Dispersionless wave propagation in nonlinear fluctuating media

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(Submitted 2 July 1987)
Zh. Eksp. Teor. Fiz. **94**, 175–185 (January 1988)

We investigate the scattering of nondispersive (acoustic) waves in nonlinear media with fluctuating parameters. Using the asymptotic method of multiple scales, we derive the evolution equation which governs the propagation of coherent pulses or periodic waves. Previously, problems of this type were solved using the so-called mean-field method, which was not fully justified mathematically. In the present paper, it is shown that this method is actually incorrect, and only gives the proper results for linear problems.

1. INTRODUCTION

The study of the scattering of nondispersive waves in fluctuating media is one of the key problems in the general theory of waves. This problem arises in acoustics[1,2] and optics,[1,3] and it describes an important hydrodynamic problem dealing with the propagation of long surface waves over an uneven sea floor,[4] as well as a number of problems in seismology[5] and plasma physics (see Ref. 6, for example). In short, whenever a uniform medium is described by a classical linear wave equation, the problem of wave evolution in the presence of random fluctuations of the parameters of the medium arises. An early (albeit incomplete and nonrigorous) solution of this problem was obtained by Kaner,[7] while the full justification and construction of a rigorous asymptotic theory, based on a diagrammatic method, was carried out in the classic works of Bouret[8] and Tatarskiĭ,[1,3] Making use of their results, many authors have fruitfully applied the approach of Kaner (now known as the mean-field method) with complete rigor to solve various wave-scattering problems in fluctuating media (see the reviews in Refs. 9 and 10).

There is a substantial amount of interest, in addition, in generalizing the existing results to nonlinear fluctuating media (all of the foregoing remarks applied to linear waves). Beginning in 1971, a large number of papers started appearing in which the mean-field method was extended to the nonlinear case. Thus, nonlinear acoustic waves in a medium with fluctuations in the speed of sound were examined in Refs. 11–13, and in Refs. 14 and 15, so were finite-amplitude long surface waves in tanks having an uneven bottom. The nonlinear mean-field method has been used in the context of plasmas,[6] and in Ref. 16 it was used to investigate weakly dispersive waves in an arbitrary fluctuating slightly nonlinear medium. Highly dispersive waves have been included in the general framework in Ref. 17, and furthermore the results obtained with the linear mean field method have been used to study the nonlinear attenuation of solitons.[18,19] They have also appeared in experimental work (see Ref. 20, for example), in which the measured mean nonlinear fields have been compared with theoretical predictions.

Note, however, that the rigorous methods of Refs. 1 and 8 have not been extended to the nonlinear case, and further justification is need if one is to use the partially flawed mean-field method. Pelinovskiĭ and Saichev[21] have presented examples of nonlinear systems with fluctuating parameters for which this method gives clearly incorrect results. Following the appearance of that paper, the systems discussed in the literature were basically those which are almost integrable by the inverse scattering method (see Refs. 22 and 23, for example), and a number of problems were solved, moreover, in the Born approximation.[24,25] Unfortunately, the first of these approaches assumes integrability of the unperturbed (homogeneous) problem, which is an extremely rare occurrence, and the Born approximation is restricted to very short times.

In the present paper, using a nonlinear wave equation as an example, we demonstrate why the mean field method is incorrect. We find that the small corrections (i.e., those proportional to a small parameter) which are discarded contain divergent integrals, which negates the method's validity. The physical reason for these divergences lies in the unbounded growth of the root-mean-square phase fluctuations ("locations") of the waves, which is due in turn to fluctuations in the speed of sound in an inhomogeneous medium. We propose an approximate method based on the asymptotically rigorous multiscale expansion procedure,[26] which eliminates phase fluctuations of waves by an appropriate coordinate transformation. Using the method we have developed, we derive a simplified equation which contains only deterministic quantities, and we investigate its elementary properties.[1)]

2. THE MEAN-FIELD METHOD

We consider the simplest model of a weakly nonlinear, nondispersive one-dimensional medium with small fluctuations in the wave propagation velocity (speed of sound):

$$u_{tt}-[1+\varepsilon\alpha(x)]^2u_{xx}=\varepsilon^2(u^2)_{xx}. \qquad (1)$$

Here $0<\varepsilon\ll 1$, and $\alpha(x)$ is a zero-mean uniform random function (the mean dimensionless speed of sound equals unity). According to the standard approach taken by the mean-field method (see Ref. 14, for example), the unknown function u is represented as a sum of coherent and fluctuating components:

$$u=\bar{u}+\varepsilon u'. \qquad (2)$$

Substituting (2) and (1) and averaging over an ensemble of realizations, we have

$$\bar{u}_{tt}-\bar{u}_{xx}=\varepsilon^2[2\overline{\alpha u'_{xx}}+\overline{\alpha^2}\bar{u}_{xx}+(\bar{u}^2)_{xx}]+\varepsilon^3\overline{\alpha^2u'_{xx}}+\varepsilon^4(\overline{u'^2})_{xx}. \qquad (3a)$$

Subtracting (3a) from (1), we obtain

0038-5646/88/010098-06$04.00

$$u'_{tt}-u'_{xx}=2\alpha\bar{u}_{xx}+\varepsilon[2(\overline{\alpha u'_{xx}}-\overline{\alpha u'_{xx}})+\bar{u}_{xx}(\alpha^2-\overline{\alpha^2})]$$

$$+\varepsilon^2[\overline{\alpha^2}u'_{xx}+(\alpha^2u'_{xx}-\overline{\alpha^2u'_{xx}})+2(\bar{u}u')_{xx}]+\varepsilon^3[(u'^2)_{xx}-\overline{(u'^2)}_{xx}]. \tag{3b}$$

Keeping only the first nontrivial terms in (3), we obtain a system of equations for u and u'.

$$\bar{u}_{tt}-\bar{u}_{xx}=\varepsilon^2[2\overline{\alpha u'_{xx}}+\overline{\alpha^2}\bar{u}_{xx}+(\bar{u}^2)_{xx}], \tag{4a}$$

$$u'_{tt}-u'_{xx}=2\alpha\bar{u}_{xx}. \tag{4b}$$

In order to obtain a closed-form equation for the deterministic function $\bar{u}$, we must express u' in the form $u'=u'[\alpha(x),\bar{u}(x,t)]$ from (4b), and substitute into (4a). This can be accomplished using a retarded Green's function, which is equivalent to setting the initial conditions for Eq. (4b) at $t\to-\infty$.

$$u'|_{t\to-\infty}=u_t'|_{t\to-\infty}=0.$$

The Green's function for the operator $\partial^2_{tt}-\partial^2_{xx}$ is

$$G(x,t)=\tfrac{1}{2}H(t-|x|)=\tfrac{1}{2}H(t-x)H(t+x), \tag{5}$$

where $H(x)$ is the Heaviside step function ($H(x\geqslant0)=1$, $H(x<0)=0$). We then have

$$u'(x,t)=\iint\limits_{-\infty}^{\infty}G(x-x_0,t-t_0)[2\alpha(x_0)\bar{u}_{x_0x_0}(x_0,t_0)]dx_0\,dt_0. \tag{6}$$

Substituting (5) and (6) into (4a), some straightforward manipulation gives

$$\bar{u}_{tt}-\bar{u}_{xx}=\varepsilon^2\Big[2\int\limits_{-\infty}^{\infty}W(\tau)\bar{u}_{xxt}(x-\tau,t-|\tau|)d\tau-3\sigma^2\bar{u}_{xx}+(\bar{u}^2)_{xx}\Big], \tag{7}$$

where $W(\tau)=\overline{\alpha(x)\alpha(x+\tau)}$ is the correlation function for fluctuations in the speed of sound, and $\sigma^2=\overline{\alpha^2}=W(0)$ is the variance of the fluctuations. We have thus obtained the desired closed-form equation for the mean field, containing deterministic coefficients only; the latter circumstance in fact constitutes its advantage over the original equation (1). There is, however, an important drawback to Eq. (7). Because of the integration over the temporal argument, it is not possible in principal to pose (7) as a Cauchy problem. This drawback can be corrected by going to the so-called single-wave approximation (see Ref. 15, for example). Specifically, let the initial condition $u(x,0)$ consist only of waves moving to the right. Then $\bar{u}(x,t)=\bar{u}(z,T)$, where

$$z=x-t,\quad T=\varepsilon^2t, \tag{8}$$

and the integral in Eq. (7) can be rewritten as

$$-\int\limits_{-\infty}^{\infty}W(\tau)\bar{u}_{zzz}(z-\tau+|\tau|,T-\varepsilon^2|\tau|)d\tau$$

$$+\varepsilon^2\int\limits_{-\infty}^{\infty}W(\tau)\bar{u}_{zzT}(z-\tau+|\tau|,T-\varepsilon^2|\tau|)d\tau$$

$$=-\int\limits_{-\infty}^{0}W(\tau)\bar{u}_{zzz}(z-2\tau,T)d\tau-\bar{u}_{zzz}\int\limits_{0}^{\infty}W(\tau)d\tau+O(\varepsilon^2). \tag{9}$$

Here we have used the obvious formula

$$\int\limits_0^{\infty}f(\tau)g(\varepsilon^2\tau)d\tau=g(0)\int\limits_0^{\infty}f(\tau)d\tau+O(\varepsilon^2).$$

Substituting (8) and (9) into (7) and neglecting small terms, we have

$$\bar{u}_T-\frac{3\sigma^2}{2}\bar{u}_z+\bar{u}\bar{u}_z-\bar{u}_{zz}\int\limits_0^{\infty}W(\tau)d\tau$$

$$-\int\limits_0^{\infty}W(\tau)\bar{u}_{zz}(z+2\tau,T)d\tau=0.$$

We may rewrite this equation in the original variables:

$$\bar{u}_t+\bar{u}_x+\varepsilon^2\Big\{-\frac{3\sigma^2}{2}\bar{u}_x+\frac{1}{2}(\bar{u}^2)_x-\beta\bar{u}_{xx}$$

$$-\int\limits_0^{\infty}W(\tau)\bar{u}_{zz}(z+2\tau,T)d\tau\Big\}=0, \tag{10a}$$

$$\beta=\int\limits_0^{\infty}W(\tau)d\tau. \tag{10b}$$

Equations (7) and (10) are the "final outcome" of the mean-field method.

In order to evaluate the applicability of the mean-field method, we use the single-wave approximation in the expression for the random component of the field u' in (6). By analogy with (9), we obtain

$$u'(x,t)=-\bar{u}_z(z,T)\int\limits_{-\infty}^{x}\alpha(\tau)d\tau$$

$$-\int\limits_0^{\infty}\alpha(z+t+\tau)\bar{u}_z(z+2\tau,T)d\tau+O(\varepsilon^2). \tag{11}$$

Note that the first term in (11) describes a small displacement of the incident wave as a whole, and the second describes a reflected wave.[2)] In fact, we may represent the total field u as the sum of an incident wave $u^{(0)}$ with fluctuating phase and a reflected wave $u^{(1)}$:

$$u=u^{(0)}(z-\varepsilon\theta(x),T)+\varepsilon u^{(1)}(x,t)$$
$$\approx u^{(0)}(z,T)+\varepsilon[-u_z^{(0)}(z,T)\theta+u^{(1)}(x,t)], \tag{12a}$$

$$\theta=\int\limits_{-\infty}^{x}\alpha(\tau)d\tau, \tag{12b}$$

where θ is the phase fluctuation (the integral of the fluctuations in the speed of sound). It is easy to see that the mean squared phase fluctuations $\overline{\theta^2}$ are infinite:

$$\overline{\theta^2}=\iint\limits_{-\infty}^{x}\overline{\alpha(\tau)\alpha(\tau')d\tau\,d\tau'}=\iint\limits_{-\infty}^{x}W(\tau-\tau')d\tau\,d\tau', \tag{13a}$$

whereupon the integral in (13a) obviously diverges for any $W(\tau)$ except the degenerate case

$$\int\limits_{-\infty}^{\infty}W(\tau)d\tau=0.$$

The appearance of an infinity is related to way in which the initial conditions for u' are posed as $t \to -\infty$ (i.e., the use of the retarded Green's function (5)). As a result, over the infinite time interval $(-\infty, t)$, the quantity $\overline{\theta^2}$ can grow to infinity. If we assume Cauchy conditions at $t = 0$, the phase fluctuations may be estimated by

$$\theta \approx \int_0^x \alpha(\tau)\,d\tau,$$

and as $x \to \infty$ we have

$$\overline{\theta^2} \to 2\beta x. \tag{13b}$$

It is also clear that $\overline{u'^2}$ is infinite, in addition to $\overline{\theta^2}$ (compare Eq. (11) and the second term on the right-hand side of (12a)), and this is just the term which was discarded in going from the exact equation (3a) to the approximate one, (4a)! It was in fact this very circumstance, which fundamentally violates the basis of the mean-field method, which remained unnoticed in Refs. 6 and 11–20, leading to erroneous results. We emphasize that nothing said above bears on linear media, in which waves do not interact among themselves and the phase with which a wave arrives at some point in space is in some sense irrelevant. From a formal standpoint, nonlinear (quadratic) terms like $\overline{u'^2}$, which lead to divergences, do not occur in linear problems.

3. CONSTRUCTION OF A VALID METHOD FOR DESCRIBING NONLINEAR WAVES IN FLUCTUATING MEDIA

It is clear from the foregoing that in constructing a valid perturbation theory for the original equation (1), it is necessary to eliminate phase fluctuations of the incident wave in some fashion. The easiest way to do this is to stretch (compress) spatial variables, so that in the new variables the wave moves equal distances in equal times.[3)] It is easily seen that terms $\sim \alpha u_{xx}$ correspond to fluctuating field transport in Eq. (1). A transformation eliminating these from (1) is thus given by

$$y = y(x, t) \quad t = t; \tag{14a}$$

$$\partial y/\partial x = 1/[1 + \varepsilon\alpha(x)] + O(\varepsilon^3) \tag{14b}$$

(terms $\sim \varepsilon^3$ respond to nonlinear transport of the incident wave by the scattered field, but this effect is negligible to the stated accuracy). In the new variables, Eq. (1) takes the form

$$u_{tt} - u_{yy} + \varepsilon\alpha_x(x) u_y = \varepsilon^2 (u^2)_{yy} + O(\varepsilon^3). \tag{15}$$

Using (14b) to express x in terms of y,

$$x = y + \varepsilon \int_0^y \alpha(\tau)\,d\tau + O(\varepsilon^2), \tag{16}$$

we may insert (16) into (15) and transform to a frame of reference moving at the mean speed of sound (equal to unity):

$$z = y - t, \quad t = t. \tag{17}$$

We have as a result

$$u_{tt} - 2u_{tz} + \left[\varepsilon\alpha_z(z+t) + \varepsilon^2\alpha_{zz}(z+t) \int_0^{z+t} \alpha(\tau)\,d\tau\right] u_z$$
$$= \varepsilon^2 (u^2)_{zz} + O(\varepsilon^3). \tag{18}$$

To simplify Eq. (18), we use the rigorous asymptotic method of multiple scales (see Ref. 26, for example), wherein along with the "rapid" time t, we introduce a hierarchy of "slow" times: $T = \varepsilon^2 t$, $T_1 = \varepsilon^3 t, \ldots$. Derivatives with respect to time in (18) are transformed to the form

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} + \varepsilon^3 \frac{\partial}{\partial T_1} + \ldots .$$

Omitting $T_1, T_2, \ldots$ from the formal list of arguments, we seek a solution in the form

$$u(z, t, T) = u^{(0)}(z, T) + \varepsilon u^{(1)}(z, t, T) + \ldots \tag{19}$$

(the form of the main term in the asymptotic series (19) corresponds to waves traveling to the right at close to the speed of sound). Zeroth-order perturbation theory is satisfied automatically, and to first order, Eq. (18) gives the following equation for $u^{(1)}$:

$$u_{tt}^{(1)} - 2u_{zt}^{(1)} = -\alpha_z(z+t)\, u_z^{(0)}(z, T), \tag{20a}$$

which must be supplemented by the condition of having no scattered waves as $t \to -\infty$:

$$u^{(1)}|_{z=\text{const}} = u^{(1)}|_{z+2t=\text{const}} = 0. \tag{20b}$$

(It is implicitly understood here that fluctuations in the medium are turned on "adiabatically," i.e, (20) must be solved by replacing $\alpha(z)$ with $\alpha(z)\exp(\nu z)$, $0 < \nu \ll 1$, and letting $\nu \to 0$. This assures convergence of improper integrals like (12b)). The Goursat problem (20) is then easy to integrate:

$$u^{(1)} = -\int_0^\infty \alpha(z+t+\tau)\, u_z^{(0)}(z+2\tau, T)\,d\tau. \tag{21}$$

Note that there is not term in (21) corresponding to phase fluctuations of the incident wave (compare (21) and (11)).

To the next order ($\sim \varepsilon^2$) of perturbation theory, we have

$$U_t = F(z, t, T), \tag{22a}$$

$$F = 2u_{Tz}^{(0)} + [(u^{(0)})^2]_{zz} - \alpha_z(z+t)\, u_z^{(1)} - \alpha_{zz}(z+t) \int_0^{z+t} \alpha(\tau)\,d\tau \cdot u_z^{(0)}, \tag{22b}$$

where $U = u_t^{(2)} - 2u_z^{(2)}$. It is easy to see that by virtue of the statistical uniformity of $\alpha(z)$, the function $F(z,t,T)$ is a statistically stationary function of the rapid time t. Consequently, the requirement for U (and $u^{(2)}$) to be bounded as $t \to \infty$ is the vanishing of the mean over t of the "forcing function" F:

$$\langle F \rangle \equiv \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} F(z, t, T)\,dt = 0. \tag{23}$$

Putting (21) and (22b) into (23), we obtain the desired equation detailing the relationship be $u^{(0)}$ and T:

$$u_T^{(0)} - \frac{\sigma^2}{2} u_z^{(0)} + u^{(0)} u_z^{(0)} = \int_0^\infty W(\tau)\, u_{zz}^{(0)}(z+2\tau, T)\,d\tau, \tag{24}$$

where $W(\tau) = \langle \alpha(x)\alpha(x+\tau) \rangle$, $\sigma^2 = \langle \alpha^2 \rangle$. Note that in the present case, the correlation function $W(\tau)$ has been averaged over space and not over an ensemble of realizations, as in the mean-field method. Note also that averaging over the rapid time t in (23) occurs in the method of multiple scales as a natural condition for boundedness of the small corrections, and in this sense it differs from averaging over an ensemble, which to some extent is put in "by hand." Furthermore, in an experiment, as a rule, one measures the characteristics of the irregularity in a medium which are averaged over space, and to interpret the measured mean fields using ensemble means, strictly speaking, requires a proof of the appropriate ergodic theorem. The advantages of the proposed method thus also include the fact that nowhere does it assume a probability distribution for the fluctuations in the speed of sound (this is also true of all higher-order approximations in perturbation theory).

It must be emphasized that Eq. (24) governs the wave profile itself, rather than the mean field. In fact $u^{(0)}(z,T)$ is independent of the rapid time t, and is therefore never averaged at any stage of the asymptotic procedure which is used. In order to find the mean field, we revert to the original variables, using Eqs. (14) and (17):

$$z=y(x)-t=x-\Theta(x)-t, \quad T=\varepsilon^2 t,$$

$$\Theta=\varepsilon\int_0^x \alpha(\tau)\,d\tau-\varepsilon^2\int_0^x \alpha^2(\tau)\,d\tau+O(\varepsilon^3).$$

The mean field is then given by

$$\bar{u}(x,t)=\int_{-\infty}^{\infty} u^{(0)}(x-t-\Theta,\varepsilon^2 t)\,\Gamma(\Theta,x)\,d\Theta, \tag{25}$$

where $\Gamma(\Theta,x)$ is the probability distribution of the phase $\Theta(x)$. According to the central limit theorem of probability theory, when $x\gg R$ (R is the correlation length of $\alpha(x)$), the distribution $\Gamma(\Theta,x)$ tends to a Gaussian, regardless of the form of $\Gamma(\alpha,x)$, and it is completely determined by its variance $\overline{\Theta^2}=2\varepsilon^2\beta x+O(\varepsilon^3)$ (see (13b)) and mean $\overline{\Theta}=-\varepsilon^2\sigma^2 x+O(\varepsilon^3)$. Accordingly,

$$\Gamma(\Theta,x)=[2\pi\cdot 2\varepsilon^2\beta x]^{-1/2}\exp[-(\Theta+\varepsilon^2\sigma^2 x)^2/2\cdot 2\varepsilon^2\beta x]. \tag{26}$$

This implies that

$$\Gamma_x=\varepsilon^2(\beta\Gamma_{\Theta\Theta}-\sigma^2\Gamma_\Theta). \tag{27}$$

If we now differentiate the expression (25) for the mean field with respect to t and take (24) into account, we obtain the correct equation for $\bar{u}$. Transforming derivatives with respect to x with the aid of (27),

$$\int_{-\infty}^{\infty} u_x^{(0)}(x-\Theta)\Gamma(\Theta,x)\,d\Theta=\bar{u}_x-\varepsilon^2(\beta\bar{u}_{xx}+\sigma^2\bar{u}_x)+O(\varepsilon^4)$$

and keeping terms $\gtrsim\varepsilon^2$, we have

$$\bar{u}_t+\bar{u}_x+\varepsilon^2\Big\{-\frac{3\sigma^2}{2}\bar{u}_x+\frac{1}{2}[\overline{(u^{(0)})^2}]_x-\beta\bar{u}_{xx} - \int_0^\infty W(\tau)\bar{u}_{xx}(x+2\tau,t)\,d\tau\Big\}=0. \tag{28}$$

Comparing (28) and Eq. (10), which was obtained with the mean field method, it is clear that the distinction lies in the fundamentally different nonlinear terms. The one exception is the special case $\beta=0$: the Gaussian distribution (26) then tends to a Dirac delta function, and $\overline{u^2}=\bar{u}^2$. We thus conclude that for a general situation ($\beta\neq 0$), the nonlinear divergences in the higher orders of the mean field method are important, and the method is therefore only useable in the linear limit. We also note that in real physical systems, the correlation function $W(\tau)$ decreases monotonically over the interval $(0,\infty)$, as a rule, and we never have $\beta=0$ (see 10b)).

4. LIMITING CASES AND GENERALIZATIONS OF THE THEORY

Let us eliminate the term $(1/2)\sigma^2 u_z^{(0)}$ in (24) by the Galilean coordinate transformation $x'=z+\sigma^2 T/2$, $t'=T$. Dropping the primes and the superscript (0), we have

$$u_t+uu_x=\int_0^\infty W(\tau)u_{xx}(x+2\tau,t)\,d\tau. \tag{29}$$

We first determine the dispersion relation for small amplitude waves, within the scope of Eq. (29). Neglecting nonlinear terms, we assume that $u\sim\exp(i\omega t-ikx)$, where

$$\omega=ik^2\int_0^\infty W(\tau)e^{-2ik\tau}\,d\tau. \tag{30}$$

It is easily seen that for a monotonically decreasing correlation function $W(\tau)$, the dispersion relation (30) is dissipative ($\mathrm{Im}\,\omega>0$). The reason for this is that the back-scattered field is not described by Eq. (29), and its energy is subtacted from the energy of waves traveling to the right, decreasing the latter. The rate of dissipation largely depends, however, on the value of the parameter λ/R (λ is the characteristic wavelength, R is the correlation length of $\alpha(x)$). We consider two limiting cases, $\lambda\gg R$ and $\lambda\ll R$.

1. When $\lambda\gg R$ (small-scale inhomogeneities), the integrand on the right-hand side of (29) can be expanded in a Taylor series:

$$u(x+2\tau,t)\approx u(x,t)+2\tau u_x(x,t).$$

As a result, one obtains the well known Korteweg–de Vries equation (see Ref. 28, for example):

$$u_t+uu_x=\beta u_{xx}+\gamma u_{xxx}, \quad \gamma=2\int_0^\infty \tau W(\tau)\,d\tau. \tag{31}$$

Dissipative effects play a central role in Eq. (31).

2. When $\lambda\ll R$ (the smooth inhomogeneity case), we may differentiate (29) with respect to x. Integrating by parts, the integral term can then easily be expanded as a series in powers of $(\lambda/R)^2$. Retaining only the first two terms of this series, we obtain

$$(u_t+uu_x)_x=-{}^1/_2\sigma^2 u_{xx}-{}^1/_8 W_{\tau\tau}(0)u. \tag{32}$$

Equation (32) is a conservative, Hamiltonian equation. This is to be expected, as the coefficeint of wave reflection from smooth inhomogeneities is exponentially small. It is also noteworthy that up to the accuracy of its coefficients, (32) is identical to the Ostrovskiĭ equation,[29] which describes long, slightly nonlinear waves in a medium with a dispersion relation

$$\omega^2(k)=\omega_0^2+\omega_1^2(k), \quad |(d\omega_1/dk)_{k=0}|<\infty$$

(for example, interior gravitational waves in a rotating tank). Equation (32) possesses stationary solutions describing periodic waves, including those with cusped "crests".[29]

We now consider some generalizations of the foregoing approach.

1. The results obtained are most simply generalized to weakly dispersive media, for which the original equation (1) must be replaced by the equation for a nonlinear string (Boussinesq equation):

$$u_{tt}-[1+\varepsilon\alpha(x)]^2u_{xx}=\varepsilon^2[(u^2)_{xx}\pm u_{xxxx}].$$

Without an alterations, the asymptotic procedure gives the following generalization of Eq. (24):

$$u_t^{(0)}-{}^1/_2\sigma^2u_z^{(0)}+u^{(0)}u_z^{(0)}\pm{}^1/_2U_{zzz}^{(0)}=\int_0^\infty W(\tau)u^{(0)}(z+2\tau,t)\,d\tau \quad (33)$$

(compare (33) with the corresponding equation in Ref. 16, which was derived via the mean-field method).

2. The present approach can also be extended to highly dispersive media, but it then gives no new results. We can clarify this with a simple model describing plasma wave scattering from density fluctuations of ions in a rarefield plasma

$$iE_t+E_{xx}+E(|E|^2+\varepsilon n)=0, \quad (34)$$

where E is the dimensionless complex electric field strength in the plasma wave, and n specifies the fluctuations in the ion density (this problem was examined in a slightly more complicated setting in Ref. 17 used the mean-field method). From the standpoint of our approach, the main difference between (34) and (1) is the complete immutability of the form of the solution to (34), even in the zeroth approximation in ε. In fact, the equation for the zeroth approximation,

$$iE_t^{(0)}+E_{xx}^{(0)}+E^{(0)}|E^{(0)}|^2=0 \quad (35)$$

completely determines the solution (soliton, periodic wave), and when there are fluctuations, one can only speak of the evolution of the parameters of this solution. Powerful universal methods have been developed for studying such problems (the averaging method of Witham,[30] the direct perturbation theory of Ostrovsky and Gorshkov,[31] and the like), so that there is simply no need to use our approach (nor the mean-field method). Note also that in the example considered, we have also employed perturbation theory for almost integrable systems. In actuality, the zeroth-order equation (35) can be integrated by the inverse scattering method, although in general this property is encountered very rarely.

3. Let us briefly discuss media having spatiotemporal (and not just spatial) fluctuations in the speed of sound: $\alpha=\alpha(x,t)$. Without dwelling on the details, we note that the proposed method can also be extended to this situation with one exception, namely if the fluctuations are produced by waves whose phase velocity is close to the speed of sound. When $\alpha\approx\alpha(x\pm t)$, the interaction of sound with inhomogeneities will be resonant and strong, and its evolution will be described by a Korteweg–de Vries type of equation containing no integral operators (but with coefficients which depend explicitly on x and t).

4. The least trivial generalization of the theory we have constructed applies to multidimensional nonlinear media described by the equation

$$u_{tt}-[1+\varepsilon\alpha(\mathbf{x})]^2\Delta u=\varepsilon^2\Delta(u^2), \quad (36)$$

where $\mathbf{x}=(x_1,...,x_N)$, and $N=2,3$ is the number of spatial variables. In order to exclude "dangerous" terms $\sim\alpha\Delta u$ from (36), which lead to divergence, the transformation $y_j=y_j(\mathbf{x})$ should satisfy the system of equations

$$\sum_{j=1}^{N}\frac{\partial y_n}{\partial x_j}\frac{\partial y_k}{\partial x_j}=[1+\varepsilon\alpha(\mathbf{x})]^{-2}\delta_{nk}, \quad (37)$$

where δ_{nk} is the Kronecker delta. It is easy to see that when $N>1$, the system (37) is indeterminate, and cannot be solved in general. We can only eliminate terms $\sim\alpha u_{x_jx_j}$, $j=1,...,N$. This means that for multidimensional fluctuations, the proposed asymptotic scheme, which can be regularly refined out to infinite times, can only be constructed for plane waves. For quasi-plane waves (i.e., $L_1\sim1$, $L_2\sim\varepsilon^{-1}$, where L_1 is the transverse scale (with respect to the wavefront), and L_2 the longitudinal), the secular terms (proportional to the rapid time T) are non-imaginary in higher-order perturbation theory, and the equation which generalizes (24) is only valid up to times $t\ll\varepsilon^{-4}(T\ll\varepsilon^{-2})$. This is sufficient for a study of the initial stages of propagation of a quasi-plane wave, but it is still not known how to describe the scattering process for $t\gtrsim\varepsilon^{-4}$. As for the evolution of nonlinear wave fields with a broad angular spectrum, this may possibly yield to investigation using a nonlinear generalization of the kinetic equation previously derived for linear fluctuating media.[32,33]

1)The results opf the present paper were briefly summarized in a previous note.[27]

2)Since the function u' describes waves traveling in different directions, it cannot be completely represented in the form $u'=u'(z,T)$.

3)In Ref. 27, phase fluctuations were eliminated by transforming to a reference frame comoving at the local speed of sound. However, such an approach generalizes poorly to three dimensions.

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Translated by M. Damashek