta\theta} = 0\,. \tag{3}$$

Equation (3) should be supplemented by the following boundary conditions:

$$u \to u_{\mathrm{w}} \quad \text{as } \theta \to \infty\,, \tag{4}$$

$$u \to 0 \quad \text{as } \theta \to -\infty\,, \tag{5}$$

where u_{w} is a periodic function which describes the oscillatory tail. If $\varepsilon \ll 1$, the amplitude of this tail is small, and (4) can be replaced by

$$u \to a \sin(kx + \phi) + \mathcal{O}(a^2) \quad \text{as } \theta \to \infty\,, \tag{6}$$

where a, ϕ and k are the amplitude, phase and wavenumber, respectively. The latter can be readily found via substitution of (6) into the linearized version of (3):

$$k = \varepsilon^{-1} + \tfrac{1}{2}\varepsilon c + \mathcal{O}(\varepsilon^3)\,. \tag{7}$$

Observe that, if $\varepsilon \ll 1$, the trailing wave is short: $k \gg 1$.

The boundary value problem (3), (5), (6) was solved asymptotically by Pomeau et al. [12]. For the central core part of the non-local solitary wave the solution was obtained by letting

$$u = \sum_{n=0}^{\infty} \varepsilon^{2n} u_n\,, \quad c = \sum_{n=0}^{\infty} \varepsilon^{2n} c_n\,, \tag{8}$$

where

$$u_0 = 2\gamma^2 \operatorname{sech}^2(\gamma\theta)\,, \quad c_0 = 4\gamma^2\,, \tag{9a}$$

$$u_1 = \gamma^4[-20 \operatorname{sech}^2(\gamma\theta) + 30 \operatorname{sech}^4(\gamma\theta)]\,,$$
$$c_1 = 16\gamma^4\,, \tag{9b}$$

and γ is the (free) parameter which characterizes the amplitude and "width" of the soliton core. The amplitude of the trailing wave, in its turn, is related to γ by

$$a = -2\pi K \varepsilon^{-2} \exp\left(-\frac{\pi k}{2\gamma}\right), \tag{10}$$

where $K \approx -19.97$ is a numerical constant. Solution (9), (10) corresponds to the solution u_+ in Grimshaw and Joshi [6], where we note that the full value of k is used rather than the approximate value $k \approx \varepsilon^{-1}$ (see (7)), and that this expression is valid to a relative error of order ε^4. This result also corrects a missing factor of $\frac{1}{2}$ in the result of Pomeau et al. [12].

From a physical viewpoint, the wavenumber k of the trailing wave is determined by the requirement that the phase speed of the latter is equal to the speed c of the central core. Indeed, calculating the dispersion relation of small-amplitude waves within the framework of (1)

$$\omega(k) = -k^3 + \varepsilon^2 k^5\,,$$

we see that the condition

$$c_{\mathrm{ph}}(k) = -k^2 + \varepsilon^2 k^4 = c$$

yields the correct value for k. This is precisely how k is calculated in (7), and in terms of γ it may be shown that

$$k = \frac{1}{\varepsilon}(1 + 4\varepsilon^2\gamma^2)^{1/2}$$

exactly (see [6]). At the same time, the *group* speed of the trailing wave is *not* equal to c:

$$c_{\mathrm{gr}} = -3k^2 + 5\varepsilon^2 k^4 = 2\varepsilon^{-2} + \mathcal{O}(1)\,, \tag{11}$$

and since the wave energy is transferred at the *group* speed, the corresponding energy flux in the moving reference frame is

$$F = \rho(c_{\mathrm{gr}} - c_{\mathrm{ph}}) \neq 0\,,$$

where ρ is the density of energy in the trailing wave. *Obviously, the solitary wave core loses energy (due to the radiation of the trailing wave) and therefore cannot be steady.*

In order to express the above argument in a more rigorous form, we multiply (3) by u and integrate it:

$$-\tfrac{1}{2}cu^2 + 2u^3 + uu_{\theta\theta} - \tfrac{1}{2}(u_\theta)^2 + \varepsilon^2[uu_{\theta\theta\theta\theta} - u_\theta u_{\theta\theta\theta} + \tfrac{1}{2}(u_{\theta\theta})^2] = P\,, \tag{12}$$

where P is the constant of integration (physically, the left-hand side of (12) represents the energy flux). Substitution of boundary conditions (5) and (6) into (12) gives

$$0 = P\,,$$
$$\tfrac{1}{4}a^2(c + 3k^2 - 5\varepsilon^2 k^4) + \mathcal{O}(a^3) = P\,.$$

These equalities are, evidently, incompatible (see also (7)), which proves that the asymmetric solutions cannot exist in the small-amplitude limit $a \ll 1$. This simple proof can be easily generalized for the case when a is not necessarily small.

2.2. Discussion

The above argument entails the following conclusions:

(i) The asymmetric non-local solutions to eq. (1) cannot be steady: the amplitude of the central core decays due to the energy loss caused by the radiation of a trailing wave;

(ii) Since the difference between the group speed c_{gr} of the trailing wave and the speed c of the central core is *positive* (see (11)), the oscillatory wave is radiated *forward* (and therefore can hardly be called "trailing");

(iii) Since all periodic solutions to eq. (1) correspond to non-zero energy flux, steady non-local *symmetric* solutions (see [4,6,12]) cannot evolve from localized initial conditions and *require a source of wave energy as* $x \to -\infty$, *as well as the sink as* $x \to \infty$.

It should also be emphasized that

(iv) Although the non-local asymmetric solitary wave does not exist as an exact steady-state solution, *it still solves eq. (1) with an error of* $\mathcal{O}(a^2)$.
Therefore, if the evolution of the solution is sufficiently slow (i.e. if the time scale of the evolution is of the order of, or greater than a^{-2}), *relationship (10) between the amplitudes of the soliton core and oscillatory wave is asymptotically correct.* We shall expand on this below in section 2.3, and then discuss our numerical results in section 3 which confirm the above asymptotic estimates.

(v) A reviewer called our attention to the observation that it could be conjectured from our result (i) above that all steady solutions of eq. (1) must be symmetric.

2.3. The evolution of solitary waves

In this sub-section we suggest a simple asymptotic approach to the problem of the evolution of a solitary wave which radiates a small amplitude oscillatory wave. In order to determine the rate of decay of the central core, we change variables in equation (1) from (x,t) to (θ,t) where

$$\theta = x - \int_0^t c(t')\,\mathrm{d}t' .$$

Now we can assume that t is "slow", i.e. $\partial/\partial t \ll c\,\partial/\partial\theta$. Observe that the speed $c(t)$ of the core is now described by an *undetermined* function of t. The solitary wave parameter γ will also be described by an undetermined function of t, but the relationship between $c(t)$ and $\gamma(t)$ described by (8) and (9a,b) is maintained to the leading order in ε. Then we construct the energy equation

$$\rho_t + P_\theta = 0 , \qquad (13)$$

where the energy density is

$$\rho = \tfrac{1}{2}u^2$$

and the energy flux P coincides with the left-hand side of (12). Since the small-amplitude oscillatory wave is radiated to the *right* (see conclusion (ii) in section 2.2), we use boundary condition (6), and then integrate (13) from $-\infty$ to θ_0 with respect to θ, where θ_0 is a fixed value of θ less than $(c_{gr}t)$, but much greater than γ^{-1} which is the effective width of the central core. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\theta_0} \rho\,\mathrm{d}\theta + P(\theta = \theta_0) = 0 . \qquad (14)$$

This equation states that the rate of loss of energy of the solitary wave core is balanced by the energy flux in the oscillatory waves. The integral term can be estimated using the unperturbed profile of the KdV soliton u_0 given by (9a). On the other hand $P(\theta = \theta_0)$ is estimated from the oscillatory waves alone (given by the boundary condition (14)). To the leading order in ε we find that

$$8\gamma^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} + \frac{a^2}{2\varepsilon^2} = 0 . \qquad (15)$$

Now we see that the time scale of the evolution of the solution is proportional to a^{-2}, and the time derivative in equation (1) is sufficiently small. Thus, in order to close equation (14), we may use the relationship (10) between a and γ

(see conclusion (iv) in section 2.2). Substitution of (10) into (15) yields

$$\frac{d\gamma}{dt}+\frac{\pi^2 K^2}{4\varepsilon^6\gamma^2}\exp\left(-\frac{\pi k}{\gamma}\right)=0\,.$$

With k given by (7), this equation can be integrated to determine $\gamma=\gamma(t)$ and the rate of decay of the core amplitude $2\gamma^2$ can be estimated.

3. Numerical results

As mentioned above, the asymptotic formula (10) was obtained for the *non-existent* steady asymmetric solution to eq. (1). Nevertheless, in section 2.2 we argued that it is correct for the *quasi*-steady non-local solitary waves. In order to verify this hypothesis, eq. (1) was integrated numerically with the initial condition

$$u(x,0)=u_0(x)\,, \tag{16}$$

where $u_0(x)$ is the unperturbed KdV solitary wave (9a). As it turned out, the traditional spectral or finite-difference methods are not accurate enough to simulate the exponentially weak effect of oscillatory-wave radiation, and hence we developed the following high-precision spectral technique.

3.1. Numerical method

The basic concept of our method consists in solving the equation in the Fourier space. In terms of the Fourier transform,

$$\hat{u}(\lambda,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)\,e^{i\lambda x}\,dx$$

we can rewrite (1) in the form

$$\hat{u}_t(\lambda)=i\omega(\lambda)\,\hat{u}(\lambda)+3i\lambda\int_{-\infty}^{\infty}\hat{u}(\lambda-\lambda')\,\hat{u}(\lambda')\,d\lambda'\,, \tag{17}$$

where $\omega(\lambda)=-\lambda^3+\varepsilon^2\lambda^5$. In order to "prepare" eq. (16) for numerical integration, we truncate $u(\lambda)$:

$$u(\lambda)\equiv 0 \quad \text{if } |\lambda|\geq\lambda_{tr}$$

and discretize the Fourier variable λ. The integral on the right-hand side of (16) is evaluated using the Simpson rule, while the time derivative is approximated by the leap-frog finite difference scheme. Thus, the accuracy of this method is

$$\delta u=\mathcal{O}(\tau^2,h^4)\,,$$

where τ and h are the steps in time t and wavenumber λ respectively. The error caused by the truncation of the integral is *exponentially* small: since $u(x,\,t)$ is a smooth function of x, $\hat{u}(\lambda,\,t)$ decreases exponentially as $\lambda\to\infty$.

It should be emphasized that this technique *is numerically stable regardless of the ratio of the steps τ and h*; which means that the accuracy with respect to h can be improved *with the time step τ being fixed*. The only constraint on τ follows from the condition of the stability of the leap-frog approximation:

$$\tau\leqslant\frac{1}{\omega(\lambda_{tr})}$$

(this constraint can be easily satisfied in all cases of practical interest). The independence of steps τ and h is a major advantage over the "traditional" spectral method where the spatial step may not be diminished without diminishing the time step. This advantage is particularly important for the fifth-order KdV equation; if it is integrated in the x-space (the traditional spectral method), the spatial step should be small enough to resolve the short-wave oscillatory tail and the stability requirement then makes this approach practically not applicable to (1) with $\varepsilon\leqslant 0.3$.

In general, the traditional spectral method is slower in all cases which require high accuracy.

3.2. Results

We integrated the initial value problem (16),

(17) using the above numerical method. In order to compute the amplitudes of the solitary wave core and the oscillatory wave, we performed the inverse Fourier transform each fifth time step. The fast Fourier transform was used to locate the crest of the core and the beginning of the oscillatory wave, after which we used the "slow" Fourier transform to compute the amplitudes of the core and oscillatory wave (the FFT was not accurate enough for this task). The amplitude of the core we denote by A and determine as the maximum of $u(x, t)$. The amplitude of the oscillatory wave was determined as

$$a = \tfrac{1}{2} \max_{1 \le j \le 5} \{u^{(j)}_{\text{crest}} - u^{(j)}_{\text{trough}}\} ,$$

where j enumerates the local maxima ($u^{(j)}_{\text{crest}}$) and minima ($u^{(j)}_{\text{trough}}$) of the function $u(x,t)$ (the minimum closest to the solitary wave and the next maximum correspond to $j = 1$). Since both A and a depend on t (see fig. 1), it is important to set a formal criterion of "quasi-steadiness" which we define as that moment of time when the solitary wave core has finished adjusting its initial form and the oscillatory wave has adjusted its amplitude to the current value of A. *We assumed that the solution is quasi-steady when $a(t)$ begins to decrease* (see fig. 1).

During the computation we monitored the energy

$$E = \int_{-\infty}^{\infty} u^2 \, dx ,$$

the Hamiltonian

$$H = \int_{-\infty}^{\infty} [u^3 - \tfrac{1}{2}(u_x)^2 + \tfrac{1}{2}\varepsilon^2(u_{xx})^2] \, dx$$

and the velocity of the center of mass

$$v = \int_{-\infty}^{\infty} x u_t \, dx .$$

All these are conserved quantities for (1). Note that the spectral method automatically conserves the mass invariant (i.e. $\hat{u}(\lambda, t)$ at $\lambda = 0$). H and v proved to be most sensitive to the computational errors and we were able to keep $\delta H/H$ and $\delta v/v$ only within the limits of 0.5%, whereas the error $\delta E/E$ was usually less than 0.01%.

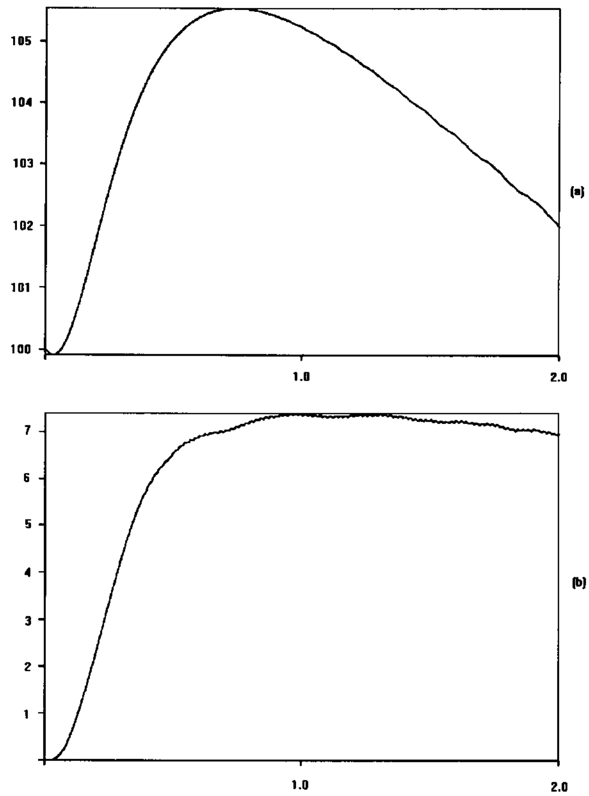

Fig. 1. The amplitudes of the solitary wave core (a) and oscillatory wave (b) versus time ($\varepsilon^2 = 0.065$).

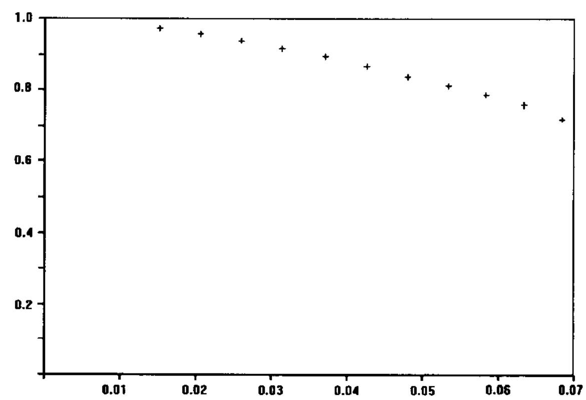

Fig. 2. The amplitude of the oscillatory wave scaled by the left-hand side of formula (10), versus the parameter $\varepsilon^2 A$.

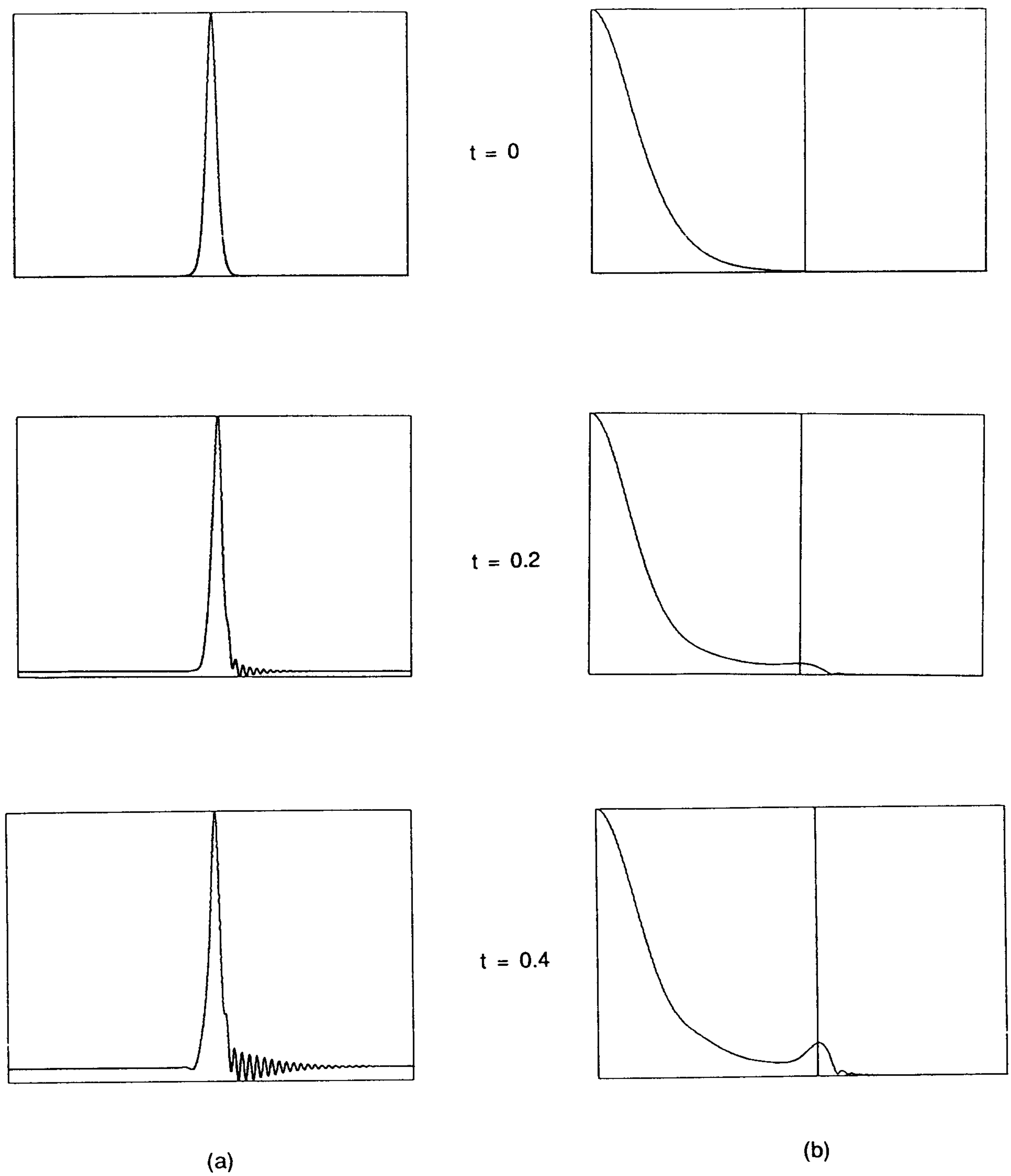

Fig. 3. The evolution of a solitary wave ($\varepsilon^2 = 0.065$, the initial amplitude $A(0) = 1$). (a) $u(x, t)$; (b) $|\hat{u}(\lambda, t)|$. The vertical line indicates the asymptotic value of the wavenumber of oscillatory wave (formula 7)).

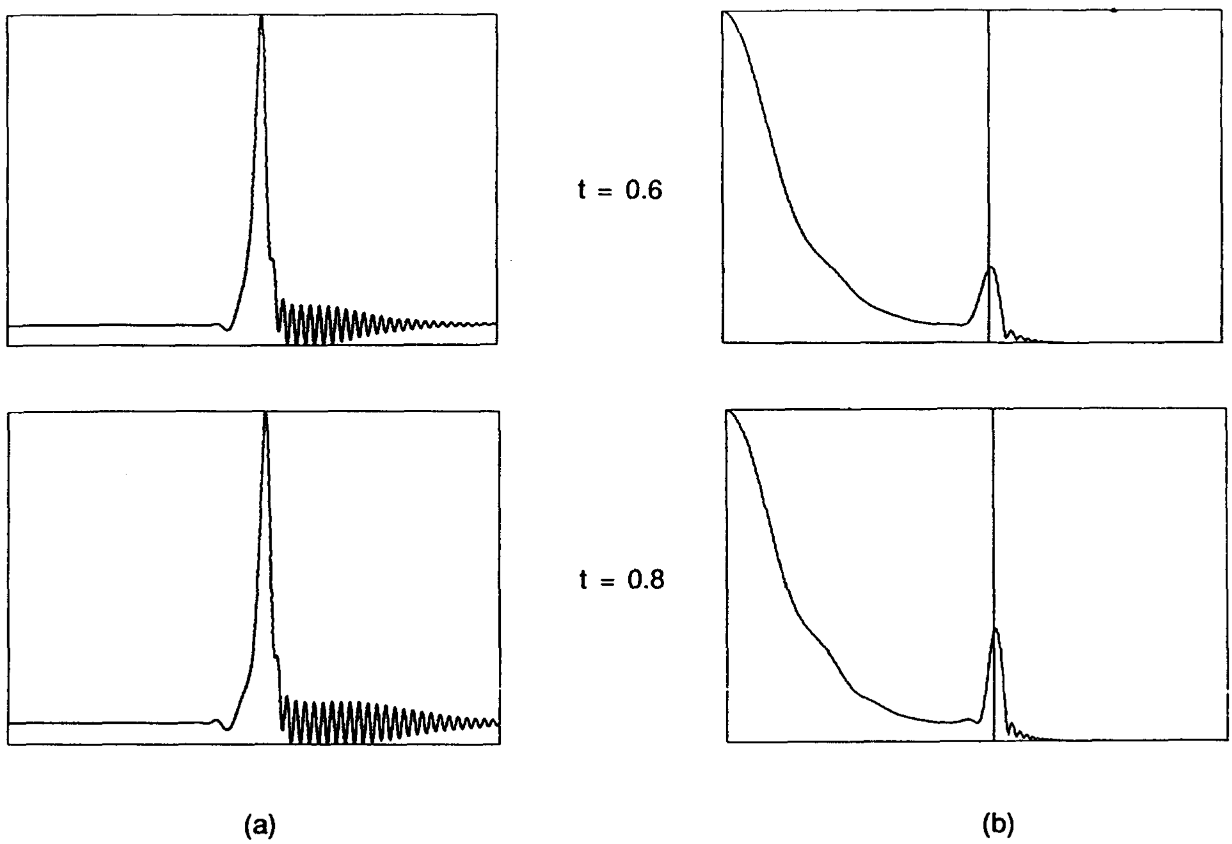

Fig. 3. Contd.

In order to verify formula (10) for the amplitude of the oscillatory wave, we shall first relate the parameter γ to the *observed* value of A, which takes into account distortion of the solitary wave form due to the perturbation. Taking into account the first correction (9b), we obtain

$$2\gamma^2 \approx A - \tfrac{5}{2}\varepsilon^2 A^2 . \qquad (18)$$

Then, the computed values of a were scaled by the right-hand side of (10) (the scaled quantity is denoted by $\tilde{a}$) and plotted versus $\delta = \varepsilon^2 A$ (see fig. 2). Evidently, (10) and (18) provide a good approximation of the numerical results for $0.015 \leqslant \delta \leqslant 0.04$. Unfortunately, we were not able to compute $\tilde{a}$ for $\delta < 0.015$, where the roundoff error was of the order of (or even greater than) the solution itself. Instead, we extrapolated $\tilde{a}(\delta)$ using the cubic parabola, passing through the last three points. Assuming that the coefficient of the quadratic term is zero (it should be recalled that formulae (14) and (22) are valid up to the quadratic term in ε), we obtain

$$\tilde{a} = 0.9991 - 136.51\,\delta^2 + 1803.1\,\delta^3 ,$$

which shows that formulae (10), (14) work increasingly well as $\delta \to 0$. The difference between the exact value $\tilde{a}(0) = 1$ and the numerical one $\tilde{a}(0) = 0.9991$ characterizes the computational error of our numerical method.

It is also worth noting that the "waves" on the graph $\tilde{a}(\delta)$ (see fig. 2) are too big to be accounted for solely by the numerical error. The most probable reason is that our criterion of quasi-steadiness does not work well when the maximum of $a(t)$ is flat, while the maximum of $A(t)$ is "sharp" (see fig. 1). As $\varepsilon \to 0$, both maxima become flat, and the waves disappear.

In conclusion, we should emphasize that the *symmetric* (steady or quasi-steady) non-local solitary wave never emerged from the localized initial condition. In all cases the oscillatory tail grew only *in front* of the solitary wave (see fig. 3a). Fig. 3b, in its turn, demonstrates that the asymptotic expression (7) for the wavenumber k of the oscillatory wave agrees well with the numerical results.

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