Stability of plane nonlinear waves in smoothly inhomogeneous random media

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The stability of plane stationary waves in inhomogeneous media to three-dimensional perturbations is investigated. The most typical (and at the same time important in practice) example is taken to be a long solitary wave propagating on the surface of an ideal liquid over an uneven bottom. It is shown that the wave is unstable to parametric resonance between flexural perturbations of its front and "collisions" with the irregularities of the bottom. The factor that limits the instability is the backward emission of unstable perturbation by the soliton, so that solitary waves of sufficiently small amplitudes become stabilized. The growth rate of the perturbation and the shape of the neutral-stability curve are calculated. The theory developed for solitary waves on water is qualitatively generalized to include arbitrary media and wave solutions of four types: 1) solitons; 2) periodic waves; 3) (Burgers) shock waves with continuous profiles; 4) discontinuous (gasdynamic) shock waves. In addition, the treatment is restricted to waves that are stable in the absence of inhomogeneities. Analysis has shown that wave solutions of the second and fourth type are apparently unstable in any medium with random or periodic smooth inhomogeneities, since they are unstable, just as in the case of solitons on water, only in a certain range of their amplitudes.

1. INTRODUCTION

Plane (one-dimensional) waves with stationary profiles play a notable role in various branches of physics. The most important examples are shock waves in an ideal gas, solitary and periodic waves on the surface (in the interior) of a homogeneous (layered) liquid, and tidal ~~gusts~~ bores (waves describing the level drop of a liquid with a "breaker" on the front). Ion-sound and magnetosonic solitons and periodic waves are of considerable interest for plasma physics, the so-called fluxons are of interest for the theory of Josephson junctions, while shock waves with continuous profiles of the Burgers type are encountered in many problems of acoustics, hydrodynamics, chemical kinetics, and combustion theory.

All the foregoing plane-wave types have an important common property, viz., stability to longitudinal (one-dimensional) and transverse (to the wave-propagation direction) perturbations[1]). In particular cases that reduce to equations of the Kadomtsev-Petviashvili type (gravitational waves in a liquid, sound in a plasma), soliton stability was demonstrated in Refs. 1–4, and stability of periodic (cnoidal) waves in Refs. 5 and 6. The stability of Burgers waves was proved in the framework of the Zabolotskaya-Khokhlov equation in Refs. 7 and 8, while a stability criterion for shock waves in a gas was obtained in Ref. 9. Note that the question of the stability of any wave solution is of importance, since only stable solutions are realized in actual physical situations.

At the same time, real wave media have as a rule spatial inhomogeneities. These can be the roughness of the basin bottom in the case of waves in a liquid, turbulent wind in shock-wave propagation, fluctuations of ion density in a plasma, and others. The inhomogeneities of the medium act apparently as a destabilizing factor, but their effect on the stability of stationary waves has so far not been investigated.

The main result of the present paper is the following statement; nonlinear waves (solitary, periodic, or shock) are unstable in a wide range of parameters if the medium contains sufficiently continuous spatial inhomogeneities (albeit small). The instability is transverse and is manifested in the case of both periodic and stochastic inhomogeneities. We shall elucidate the instability mechanism with a Korteweg–de Vries soliton and periodic fluctuations of a wave medium as the example.

It is known[2,3] that "secondary" waves of flexural type can propagate along a soliton front. If the length of these waves exceeds greatly the longitudinal spatial scale of the soliton, their dispersion equation has an "acoustic" form:

$$\omega=c|k|, \tag{1}$$

where ω and k are the frequency and wave number of the second sound, and c is its velocity (which depends on the soliton parameters). Note that relation (1) is valid only for media with negative dispersion (gravitational waves in a liquid, sound in a plasma). For media with positive dispersion (capillary waves, etc.) the second-sound frequency is pure imaginary ($\omega = ic|k|$), meaning that the soliton is unstable even in a homogeneous medium (in the absence of inhomogeneities).[1-3] We do not deal with positive dispersion in the present paper. Consider now periodic, for simplicity one-dimensional, fluctuations of the parameters of the wave medium (the soliton propagation direction coincides with the parameter-variation direction, see Fig. 1). We denote by T_0 the time in which the wave negotiates one period of the fluctuations and introduce the frequency $\Omega = 2\pi/T_0$. It is easily seen that a secondary wave with wave number $k_* = \Omega/2c$ enters into parametric resonance with the periodic inhomogeneities of the medium:

$$\Omega=2\omega_*. \tag{2}$$

The amplitude of the secondary wave increases exponentially, and it is this which leads to instability of the initial soliton. In the case of stochastic fluctuations we have not one

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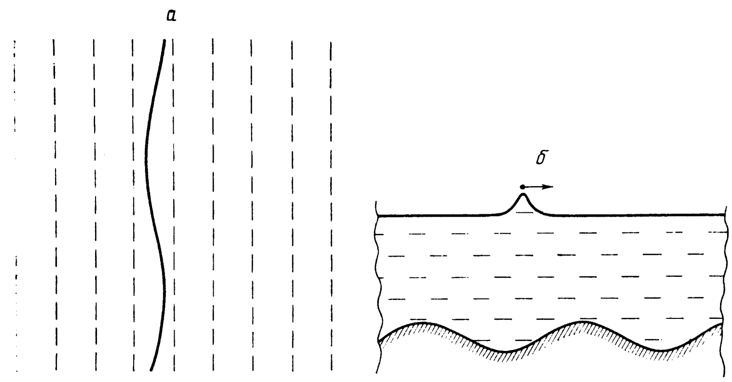

FIG. 1. Formulation of problem: propagation of a solitary wave over the surface of an ideal liquid in a basin with an uneven bottom: a) top view (solid line—soliton ridge, dashed—equal-depth line); b) side view.

frequency Ω but a continuous spectrum, so that the spectrum of the unstable secondary waves is no longer pointlike and occupies a rather broad region. Nor does the instability vanish in the other aforementioned wave media, as well as for other solutions (periodic or shock waves). Indeed, all that is needed for instability development is the presence of second sound, and this is a property of all wave solutions that are stable in homogeneous media, and not only of the soliton solution (see, e.g., Refs. 10–12). Moreover, it can be stated that all the characteristic features of the evolution of second sound depend primarily on the type of solution, and not on the specifics of the actual wave medium. We shall distinguish between four types of solution: solitons, periodic waves, and discontinuous and distributed (Burgers) shock waves.

We emphasize that for the indicated instability to set in it is of principal importance that the inhomogeneities of the medium be *smooth*, i.e., that the characteristic spatial scale of the inhomogeneity be many times larger than the length of the nonlinear wave. This requirement is imposed by the long-wave character of second sound, and accordingly with low frequencies in relations (1) and (2) ($k, \omega, \Omega \to 0$, hence $T_0 \to \infty$).

Note also that the expected effect is of interest for one more reason, namely, the recent[13] investigation of the propagation of nonlinearly weakly dispersive waves in media with small random fluctuations of the parameters. In the case of one spatial variable, a simplified equation was derived, containing only determined coefficients. However, a multidimensional generalization of the proposed asymptotic method turned out to contain nonremovable divergences, due precisely to the long-wave (smooth) component of the fluctuations (see Ref. 13). This circumstance is now seen in a somewhat different light: it is most likely connected with the instability investigated in the present paper. Indirect evidence is provided by the fact that the characteristic instability development time coincides with the time scale of the onset of instabilities in the asymptotic theory of Ref. 13. This question seems worthy of further investigation.

The present paper is devoted to a calculation of the instability parameters of nonlinear waves in inhomogeneous media, using as an example long surface solitons in a liquid over a randomly rough bottom. This example is chosen not only because of its practical importance,[2] but also because of the universality of the equation that describes this case (generalized Kadomtsev-Petviashvili equation). In Sec. 2 we deduce from this equation an asymptotic relation that describes second sound on a soliton propagating above a smoothly uneven bottom. The relation derived is used in Sec. 3 to investigate the expected instability, whose growth rate is calculated. In Sec. 4 we consider effects that limit the instability in the case of solitons and distributed shock waves, but are not accounted for by the theory developed. The equation for the growth rate will be accordingly refined. All the technical details are relegated to Appendices 1 and 2.

2. DERIVATION OF BASIC EQUATIONS

The nondimensionalized equation for long-wave gravitational waves on the surface of an ideal liquid above an uneven bottom are (the derivation is in Appendix 1):

$$(2u_x+3H^{-3/2}uu_\theta+{}^1/_3H^{1/2}u_{\theta\theta\theta}+{}^1/_2H^{-1}\times H_x u)_\theta+H^{1/2}u_{yy}=0. \tag{3}$$

Here u is the deviation of the liquid surface from the unperturbed level and H is the basin depth which depends, generally speaking, on both horizontal spatial coordinates x and y. The "running" coordinate is defined as

$$\theta=\int_{x_0}^{x} H^{-1/2}(x',y)\,dx'-t, \tag{4}$$

where t is the time. Thus, Eq. (3) is written in a coordinate frame that moves with the velocity $H^{1/2}$ of the linear waves. Note that all the quantities (x, y, t, u, and H) have been made dimensionless with the aid of the free-fall acceleration and the average basin depth.

Equation (3) is a generalization of the known Kadomtsev-Petviashvili equation, written in a somewhat unusual (evolutional in x) form. The variables (x, θ, y) are nonetheless the most natural for spatially inhomogeneous media (see, e.g., Refs. 14–17). We note also that an equation of type (3) can describe also ion-sound or magnetosonic waves in an inhomogeneous plasma, but the connection between its coefficients and the parameters of the inhomogeneities requires additional refinement in these cases.

We shall assume that the characteristic spatial scale of the wave field *along the y axis* is much smaller than the horizontal scale of the roughnesses of the bottom. We can then neglect the dependence of the basin depth H on y and solve the problem locally, in the vicinity of a certain straight line $y=y_0$. Such a dependence follows in fact from the conditions for the validity of Eq. (3), and thus does not "lower" the accuracy of the latter (see Appendix 1 for details). At the same time, we assume the roughnesses to be smooth and put accordingly $H=H(\varepsilon x)$, $0<\varepsilon\ll 1$. We now make a change of variables

$$\tilde{x}=\varepsilon x,\ \tilde{y}=\varepsilon y, \quad (5a)$$

$$\tilde{\theta}=\theta-\frac{S(\varepsilon x,\varepsilon y)}{\varepsilon}, \quad (5b)$$

where S has the meaning of the phase (the ridge displacement) of the soliton. The change (5) is based on the hypothesis that the solution can be represented in the form of a *smoothly* bent soliton with a smoothly modulated amplitude (it will be made clear that the amplitude modulations are small compared with the inflection, meaning the modulation of the phase). A simplified equation for the phase function S is usually derived by Whitham's geometric method (see Refs. 10–12), which yields good results in similar problem, but is mathematically not fully rigorous. We, however, use direct perturbation theory, which enables us to develop a systematic asymptotic procedure.

Substituting (5) in (3) and omitting the tildes we get

$$(-2vu_\theta+3H^{-3/2}uu_\theta+{}^1/_3H^{1/2}u_{\theta\theta\theta})_\theta$$
$$=\varepsilon[H^{1/2}(S_{yy}u+2S_yu_y)-2u_x-{}^1/_2H^{-1}H_xu]_\theta+O(\varepsilon^2), \quad (6)$$

where

$$v(x,\ y)=S_x-{}^1/_2H^{1/2}(S_y)^2 \quad (7)$$

is the local velocity of the soliton. (Actually v is a nonlinear correction to the phase velocity of waves of small amplitude $H^{1/2}$. The total soliton velocity in the lab is $H^{1/2}+v$.) We seek the solution in the form of the asymptotic series

$$u=u^{(0)}+\varepsilon u^{(1)}+\dots . \quad (8)$$

Substituting (8) in (6) we obtain in zeroth order of perturbation theory

$$-2vu^{(0)}+{}^3/_2H^{-3/2}(u^{(0)})^2+{}^1/_3H^{1/2}u^{(0)}_{\theta\theta}=0. \quad (9)$$

Integration of (9) is trivial and yields

$$u^{(0)}=2H^{3/2}v\,\mathrm{ch}^{-2}[({}^3/_2H^{-1/2}v)^{1/2}\theta]. \quad (10)$$

It is seen that Eq. (10) describes a smoothly bent Korteweg–de Vries soliton with parameters (amplitude, velocity, and length) that depend on x and y. In fact, all the wave parameters are expressed in terms of its local velocity v, which is connected in turn with the phase S by the relation (7). The evolutional equation for S is deduced from the next order of perturbation theory

$$\hat{L}u^{(1)}\equiv\frac{\partial}{\partial\theta}\left(-2v+3H^{-3/2}u^{(0)}+{}^1/_3H^{1/2}\frac{\partial}{\partial\theta^2}\right)u^{(1)}$$
$$=H^{1/2}(S_{yy}u^{(0)}+2S_yu^{(0)}_y)-2u^{(0)}_x-{}^1/_2H^{-1}H_xu^{(0)}. \quad (11)$$

Equation (11) must be understood as an ordinary differential equation (with a right-hand side) for $u^{(1)}(\theta)$. For this equation to be solvable with respect to $u^{(1)}$, its right-hand side must be orthogonal to the eigenfunctions of the adjoint operator $\hat{L}^+$. It is easy to verify that the equation

$$\hat{L}^+f=0$$

has only one solution $f=u^{(0)}$, so that by multiplying (11) by $u^{(0)}$ and integrating with respect to θ from $-\infty$ to ∞ we have

$$H(S_yP)_y=(H^{1/2}P)_x,\ P=H^{13/4}v^{3/2}, \quad (12)$$

where P is proportional to the soliton momentum:

$$P\propto\int_{-\infty}^{\infty}(u^{(0)})^2\,d\theta.$$

Equations (7) and (12) constitute a closed system for the determination of the unknowns $S(x,y)$ and $v(x,y)$. It is easily seen that it is of the hyperbolic type and describes nonlinear oscillations propagating along the soliton front, i.e., second sound.

Let us make more precise the physical meaning of S. It is seen from (4) and (5b) that $S(x,y)$ is the difference between the dimensionless times at which a soliton and a wave of infinitesimally small amplitude arrive at the point (x,y) after starting from the same initial position. We note also that the parameter ε has already performed its role as the "indicator" of the small terms in the initial equations, and we can set it equal to unity. Our variables now coincide with the physical (nondimensionalized) variables.

Let us examine the generalization of the "soliton" equations (7) and (12) in the case of a periodic (cnoidal) wave propagating in an inhomogeneous medium. It is well known[10] that for cnoidal waves there exist three modes (and not one) of second sound: longitudinally modulational, transversely modulational,[3)] and flexural. Accordingly, the analog of the system (7), (12) should be a hyperbolic system of equations of second order in x, and in addition, should contain derivatives with respect to $\vartheta=\varepsilon\theta$. In turn, equations for second sound in shock waves of both types (in inhomogeneous media) are close to those for the soliton.

Equations (7) and (12) have an exact solution that is independent of the transverse variable y:

$$v(x)=v_0H^{-5/2},\qquad S(x)=v_0\int_{x_0}^{x}H^{-5/2}(x')\,dx', \quad (13)$$

meaning inverse proportionality of the soliton amplitude to the basin depth

$$A(x)=2H^{3/2}v\propto H^{-1}(x),$$

obtained in Refs. 17 and 18. Our next task is to investigate the transverse stability of the solution (13).

3. SOLITON INSTABILITY

To investigate the stability of a plane soliton we linearize Eqs. (7) and (12) against the background of the solution (13)

$$S=v_0\int_{x_0}^{x}H^{-5/2}(x')\,dx'+\tilde{S},$$

and assume a harmonic dependence of $\tilde{S}$ on the transverse variable y:

$$\tilde{S}(x,\ y)=H^{-5/4}(x)\tilde{\tilde{S}}(x)e^{-iky}.$$

Here k is the second-sound wave number, and the factor $H^{-5/4}$ was introduced for convenience. Omitting the tildes, we have

$$S_{xx}+\left[\frac{\omega^2}{H^2}-\frac{5}{4}\frac{H_{xx}}{H}-\frac{5}{16}\left(\frac{H_x}{H}\right)^2\right]S=0, \quad (14)$$

where

$$\omega=({}^2/_3 v_0)^{1/2}|k| \tag{15}$$

is the dispersion dependence of the second sound in the absence of inhomogeneities [it is seen from a comparison of (15) and (1) that the second sound velocity is $c=(\frac{2}{3}v_0)^{1/2}$).

With the aid of (14) we can investigate the case of periodic bottom irregularities of arbitary amplitude, and calculate the corresponding instability zones. We, however, are interested in small stochastic inhomogeneities, since they are the most typical for real wavy media. We put

$$H=1+h(x), \tag{16}$$

where $h(x)$ is a spatially homogeneous random centered function with a small mean squared value (variance) σ: $\sigma^2=\langle h^2\rangle \ll 1$. From (14) we have

$$S_{xx}+\omega^2 S=(2\omega^2 h+{}^5/_4 h_{xx})S. \tag{17}$$

Equation (17) is a second-order linear differential equation with a small random potential. An asymptotically rigorous theory of such equations, based on the method of many scales (see Ref. 19) is contained in Appendix 1. Here we note only that the leading term of the asymptotic expansion of S in powers of σ satisfies the equality

$$|S(x)|=|S(0)|e^{\gamma_+ x}, \tag{18a}$$

where the growth rate γ_+ for the potential in (17) is given by

$$\gamma_+(\omega)={}^9/_8\omega^2 N(2\omega). \tag{18b}$$

Here $N(\Omega)$ is the Fourier transform of the correlation function of the fluctuations $h(x)$

$$N(\Omega)=\int_{-\infty}^{\infty}\langle h(x+x')h(x)\rangle e^{i\Omega x'}dx'$$

(the angle brackets denote averaging over an ensemble of realizations). A schematic plot of $\gamma_+(k)$ is shown in Fig. 2a. As expected, expression (18b) points to a patently parametric character of the instability. Indeed, the growth rate of the harmonic with frquency ω depends on the spectral density of the fluctuations at the doubled frequency 2ω. We note one characteristic feature of the question: the instability growth rates of solitons of any amplitude (velocity) *are equal*. This follows from the fact that the parameter v_0 enters in (18b) not explicitly, but in the product $\omega=(2v_0/3)^{1/2}|k|$. Thus, when the soliton amplitude is varied only the wave number of the most unstable harmonic changes:

$$k_{max}\propto(v_0)^{-1/2}.$$

We point out one more important feature of soliton instability in an inhomogeneous medium: it is easily seen that the growth rate $\gamma_+\to 0$ as $k\to 0$ [see (15) and (18)]. This follows inevitably from the non-one-dimensionality of the effect. On the other hand, expression (18b) is valid only for small k and must not be changed by the substitution $k\to -k$ (isotropy of the y axis). Thus, the instability growth rate should be proportional to the lowest even power of the perturbation wave number, i.e., to its square. These quantitative arguments indicate that the growth rate of the transverse instability of *any* wave solution in *any* medium with smooth inhomogeneities is of the form

$$\gamma_+(k)=\nu k^2 N(\beta k). \tag{19}$$

Here ν and β characterize the specific properties of the actual medium and the parameters of the investigated solution, and $N(\Omega)$ is the spectral density of the fluctuations.

4. "LAG" OF THE UNSTABLE PERTURBATION RELATIVE TO THE SOLITON

As already noted, the second-sound wavelength ($\sim 1/k$) should be much larger than a certain power of the dimensionless width of the soliton ($\sim v_0^{-1/2}$). In other words, the dispersion relation (15) is only the first term of the asymptotic expansion of the second sound in terms of the parameter kv_0^{-1}. We can calculate also the second term of this expansion (see Refs. 2 and 3):

$$\omega=({}^2/_3 v_0)^{1/2}|k|(1+{}^1/_3 i|k|v_0^{-1}).$$

One can see readily that the imaginary correction to the frequency corresponds to damping of flexural waves on a solition, with a decrement

$$\gamma_-=({}^2/_{27})^{1/2}v_0^{-1/2}k^2. \tag{20}$$

This damping is due to the lag (backward emission) of the perturbations relative to a solitary wave, and is naturally independent of the fluctuations of the medium. We can therefore expect the lag of the unstable perturbations to be able to suppress the instability if the fluctuations are small enough.[4)] To derive an equation that takes second-sound damping into account it is necessary to include in the initial equation (3) from the very outset the hypothesis (16) that the roughnesses of the bottom are small, to assume the perturbation amplitude to be likewise small, and to include one more order of perturbation theory. While the idea of this program is simple, the implementation is fraught with appreciable technical difficulties. On the other hand, in view of the aforementioned independence of the two effects, it is perfectly clear that the growth rate (18b) and the decrement (20) enter in the general expression *additively*. Taking (15) into account, we have

$$\gamma(k)={}^3/_4 v_0 k^2 N[({}^8/_3 v_0)^{1/2}k]-({}^2/_{27})^{1/2}v_0^{-1/2}k^2. \tag{21}$$

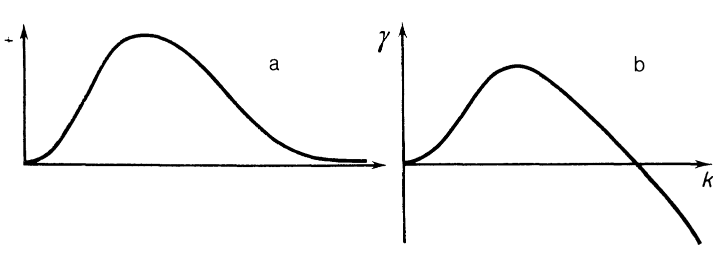

FIG. 2. Plots of the growth rate of flexural perturbations on a soliton: a—without allowance for perturbation damping; b—with allowance for the damping.

This expression is the final formulation of the main result of the present paper. It is easily seen that the criterion for the instability of the soliton velocity v_0 in a basin with given bottom roughnesses is of the form

$$\max\{N(\Omega)\}>(32/243)^{1/2}v_0^{-3/2}. \tag{22}$$

To estimate the instability parameters, we specify the model correlation function of the bottom roughnesses in the form

$$\langle h(x)h(x+x')\rangle=\sigma^2\exp(-x'^2/R^2), \tag{23}$$

where σ and R have the meaning of the variance and correlation radius of the roughnesses. In this case

$$N(\Omega)=\pi^{1/2}\sigma^2 R\exp(-{}^1/_4\Omega^2R^2)$$

and the criterion (22) yields

$$\sigma^2Rv_0^{3/2}>{}^4/_9(2/3\pi)^{1/2}\approx0.2. \tag{24}$$

From the criteria for the applicability of the employed perturbation theory we have $R\gg1$ and $\sigma^2\ll1$ (smooth small irregularities). Thus only solitons in basins with very small bottom irregularities are stable. A $\gamma(k)$ plot for the case when the criterion (24) is met is shown in Fig. 2b. For the instability boundary [in the model case (23)] we have

$$k_r=({}^3/_2)^{1/2}v_0^{-1/2}R^{-1}\{\ln[(243\pi/32)^{1/2}\sigma^2Rv_0^{3/2}]\}^{1/2} \tag{25}$$

(the neutral-stability curve on the plane of the parameters (v_0, k) is shown in Fig. 3). The maximum growth rate satisfies the estimate

$$\gamma_{max}\sim\sigma^2/R. \tag{26}$$

Relations (24)–(26) give a quantitative idea of the parameters of the observed instability.

Note a very important circumstance from the standpoint of generalizations. Namely, the unstable perturbations lag the wave only in the case of solitons and distributed shock waves (with continuous profiles). For periodic solutions, however, the perturbations are not damped, and accordingly *all* the terms of the frequency expansion in kv_0^{-1} (or in the analog of this parameter) are real.[5,6] This fact has a clear physical intepretation: perturbations that lag some "crest" of a periodic wave land eventually on another and do not leave the region of our analysis. Thus, the instability due to the inhomogeneity of the medium is left without a "competitor" and it becomes manifested for all amplitude fluctuations, even the smallest ones. As noted above, the qualitative results of our analysis depend not on the initial equations but on the type of investigated solution. It can be assumed therefore that all periodic waves in smoothly inhomogeneous media are unstable, with an increment determined by Eq. (19).

The situation is exactly the same, although for another reason, for discontinuous shock waves. In this case the perturbations are connected primarily with the discontinuity, which prevents them from lagging and propagating outside the discontinuity. Their frequency is accordingly real—see the exact dispersion relation for flexural ("corrugated") waves in Ref. 9. As to the qualitative conclusions concerning the evolution of discontinuous shock waves, they are perfectly analogous to the case of periodic solutions [instability with growth rate (19)]. All the foregoing applies also to tidal breakers (discontinuous "shock waves" on shallow water) over an uneven bottom.

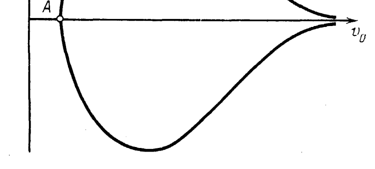

FIG. 3. Neutral-stability curve of a solitary surface wave over an uneven bottom. The point A corresponds to the the value $v_0 = 0.2\,\sigma^{-4/3}R^{-2/3}$.

5. CONCLUSION

We have thus investigated the stability of plane nonlinear waves in smoothly inhomogeneous media. The case of a long-wave soliton on the surface of an ideal liquid in a basin with an uneven bottom was analyzed in detail. The transverse instability observed in this case has a clear physical meaning (parametric resonance), so that the results can be generalized to include wave solutions of other types in other inhomogeneous systems. This paper does not contain a quantitative investigation of these cases, but the general qualitative consideration suggests the following equation for the growth rate of transverse perturbations on a nonlinear wave

$$\gamma=k^2[\nu N(\beta k)-\mu], \tag{27}$$

where ν, β, and μ depend on the parameters of the investigated solution and the wave medium, and $N(\Omega)$ is the spectral density of the fluctuations [Eq. (27) summarizes Eqs. (19) and (20)]. The first term in (27) describes the instability proper, and the second the competing lag of the unstable perturbation relative to the wave. A criterion of the instability of any specific solution is the inequality

$$\max\{N(\Omega)\}>\mu/\nu.$$

An important circumstance here is the vanishing of the coefficient μ for discontinuous shock waves and periodic solutions. Solutions of this type are thus unstable in all smoothly inhomogeneous random media.

The author is indebted to V. E. Zakharov and V. I. Shrira for valuable remarks.

APPENDIX 1

Derivation of Eq. (3). Long potential waves over an uneven bottom are described by the following system of Boussinesq equations[14];

$$\eta_t+\operatorname{div}(H\nabla\Phi+\delta^2[\operatorname{div}(\eta\nabla\Phi)+{}^1/_3H^3\Delta\Delta\Phi]=0,$$
$$\Phi_t+\eta+\delta^2[{}^1/_2(\nabla\Phi)^2]=0. \tag{1.1}$$

Here δ is a small parameter indicative of the weakness of the dispersion and of the nonlinearity,

$$x=\delta x'/H_0,\quad y=\delta y'/H_0,\quad t=\delta t'/(H_0/g)^{1/2},$$
$$\eta=\delta^{-2}\eta'/H_0,\quad \Phi=\delta^{-1}\Phi'/H_0(gH_0)^{1/2},\quad H=H'/H_0,$$

where H_0 is the average depth of the basin, and the prime denotes the dimensional variables (η' is the rise above the liquid surface and Φ' is the hydrodynamic potential).

The presence of a small parameter notwithstanding, the system (1.1) is difficult to analyze without resorting to some hypothesis concerning the character of the irregularities $H(x,y)$ of the bottom. The most interesting particular case is

$$H=H(\delta^2 x, \delta^2 y).$$

In this case the characteristic scale of the inhomogeneity ($\sim\delta^{-2}$) is equal to the nonlinearity and dispersion lengths, so that the interaction of the three effects is the strongest. In addition, we are interested in "quasi-plane" waves that travel with "near-sonic" velocity (close to unity). It is therefore convenient to change over to the variables

$$\theta=\int_{x_0}^{x}[H^{-1/2}(x',y)-1]dx'+x-t,\quad \tilde{x}=\delta^2 x,\quad \tilde{y}=\delta y+y_0$$

(note that the longitudinal variable enters in θ *without* the multiplier δ, since the spatial scale of the field along the x axis remains ~ 1, despite the "stretching" $\tilde{x}$). Accurate to $O(\delta^2)$ the system (1.1) yields (the tildes are omitted)

$$-\eta_\theta+\Phi_{\theta\theta}+\delta^2[2H^{1/2}\Phi_{x\theta}+{}^1/_2H^{-1/2}H_x\Phi_\theta+H^{-1}(\eta\Phi_\theta)_\theta$$
$$+{}^1/_3H\Phi_{\theta\theta\theta\theta}+H\Phi_{yy}]=0,$$
$$-\Phi_\theta+\eta+\delta^2[{}^1/_2(H^{-1/2}\Phi_\theta)^2]=0.$$

We have then in the new variables

$$H=H(x,\delta y+y_0). \qquad (1.2)$$

We seek the solution in the form of asymptotic series

$$\eta=\eta^{(0)}+\delta^2\eta^{(1)}+\dots,\qquad \Phi=\Phi^{(0)}+\delta^2\Phi^{(1)}+\dots.$$

In the zeroth order of perturbation theory we have the trivial equality $\eta^{(0)}=\Phi_\theta^{(0)}\equiv u$, and in the first order we have equations $\eta^{(1)}$ for and $\Phi^{(1)}$.

$$-\eta_\theta^{(1)}+\Phi_{\theta\theta}^{(1)}+[2H^{1/2}u_x+{}^1/_2H^{-1/2}H_xu+2Huu_\theta$$
$$+{}^1/_3Hu_{\theta\theta\theta}+H\Phi_{yy}^{(0)}]=0, \qquad (1.3)$$
$$-\Phi_\theta^{(1)}+\eta^{(1)}+[{}^1/_2H^{-1}u^2]=0. \qquad (1.4)$$

We differentiate (1.3) once with respect to θ, (1.4) twice, and add them. The result is the desired evolutional equation for u:

$$(2u_x+3H^{-3/2}uu_\theta+{}^1/_3H^{1/2}u_{\theta\theta\theta}+{}^1/_2H^{-1}H_xu)_\theta+H^{1/2}u_{yy}=0. \quad (1.5)$$

It is evident that (1.5) depends on δ only via the coefficient H [see (1.2)]. Without using *supplementary* assumptions other than $\delta\ll 1$, we have

$$H(x,\delta y+y_0)=H(x,y_0)+O(\delta).$$

In other words, all the processes described by Eq. (1.5) can be studied locally, say in the vicinity of the line $y=y_0$. We emphasize once more that there is no additional loss of accuracy in Eq. (1.5)

APPENDIX 2

Asymptotic analysis of Eq. (17). We express (17) in the form

$$S_{xx}+\omega^2S=\sigma\alpha(x)S, \qquad (2.1a)$$
$$\sigma\alpha=2\omega^2h+{}^5/_4h_{xx}, \qquad (2.1b)$$

where $\sigma=\langle h^2\rangle^{1/2}, \sigma\ll 1$ is the variance of $h(x)$. Equation (2.1a) is frequently encountered in various branches of physics (in particular, it describes parametric excitation of a mathematical pendulum). As a result, (2.1a) was investigated many times both by physicists (e.g., [20]) and by mathematicians (see [21] and the references therein). The small-perturbation limit ($\sigma\ll 1$) of interest to us, however, was considered only in [21], where the authors paid principal attention to calculation of the "Lyapunov exponent" γ_+, without calculating the solution of the equation. In addition, they had to impose rather stringent constraints on the statistical properties of the potential $\alpha(x)$. We determine below γ_+ and solve also Eq. (2.1a) with only one constraint, $\langle\alpha^2\rangle\lesssim 1$.

We introduce the (as yet) undetermined real function $\varphi(x)$

$$S=\tilde{S}\exp\left[i\int_{x_0}^{x}\varphi(x')dx'\right] \qquad (2.2)$$

(the purpose of this transformation will be made clear below). Equation (2.1a) yields (we omit the tildes)

$$S_{xx}+2i\varphi S_x+(\omega^2-\sigma\alpha-\varphi^2+i\varphi_x)S=0. \qquad (2.3)$$

We analyze (2.3) by the asymptotic many-scale method.[19] We introduce besides the "fast" variable a hierarchy of "slow" ones: $X=\sigma^2x$, $X_j=\sigma^{2+j}x$, $j=1,2,3\dots$. Accordingly,

$$\frac{d}{dx}\to\frac{\partial}{\partial x}+\sigma^2\frac{\partial}{\partial X}+\sigma^3\frac{\partial}{\partial X_1}+\dots.$$

Leaving X_j out of the formal lists of arguments, we seek the solution in the form

$$S=A(X)+\sigma S^{(1)}(x,X)+\dots,\qquad \varphi=\omega+\sigma\varphi^{(1)}+\dots, \qquad (2.4)$$

where $A(x)$ is a definite function. The principal terms of the series (2.4) describe a coherent wave with smoothly varying amplitude, propagating in the positive x direction. Substitution of (2.4) in (2.3) is satisfied identically in zeroth order, and yields in first order

$$\frac{\partial^2S^{(1)}}{\partial x^2}+2i\omega\frac{\partial S^{(1)}}{\partial x}=\left[\alpha(x)+2\omega\varphi^{(1)}-i\frac{\partial\varphi^{(1)}}{\partial x}\right]A(X). \quad (2.5)$$

In terms of the Fourier transforms

$$\alpha(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\alpha(\varkappa)e^{-i\varkappa x}d\varkappa, \qquad (2.6a)$$

$$\varphi^{(1)}(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\varphi^{(1)}(\varkappa)e^{-i\varkappa x}d\varkappa \qquad (2.6b)$$

we can express the solution of Eqs. (2.5) in the form

$$S^{(1)}=(2\pi)^{-1/2}A(X)$$

$$\times\int_{-\infty}^{\infty}\frac{\varkappa\varphi^{(1)}(\varkappa)-[2\omega\varphi^{(1)}(\varkappa)+\alpha(\varkappa)]}{\varkappa(\varkappa-2\omega)}e^{-i\varkappa x}d\varkappa. \quad (2.7a)$$

It is evident that the integrand in (2.7a) has singularities, generally non-integrable, at $\varkappa=0$ and $\varkappa=2\omega$. The first can be eliminated by stipulating satisfaction of the equality

$$\varphi^{(1)}=-\alpha/2\omega$$

(this was in fact the reason for introducing the function φ). This singularity is due to the geometric-optical fluctuation of the incident-wave phase

$$(\omega^2-\sigma\alpha(x))^{1/2}\approx\omega-\sigma\alpha/2\omega=\varphi^{(0)}+\sigma\varphi^{(1)}$$

[see (2.2)]. Note that $\varphi^{(2)}$, $\varphi^{(3)}$, etc., correspond to phase fluctuations of not only the incident but also the reflected waves. From the formal viewpoint, however, they are determined from the conditions for regularization of the analogous integrals in the higher approximations of the employed perturbation theory. The singularity in (2.7a) at $\varkappa=2\omega$ corresponds in turn to single resonant scattering and will be regularized below. As a result we have

$$S^{(1)}=-(8\pi)^{-1/2}\frac{A(X)}{\omega}\int_{-\infty}^{\infty}\frac{\alpha(\varkappa)}{\varkappa-2\omega}e^{-i\varkappa x}d\varkappa. \quad (2.7b)$$

In the next order of perturbation theory we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial S^{(2)}}{\partial x}+2i\omega S^{(2)}\right)=F(x,X), \quad (2.8a)$$

$$F=-2i\omega\frac{\partial A}{\partial X}+{}^1/_2 i\omega^{-1}\left[\frac{\partial\alpha}{\partial x}S^{(1)}+2\alpha\frac{\partial S^{(1)}}{\partial x}\right]+\ldots. \quad (2.8b)$$

The ellipsis in (2.8b) stands for terms proportional to $\varphi^{(2)}$ (they will be shown below to be immaterial). It is easily seen that the solution of (2.8a) is bounded only if the "driving force" F averaged over x is zero:

$$\langle F\rangle=0. \quad (2.9)$$

Using the ergodicity theorem, we replace in (2.9) the spatial averaging by an averaging over an ensemble of realizations. Substituting now (2.7b) and (2.8b) in (2.9):

$$\frac{\partial A}{\partial X}=\Gamma A,$$

$$\Gamma=\frac{i}{16\pi}\omega^{-3}\int_{-\infty}^{\infty}\frac{\varkappa n(\varkappa)}{\varkappa-2\omega}d\varkappa+\ldots, \quad (2.10)$$

where $n(\varkappa)$ is defined by

$$\langle\alpha(\varkappa)\alpha(\varkappa_1)\rangle=n(\varkappa)\delta(\varkappa+\varkappa_1),$$

and the ellipsis corresponds to pure imaginary terms, which are unimportant, we get

$$|A(X)|=|A(0)|e^{\gamma_+X},\quad \gamma_+=\mathrm{Re}\,\Gamma. \quad (2.11)$$

The integrand in (2.10) is formally real, but it has the aforementioned singularity at $\varkappa=2\omega$. To regularize it, we recall that the incident wave propagates to the right, and assume that the scattering potential $\alpha(x)$ decreases infinitely slowly as $x\to-\infty$ (this device is called adiabatic switching). Accordingly, returning to (2.6a), we put

$$\varkappa\to\varkappa+i0 \quad (2.12)$$

and using the known equation of the theory of generalized functions[22]

$$\frac{1}{\varkappa+i0-2\omega}=\mathcal{P}\left(\frac{1}{\varkappa-2\omega}\right)-i\pi\delta(\varkappa-2\omega),$$

we substitute (2.12) in (2.10):

$$\Gamma={}^1/_8\omega^{-2}n(2\omega)+\frac{i}{16\pi}\omega^{-3}⨍_{-\infty}^{\infty}\frac{\varkappa n(\varkappa)}{\varkappa-2\omega}d\varkappa+\ldots, \quad (2.13)$$

where the integral $⨍$ is taken in the sense of principal value. The first (real) term in (2.13) is the sought growth rate

$$\gamma_+={}^1/_8\omega^{-2}n(2\omega). \quad (2.14)$$

Thus, Eq. (18a) in the main text coincides with (2.11), and expression (18b) for γ_+ is obtained by substituting in (2.14) the connection (2.1b) between α and h.

1) Transverse perturbations are subdivided into "flexural" (corresponding to bending of the wave front) and "modulational" (modulating the wave amplitude along its front).

2) Long waves in a basin with an uneven bottom are the simplest model of tsunami wave propagation in the ocean.

3) The second sound of this mode is anisotropic and propagates only forward in the direction of the cnoidal wave.

4) This circumstance was pointed out to the author by V. I. Shrira and V. E. Zakharov.

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