Stability of plasma solitons

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We consider the problem of the stability of solitary waves in a collisionless plasma. Using the Zakharov set of equations which describes the interaction between Langmuir and ion-sound waves in the "hydrodynamic limit" we show the stability of solitons under one-dimensional perturbations. However, the solitons turn out to be unstable against three-dimensional perturbations. We obtain estimates for the growth rate and the spatial dimensions of this instability.

INTRODUCTION

It is well known that localized regions of strong Langmuir oscillations—standing or travelling plasma solitons—can exist in a collisionless plasma. In these regions the plasma density is lowered and they are essentially resonators for plasma waves. We shall consider one-dimensional solitons which were, apparently, first considered in 1963 by Gurevich and Pitaevskiĭ.[1] A number of subsequent papers have been devoted to the study of the stability of different kinds of plasma solitons. One should note that the study of the stability of solitons is of great importance for plasma physics: one can use stable solitons as building blocks to construct strong turbulence models, while the instability indicates that the turbulence is not the usual kind and that Langmuir collapse plays an important role. For instance, in 1973 Zakharov and Rubenchik[2] showed that a standing plasma soliton was unstable against three-dimensional long-wavelength perturbations. This result was later generalized by Degtyarev, Zakharov, and Rudakov.[3] We note that these studies[2,3] were carried out in the "static limit" approximation, i.e., under the assumption that the mean square ion thermal velocity u_T was larger than the hydrodynamic velocity u, i.e., $u_T \gg u$ (for details see Ref. 4). In 1975 a paper by Schmidt[5] appeared in which the instability of solitons moving with an arbitrary velocity in the "hydrodynamic limit" ($u_T \ll u$) was proven. However, this result encountered serious objections because of the incorrect use in Ref. 5 of an approximate method. For instance, in Infeld and Rowland's 1977 paper[6] it was shown that the result of Ref. 5 can not guarantee several necessary "compatibility conditions" which in the linear formulation are exact properties of this problem. Hojo[7] in 1978 showed that Schmidt's assumptions (in Ref. 5 the stability of large amplitude solitons which are physically not described by the initial equations were studied, and perturbations with an electrostatic field strength which did not satisfy the condition curl $\mathbf{E} = 0$ were considered) were unphysical. Further using various modifications of the multiple scale method the authors of Refs. 6, 7 came to the conclusion that the plasma soliton was stable against long-wavelength perturbations.

One should, however, note that the equations describing solitons with small amplitudes and with small propagation velocities are asymptotically equivalent in the hydrodynamic and the static limits (we shall explain this in what follows). In that sense the results of Infeld and Rowlands and of Hojo contradict the conclusion about the instability of a standing plasma soliton reached in Refs. 2, 3. Later Laedke and Spatschek[8,9] showed the instability of a standing plasma soliton directly in the hydrodynamic limit.[1)] We note also the numerical experiment by Pereira *et al.*[10] showing the qualitative presence of an instability without, however, confirming quantitatively any of the above-mentioned theoretical calculations. The reasons for these discrepancies were indicated in the paper by Wardrop and ter Haar[11] which was devoted to the study of the stability of solitons in the framework of a somewhat different system of equations. In fact, in Ref. 6 the stability of a soliton was proven relative to only a narrow class of perturbations and the author of Ref. 7 used an incorrect approximation method. As to the numerical calculation of Ref. 10, the reason for the lack of agreement between the the value of the instability growth rate and the results of theoretical papers was an inadequate choice of the initial conditions for the perturbations (for details see Ref. 11).

We discuss briefly the methodological side of the problem. In Refs. 8, 9 an unwieldy formal method was used which is based upon the exact relation between the eigenfunctions of the corresponding boundary value problem and the extremals of a certain functional. However, the magnitude of the instability growth rate must be guessed by substituting test functions into that functional. Moreover, this method gives little information about the nature of the instability which makes it difficult to compare it with the results obtained in the framework of other methods. The variational method used in Ref. 3 was discussed in detail in Ref. 12. In this paper it was shown that the application of that method to the problem of the stability of stationary states within the framework of the nonlinear Schroedinger equation and of the Kadomtsev-Petiviashvili equation can lead to incorrect results. Further the authors of Ref. 12 reach a conclusion about the limited applicability of this approach to the problem of soliton stability. As to the approach used in Ref. 11 (which is an improved variant of the method of Ref. 7) it does not always give good agreement with the results from the Zakharov-Rubenchik method[2] which in some sense is rigorous. Namely, in the framework of the procedure adopted in Ref. 2 one can successively calculate exact values of the coefficients of Taylor expansions of the eigenfunctions and the

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dispersive dependences of the corresponding boundary value problem in the vicinity of the point $\mathbf{k} = 0$ ($\mathbf{k}$ is the wavevector of the waves of the perturbation). This advantage is also common to the methods proposed by Han[13] and by Janssen and Rasmussen[14] which are close to that of Ref. 2.

The stability of solitons against one-dimensional perturbations was discussed in Refs. 15 to 17. In the static limit the one-dimensional evolution equations for the solitary wave reduce to the nonlinear Schroedinger equation which is integrable by the inverse scattering method;[15] such solitons are absolutely stable (i.e., stable against perturbations of any amplitude). Using equations close to the equations of the hydrodynamic limit it was shown in Ref. 16 that the solitons are stable against infinitesimal perturbations, while in Ref. 17 the absolute stability of standing solitons in the hydrodynamic approximation was proved.

The present paper is devoted to a study of the stability of plasma solitons against one- and three-dimensional perturbations in the hydrodynamic limit. We prove the absolute stability of solitary waves of any amplitude and propagation speed against one-dimensional perturbations. For the three-dimensional problem we use the Zakharov-Rubenchik method to evaluate and analyze the dispersion relation for unstable transverse perturbations of the soliton front. We discuss the nature of the instability and obtain estimates for its growth rate and spatial scale. The results obtained agree for the particular case of a small-amplitude soliton with the conclusions of Ref. 2.

1. BASIC EQUATIONS. STATIONARY SOLUTIONS

To describe the interaction of Langmuir waves and low-frequency oscillations of the ion density we shall use the dimensionless set of Zakharov equations (see Ref. 4) for the relative deviation of the ion density from its equilibrium value, $n = \delta n/n_0$, and the complex potential Ψ of the electric field of the Langmuir wave:

$$n_{tt}-\Delta n=\Delta|\nabla\Psi|^2, \tag{1a}$$

$$\Delta(2\alpha i\Psi_t+3\Delta\Psi)-\operatorname{div}(n\nabla\Psi)=0. \tag{1b}$$

The spatial coordinates $(x,\mathbf{r}_\perp) = (x,y,z)$ are here made dimensionless by dividing by the Debye radius r_d, and the time t by dividing by the time scale for ion-sound waves, r_d/c_s, where c_s is the ion sound speed. The dimensionless constant α is given by $(m/M_i)^{1/2}$, where m and M_i are the electron and ion masses. The real potential of the electric field $\tilde{\Psi}$ can, in dimensional variables, be evaluated from the formula

$$\tilde{\Psi}=2T_e q^{-1}\operatorname{Re}(\Psi e^{i\omega_p t}),$$

where ω_p is the electron plasma frequency, T_e and q the electron temperature and charge. The set of Eqs. (1) is valid for conditions of weak nonlinearity and small ion thermal velocity. We note that both unknown functions are assumed to be smoothly varying and small in magnitude (which means weak nonlinearity and dispersion):

$$|\nabla n|\ll|n|\ll 1, \quad |\nabla\Psi|\ll|\Psi|\ll 1.$$

One can find a detailed discussion of all restrictions imposed upon the solution of Eqs. (1) and on the plasma parameters in Ref. 4.

The set (1) has a solitary-wave stationary solution:

$$n(x,\mathbf{r}_\perp,t)=n(\xi), \quad \xi=x-ct,$$
$$\Psi(x,\mathbf{r}_\perp,t)=\Psi(\xi)\exp[i(\Omega t-\boldsymbol{\varkappa}\mathbf{r}_\perp)], \quad \boldsymbol{\varkappa}=(0,\varkappa_y,\varkappa_z),$$

where $n(\xi)$ and $\Psi(\xi)$ satisfy ordinary differential equations

$$(1-c^2)n+\varkappa^2|\Psi|^2+|\Psi'|^2=0, \tag{2a}$$

$$[-(2\alpha\Omega+3\varkappa^2)\Psi'-2\alpha ic\Psi''+3\Psi'''-n\Psi']'$$
$$-\varkappa^2[-(2\alpha\Omega+3\varkappa^2)\Psi$$
$$-2\alpha ic\Psi'+3\Psi''-n\Psi]=0 \tag{2b}$$

and the boundary conditions that as $\xi\to\pm\infty$, $n\to 0$, $\Psi\to 0$. We introduce the characteristic longitudinal scale l of the soliton field and distinguish two limiting cases: $\varkappa l\gg 1$ and $\varkappa = 0$. Neglecting the last term in (2a) when $\varkappa l\gg 1$ and the first group of terms in the square brackets in (2b) we have

$$n=2\lambda\,\mathrm{ch}^{-2}(\xi/l), \tag{3a}$$

$$\Psi=2\varkappa^{-1}(\lambda\mu)^{1/2}\,\mathrm{ch}^{-1}(\xi/l)\exp(i\alpha c\xi/3). \tag{3b}$$

Here

$$l^2=3/\lambda, \quad \lambda=2\alpha\Omega+3\varkappa^2+\alpha^2c^2/3>0, \quad 2\mu=1-c^2>0.$$

The solution (3) is a Langmuir wave with wavevector $(\alpha c/3,\varkappa)$, trapped in a region of ion depletion and moving in a transverse direction, while λ is proportional to the nonlinear correction to the Langmuir frequency Ω of the wave.

When $\varkappa = 0$ the profile $n(\xi)$ of the ion density is the same as the profile (3a), while the field strength E of the Langmuir wave is given by the formula

$$E=\Psi'=2(\lambda\mu)^{1/2}\,\mathrm{ch}^{-1}(\xi/l)\exp(i\alpha c\xi/3). \tag{4}$$

We shall call the subsonic ($|c|<1$) solitons which are described by Eqs. (3) and (4), respectively, quasiplanar and planar. We note that the planar soliton in contrast to the quasiplanar one contains a drop in potential, i.e., $\Psi(\infty)\neq\Psi(-\infty)$.

When describing the nonstationary evolution of a quasiplanar soliton we can simplify Eq. (1b). Making the substitution

$$\Psi=\varkappa^{-1}\psi\exp[-i(3\varkappa^2t/2\alpha+\boldsymbol{\varkappa}\mathbf{r}_\perp)]$$

and expanding in the small parameter $(\varkappa l)^{-2}$ we get

$$2\alpha i\psi_t-6i(\boldsymbol{\varkappa},\nabla_\perp\psi)+3\Delta\psi-n\psi=0, \quad |\nabla\Psi|^2\approx|\psi|^2. \tag{5}$$

We note that in the one-dimensional case Eq. (5) is formally (i.e., by renaming the unknown function, $\Psi\to E=\Psi_x$) the same as Eq. (1b).

Note that when one studies the stability of a small-amplitude soliton with a small propagation speed ($\lambda\ll\alpha^2$, $|c|\ll 1$) one can in Eq. (1a) neglect the term n_{tt} (see Ref. 4). Substituting $n\approx-|\nabla\Psi|^2$ into (1b) we get an equation which is the same as the dimensionless equation in the static limit [one can justify this procedure by carrying out a rigorous asymptotic analysis of Eqs. (1)]. The results obtained in the hydrodynamic approximation in the limit of small soliton amplitudes must thus be the same or must be asymptotically equivalent to the results of Ref. 2.

2. STABILITY OF SOLITONS AGAINST ONE-DIMENSIONAL PERTURBATIONS

In the one-dimensional geometry Eqs. (1) take a very simple form:

$$n_t+u_x=0, \quad u_t+n_x+(|E|^2)_x=0, \qquad 2\alpha iE_t+3E_{xx}-nE=0. \tag{6}$$

The quantity u can here be understood to be the hydrodynamic ion velocity. The set of Eqs. (6) conserves four integrals of motion which have a physical meaning—the mass M, the plasmon number N (wave action), the momentum I, and the energy H:

$$M=\int n\,dx, \qquad N=\int|E|^2\,dx,$$

$$I=\int[nu+\alpha i(EE_x^*-E^*E_x)]dx,$$

$$H=\int\left[\frac{u^2}{2}+\frac{n^2}{2}+n|E|^2+3|E_x|^2\right]dx.$$

We note that in the expression for the momentum I the term $I_0=\int u dx$ has been dropped (bear in mind that the total dimensionless ion density is $n_{\text{tot}}=1+n$). This is allowed because I_0 is also an integral of the motion which has, however, no special physical meaning.

One can write Eqs. (6) in Hamiltonian form:

$$n_t+\frac{\partial}{\partial x}\frac{\delta H}{\delta u}=0, \qquad u_t+\frac{\partial}{\partial x}\frac{\delta H}{\delta n}=0, \qquad E_t+\frac{i}{2\alpha}\frac{\delta H}{\delta E^*}=0.$$

We note also that the stationary Eqs. (2) (when $\varkappa=0$) can be written in the form

$$\delta(H-cI+2\alpha\Omega N)=0. \tag{7}$$

This way of writing them means that all stationary solutions of the set (6) which decrease as $x\to\pm\infty$ are also stationary points of the Hamiltonian for fixed I and N. In that case the parameters c and Ω have the meaning of Lagrangian multipliers.

To prove the stability of the soliton we use directly Lyapunov's theorem which is often applied for the solution of such problems (see, e.g., Refs. 18, 19), namely, if we show that one of the integrals of motion is bounded (from above or below) for fixed values of the others, the solution of the original equations which achieves this extremum [in the present case which satisfies condition (7) or, what comes to the same thing, Eqs. (2) for $\varkappa=0$] is then stable. We show that the Hamiltonian is bounded from below for fixed momentum and plasmon number. To do this we at once drop the first (clearly positive) term in the expression for H:

$$H\geqslant P/2+\mathscr{F}+3J,$$

where P, $\mathscr{F}$, and J are, respectively, the other terms. The only possible negative term is $\mathscr{F}$. Using the Hölder inequality we find a lower limit for it:

$$\mathscr{F}\geqslant-\left[P\int|E|^4\,dx\right]^{1/2}\geqslant-[P(|E|^2)_{max}N]^{1/2}.$$

Using then the obvious inequality

$$(|E|^2)_{max}\leqslant\int|(|E|^2)_x|dx\leqslant2\int|E||E_x|dx$$

and applying the Hölder inequality once again we have

$$\mathscr{F}\geqslant-[2PN(NJ)^{1/2}]^{1/2},$$

$$H\geqslant P/2-2^{1/2}N^{3/4}P^{1/2}J^{1/4}+3J\geqslant-N^3/12.$$

We have thus proved an even stronger statement: the Hamiltonian is bounded from below already for fixed plasmon number N and arbitrary momentum P.

3. SOLITON INSTABILITY AGAINST THREE-DIMENSIONAL PERTURBATIONS

The problem of the three-dimensional stability of both kinds of solitons reduces to analysis of the spectra of complicated non-Hermitean operators of very high order. In the general case such a problem is insoluble analytically. We use an approximate method based on Zel'dovich and Barenblatt's approach[20] to a study of combusion waves and formulated in a universal form by Zakharov and Rubenchik.[2] The method consists in an expansion of the eigenvalues and eigenfunctions of the operators studied in powers of the wavevector of the perturbation in the long-wavelength approximation. The calculations then simplify considerably thanks to a knowledge of neutrally stable modes corresponding to small variations in the form of the soliton as a whole. We consider first a planar soliton.

1. We introduce a simplified equation describing the instability of a planar soliton. To do this we differentiate (1b) with respect to x and act upon it with the operator which is the inverse of the Laplacian. We use the symbolic formula

$$\Delta^{-1}=(\partial_{xx}{}^2+\Delta_\perp)^{-1}\approx\partial_{xx}{}^{-2}(1-\Delta_\perp\partial_{xx}{}^{-2}),$$

which is valid when $\partial_{xx}^2\gg\Delta_\perp$. The operator ∂_x^{-1} is, as usually, defined here by the equation

$$\partial_x{}^{-1}f(x)=\int_{-\infty}^{x}f(x')\,dx'.$$

Taking into account only the first corrections connected with spatial variations we get

$$2\alpha iE_t+3\Delta E-nE+\Delta_\perp\partial_{xx}{}^{-2}(nE)-\text{div}_\perp\,\partial_x{}^{-1}(n\nabla_\perp\partial_x{}^{-1}E)=0. \tag{8}$$

As before E is here the x-component of the field strength: $\Psi=\partial_x^{-1}E$. Changing to a system of coordinates moving along the x-axis with velocity c, viz. $(x,\mathbf{r}_\perp,t)\to(\xi,\mathbf{r}_\perp,t)$, $\xi=x-ct$, we linearize the set of Eqs. (1a), (8) on the background of the solution (4):

$$E(\xi,\mathbf{r}_\perp,t)=[E(\xi)+\varphi(\xi,\mathbf{r}_\perp,t)+i\varphi_1(\xi,\mathbf{r}_\perp,t)]e^{i\Omega t},$$

$$n(\xi,\mathbf{r}_\perp,t)=n(\xi)+w(\xi,\mathbf{r}_\perp,t).$$

Separating the imaginary and real parts in Eq. (8) we Fourier transform with respect to the time t and the transverse coordinates $\mathbf{r}_\perp$. Retaining for the Fourier transforms of φ, φ_1, and w the same notation we have

$$2\alpha i\omega\varphi-2\alpha c\varphi'-\hat{L}\varphi_1-k^2\hat{F}\varphi_1=0, \tag{9a}$$

$$-2\alpha i\omega\varphi_1+2\alpha c\varphi_1'-\hat{L}\varphi-Ew-k^2(\hat{F}\varphi+\hat{G}w)=0, \tag{9b}$$

$$-\omega^2w-2ic\omega w'-2\mu w''+k^2w-2(E\varphi)''+2k^2E\varphi=0, \tag{9c}$$

where ω and $\mathbf{k}$ are the parameters of the Fourier transformation with respect to t and $\mathbf{r}_\perp$; $\lambda = 2\alpha\Omega$, $\mu = (1-c^2)/2$; the operators are given by

$$\hat{L}=-3\partial_{\xi\xi}^2+\lambda+n(\xi),$$

$$\hat{F}=3+\partial_{\xi\xi}^{-2}n(\xi)-\partial_\xi^{-1}n(\xi)\partial_\xi^{-1},$$

$$\hat{G}=\partial_{\xi\xi}^{-2}E(\xi)-\partial_\xi^{-1}[\partial_\xi^{-1}E(\xi)].$$

In obtaining Eqs. (9) we put $\operatorname{Im} E(\xi)\approx 0$. This is admissible when we consider solitons with not too small an amplitude for which $l \ll (\alpha c)^{-1}$ or, equivalently,

$$\lambda \gg \alpha^2 c^2. \tag{10}$$

Indeed, neglecting in (2b) the term $2i\alpha c E'$, we have when $\varkappa = 0$

$$\hat{L}E=0, \tag{11}$$

and the electric field strength of a planar soliton is given by Eq. (4) with $\alpha c = 0$. We shall use the approximation (10) in all the calculations which follow. We note also that the parameter $\alpha \ll 1$ is the ratio of the time scales of the Langmuir and the ion-sound waves; its maximum value $\alpha_{\max} = 2.34\times 10^{-2}$ is reached for a hydrogen plasma. Inequality (10) therefore does not restrict our considerations too much, but it simplifies calculations considerably. Using (10) we neglect the second terms in (9a) and (9b); eliminating φ_1 and dropping terms $\propto k^4$, we get

$$\hat{L}(\hat{L}\varphi+Ew)-4\alpha^2\omega^2\varphi+k^2[(\hat{L}\hat{F}+\hat{F}\hat{L})\varphi+(\hat{F}E+\hat{L}\hat{G})w]=0,$$

$$\mu w+E\varphi+ic\omega\partial_\xi^{-1}w+{}^1/_2(\omega^2-k^2)\partial_{\xi\xi}^{-2}w-k^2\partial_{\xi\xi}^{-2}E\varphi=0. \tag{12}$$

We expand the eigenfunctions and the dispersive dependence of the boundary value problem (12) in a power series of the wavenumber of the perturbation:

$$w=w_0+kw_1+k^2w_2+\dots,\quad \varphi=\varphi_0+k\varphi_1+k^2\varphi_2+\dots,$$

$$\omega=k\omega_1+\dots.$$

In zeroth order of perturbation theory we have

$$\hat{\mathscr{L}}\begin{pmatrix}\varphi_0\\ w_0\end{pmatrix}=0,$$

$$\hat{\mathscr{L}}=\hat{\mathscr{L}}_0\hat{\mathscr{L}}_1,\quad \hat{\mathscr{L}}_0=\begin{pmatrix}\hat{L} & 0\\ 0 & 1\end{pmatrix},\quad \hat{\mathscr{L}}_1=\begin{pmatrix}\hat{L} & E\\ E & \mu\end{pmatrix}.$$

By virtue of the translational invariance of the original equations we can put (we have introduced a factor c for the sake of convenience)

$$w_0=cn',\quad \varphi_0=cE'.$$

Such a perturbation corresponds to a bending instability of the soliton front ("balloon" type instability). In first order of perturbation theory we get

$$\hat{\mathscr{L}}\begin{pmatrix}\varphi_1\\ w_1\end{pmatrix}=\begin{pmatrix}0\\ -ic^2\omega_1 n\end{pmatrix}$$

or, using (11)

$$\hat{\mathscr{L}}_1\begin{pmatrix}\varphi_1\\ w_1\end{pmatrix}=\begin{pmatrix}-i\eta_1 E\\ -ic^2\omega_1 n\end{pmatrix},$$

where η_1 is a constant, determined below, while the factor $-i$ is also introduced for the sake of convenience. The solution of this set of equations can be found explicitly. Indeed, starting from Eqs. (2a) and (11) one can check through straightforward calculations that

$$\varphi_1=ic^2\omega_1\frac{\partial E}{\partial\mu}+i\eta_1\frac{\partial E}{\partial\lambda},\quad w_1=ic^2\omega_1\frac{\partial n}{\partial\mu}+i\eta_1\frac{\partial n}{\partial\lambda}.$$

The even perturbations φ_1 and w_1 describe a modulation of the soliton amplitude in the transverse direction—a "sausage" type instability. After a few transformations we get in the next order

$$\hat{\mathscr{L}}\begin{pmatrix}\varphi_2\\ w_2\end{pmatrix}=c\begin{pmatrix}4\alpha^2\omega_1^2E'-\hat{L}(\hat{F}E'+\hat{G}n')\\ \partial_\xi^{-1}\left[\omega_1\left(c^2\omega_1\dfrac{\partial n}{\partial\mu}+\eta_1\dfrac{\partial n}{\partial\lambda}\right)+\dfrac{1-\omega_1^2}{2}n+\dfrac{E^2}{2}\right]\end{pmatrix}. \tag{13}$$

In order that the set (13) be soluble it is necessary that the vector of the right-hand sides be orthogonal to all eigenfunctions of the operator adjoint to $\mathscr{L}$ with zero eigenvalue. As $\hat{\mathscr{L}}_0$ and $\hat{\mathscr{L}}_1$ are self-adjoint we have $\hat{\mathscr{L}}^+ = \hat{\mathscr{L}}_1\hat{\mathscr{L}}_0$; one can easily find the eigenfunctions of this operator:

$$\begin{pmatrix}E\\ 0\end{pmatrix}\quad\begin{pmatrix}\hat{L}^{-1}E'\\ n'\end{pmatrix}.$$

Here $\hat{L}^{-1}$ is the operator which is the inverse of $\hat{L}$. The condition that the right-hand side of (13) is orthogonal to the first eigenfunction of $\hat{\mathscr{L}}$ is automatically satisfied, and the second eigenfunction gives a condition connecting ω and η (from now on we drop the index 1):

$$-4\alpha^2\omega^2\langle E'|\hat{L}^{-1}|E'\rangle+\langle E'|\hat{F}|E'\rangle+\langle E'|\hat{G}|n'\rangle+\frac{\omega}{2}\eta\frac{\partial P}{\partial\lambda}+\frac{1-\omega^2}{2}P+\frac{\langle nE^2\rangle}{2}=0. \tag{14a}$$

Here

$$P=\langle n^2\rangle=\int_{-\infty}^{\infty}n^2\,d\xi.$$

In the calculations we have used Eq. (11), the fact that the operator $\hat{L}$ is self-adjoint, $\langle f|\hat{L}|g\rangle = \langle g|\hat{L}|f\rangle$, and also the fact that n does not depend on μ.

The missing equation for ω and η follows from the condition that the right-hand side of the equations of the third approximation must be orthogonal to the first eigenfunction of the operator $\hat{\mathscr{L}}^+$. After straightforward transformations we get

$$2\alpha^2\omega^2\left(c^2\omega\frac{\partial N}{\partial\mu}+\eta\frac{\partial N}{\partial\lambda}\right)+\eta\langle E|\hat{F}|E\rangle=0, \tag{14b}$$

where $N = \langle E^2\rangle$.

We consider the dispersion relations (14) for $c\sim 1$. They have four roots corresponding to two modes, one of which (the high-frequency one) corresponds to pure Langmuir oscillations ($\omega_L \sim \lambda^{1/2}/\alpha$), the other (the low-frequency one) to ion-sound oscillations ($\omega_s \sim 1$). Making appropriate approximations in (14) and using inequality (10) ($c\sim 1$, $\lambda \gg \alpha^2$) we have

$$\omega_L^2 \approx -P\langle E|\hat{F}|E\rangle \left[2\alpha^2 \left(c^2 \frac{\partial N}{\partial \mu} \frac{\partial P}{\partial \lambda} + \frac{\partial N}{\partial \lambda} P \right) \right]^{-1}, \qquad (15a)$$

$$\omega_s^2 \approx 1 + P^{-1} [2(\langle E'|\hat{F}|E'\rangle + \langle E'|\hat{G}|n'\rangle) + \langle nE^2\rangle]. \qquad (15b)$$

Performing the very time-consuming integration we get for the high-frequency mode

$$\omega_L^2 = -[12 - 7\zeta(3)]\lambda/8\alpha^2(3c^2+\mu) \approx -0.9\lambda/\alpha^2(5c^2+1), \qquad (16)$$

and it is thus unstable [$\zeta(x)$ is the Riemann zeta-function]. Similar calculations for the low-frequency mode give $\omega_s^2 > 0$. Estimates of the maximum instability growth rate and of the wavevector of the perturbation for which it occurs are of interest. To find this it is sufficient to note that for $k \gg l^{-1} \sim \lambda^{1/2}$ the frequency of the perturbing waves must tend to the frequency of free Langmuir waves, $\omega_L \approx -3k^2/2\alpha$ and the perturbations become stable. One concludes that the maximum growth rate is reached for $k \sim l^{-1}$ and estimate its order of magnitude:

$$\gamma_m \sim \omega l^{-1} \sim \lambda/\alpha, \qquad (17)$$

which is the same as the estimate of the growth rate of the modulational instability of free Langmuir waves (we remind ourselves that λ is proportional to the non-linear correction to the frequency of the Langmuir wave trapped in the soliton; see Ref. 21). We note also that when $k \ll l^{-1}$ the "hose" type interchange instability dominates and when $k \sim l^{-1}$ the "sausage" type instability (modulational instability) dominates. This follows from the estimates

$$k \sim l^{-1}, \quad \varphi_0 \sim \lambda \ll k\varphi_1 \sim \lambda^{3/2}/\alpha.$$

The presence of a weak interchange instability of the soliton distinguishes this case from the soliton in the static limit. To all appearances it arise because of the inertial nature of the nonlinearity in Eqs. (1)—low-frequency changes in the plasma density do not manage to follow the fast oscillations of the electric field of the Langmuir wave when the soliton front is bent, which leads to an imbalance of the quantities n and E. We note also that the dispersion relation given by Eq. (17) differs considerably from Schmidt's result.[5]

As $c \to 0$ only the dispersive dependence of the stable low-frequency mode (15b) changes, while Eqs. (15a) and (16) remain valid, now even for solitons of arbitrary amplitude [inequality (10) is satisfied identically when $c = 0$] while the instability is purely modulational in character. Summarizing the results we write Eq. (17) in dimensional form:

$$\tilde{\gamma}_m \sim \frac{q^2}{T_e^2} c_s r_d \left(\frac{M_i}{m} \right)^{1/2} |\tilde{E}_0|^2, \qquad (18)$$

where $|\tilde{E}_0|$ is the dimensional amplitude of the electric field of the soliton. Equation (18) is valid for $\tilde{c} \sim c_s$, $|\tilde{E}_0|^2 \gg T_e^2 m/q^2 r_d^2 M$ and when $\tilde{c} = 0$ for any value of $|\tilde{E}_0|$ ($\tilde{c}$ is the dimensional soliton speed). One must note that the dispersion relation given by Eq. (15a) is the same for $c = 0$ as the result obtained in Ref. 2 for a soliton in the static limit due to stochastic effects in the whole range of soliton amplitudes (and not only when $\lambda \ll \alpha^2$ as might be expected). This fact can be traced when one studied both problems directly using the eigenvalues for $c = 0$.

As a result of the instability the soliton breaks up into collapsing blobs with transverse dimensions of the order of the longitudinal one which finally leads to the collapse of Langmuir waves (see Refs. 3, 4).

2. The case of the quasiplanar soliton is physically less interesting and we shall not dwell on it in detail. When $\lambda \gg \alpha^2$ for perturbations moving in a direction at right angles to the wavevector $\varkappa$ of the Langmuir wave ($\theta = \pi/2$, θ is the angle between $\varkappa$ and $\mathbf{k}$) the high-frequency mode is unstable:

$$\omega^2 \approx -6\lambda\mu/\alpha^2(5c^2+1) < 0, \quad \gamma_m \sim \lambda/\alpha.$$

When the angle θ differs from the value $\theta = \pi/2$ the low-frequency mode vanishes altogether–there are two branches of unstable high-frequency oscillations (we remember that $\varkappa \gg \lambda^{1/2}$):

$$(\alpha\omega_I + 3\varkappa\cos\theta)^2 \approx -6\lambda\mu/(5c^2+1) < 0, \quad \gamma_m \sim \lambda/\alpha,$$
$$\omega_{II}^2 \approx -18\varkappa^2\cos^2\theta(1-c^2)^2/\lambda(5c^2+1) < 0, \quad \gamma_m \sim \varkappa.$$

When $\lambda \ll \alpha^{1/2}$, $c \neq 0$ the quasi-planar soliton is the usual envelope soliton; this case was analyzed in Ref. 2.

As a result of the instability a collapse of the envelopes of the blobs with monochromatic filling takes place which distinguishes this case from the case of the planar soliton in which the collapse strictly of Langmuir waves takes place. The qualitative nature of the instability is completely similar to that of the proceding case.

We have thus established that although plasma solitons are stable against one-dimensional pertrubations they turn out to be unstable in the three-dimensional case.

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[1)]Most authors have considered the particular case of a standing soliton as it is mathematically somewhat simpler than the general case.

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