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THE EFFECT OF CONTINUOUS SHEAR UPON THE TWO-LAYER MODEL OF BAROCLINIC INSTABILITY

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The two-layer model of baroclinic instability is modified to include small continuous variations of the velocity/density profile in the upper layer. It is demonstrated that, if the difference between the average upper-layer velocity and the velocity in the lower layer is negative (westward), the flow is unstable. The instability takes place in the spectral region of waves with critical levels, but the unstable disturbances do not have the logarithmic singularity, which is commonly believed to destabilize the flow through rapid variation of heat flux. The instability is interpreted as a resonance between Rossby waves and a certain "shear mode" supported by the vertical shear of the mean flow.

KEY WORDS: Baroclinic instability, two-layers model, critical levels.

1. INTRODUCTION

The two-layer model of stratification is useful only if it provides a reasonable approximation of continuously stratified flows; but, unfortunately, the complexity of the continuous model makes the comparison difficult, if not impossible. Furthermore, rare tractable particular cases of continuously stratified flows usually assume the stratification profile to be linear, which is very far from the two-layer model (and the real-life currents, for that matter). As a result, it is often impossible to distinguish a qualitative (fundamental) discrepancy of the two models (if any) from an inconsistency of parameters.

In the baroclinic-instability context, the only tractable continuous models (Charney, 1947; Eady, 1949; Green, 1960) are based on linear profiles of both velocity and density. It was demonstrated (Burger, 1962; Green, 1960) that all linearly-stratified flows with non-zero shear are unstable, which strongly disagrees with the two-layer results (Phillips, 1954): the two-layer model is stable for a sufficiently weak shear. This and other[1] discrepancies are usually attributed to the so-called critical levels, where the real part of the phase speed of a disturbance matches the velocity of the flow. Critical levels are believed to destabilize the disturbance and, since a continuously stratified flow admits a wide spectral region of disturbances with critical levels, it is unstable even for a weak shear.

[1] For example, Phillip's model predicts a short-wave cutoff of the instability, while Charney's model does not (Burger, 1962).

This explanation, however, is hardly satisfactory: for example, critical levels do not destabilize the plane inviscid Couette flow (e.g. Dikey, 1976): as it turns out, the boundary-value problem for normal modes, in this case, has neither stable nor unstable solutions. Another, even more relevant, counter-example can be seen in a two-layer flow with a westward shear, which can be readily shown to support two waves with critical levels (whose phase speeds match the velocities in the layers). The solution exists for both of them, but, if the shear is sufficiently weak, they are stable!

In this paper, the two-layer model is modified to include small continuous variations of the density/velocity profile in the upper layer. The main question to be asked in this paper is:

Does continuous shear always make the flow unstable?

Surprisingly, any continuous variation of the two-layer model is found to destabilize all flows with westward shear, no matter how small this perturbation is.

Another unusual characteristic feature of the instability is that the unstable disturbances do not have the logarithmic singularity at the critical level. Such singularities are believed to arise for all disturbances, whose imaginary part of the phase speed is much smaller than the real part, and play an important role in physical interpretation of baroclinic instability (e.g. Pedlosky, 1987, p. 536).

In order to understand the mechanism of the instability discovered, we note that the unperturbed two-layer model supports three modes of harmonic disturbances: two Rossby-wave modes and a certain mode, which exists owing to the vertical shear of the mean flow. It can be readily demonstrated that the dispersion curve of one of the Rossby-wave modes intersects that of the shear mode, and the point of intersection corresponds to an inter-mode resonance. Within the framework of the unperturbed (two-layer) system the resonant waves do not interact with each other, but in the presence of a perturbation the resonance causes an instability.

In order to calculate the characteristics of this instability, we derive (Section 2) and asymptotically solve (Sections 3 and 4) the boundary-value problem which describes the two-layer model with small continuous perturbations.

2. FORMULATION OF THE PROBLEM

Let $\tilde{\rho}(\tilde{z})$ describe the density stratification of a fluid layer bounded by two rigid planes at

$$\tilde{z}=0 \quad \text{and} \quad \tilde{z}=-H,$$

where $\tilde{z}$ is the vertical spatial variable and H is the depth of the layer. Assuming that $\tilde{\rho}(\tilde{z})$ is a monotonic function, we have

$$\min[\tilde{\rho}(z)]=\tilde{\rho}(0)=\rho_0.$$

We also assume that $\tilde{\rho}(\tilde{z})$ has a discontinuity at $\tilde{z}=-\tilde{h}$:

$$\rho(-\tilde{h}+0)-\rho(-\tilde{h}-0)=-\Delta\rho.$$

Now we can define the deformation radius:

$$R_d = \frac{1}{f}\sqrt{gH\frac{\Delta\rho}{\rho_0}},$$

where g is the acceleration due to gravity and f is the Coriolis parameter. Using R_d, H and f, we introduce the non-dimensional variables

$$\Psi = \tilde{\Psi}/(fR_d^2),$$

$$x = \tilde{x}/R_d, \qquad y = \tilde{y}/R_d, \qquad z = \tilde{z}/H, \qquad t = f\tilde{t} \qquad \rho = (\tilde{\rho} - \rho_0)/\Delta\rho;$$

where $\tilde{\Psi}$, $(\tilde{x}, \tilde{y})$ and $\tilde{t}$ are the dimensional stream function, spatial horizontal variables and time, respectively.

The non-dimensional streamfunction of a quasi-geostrophic motion on the β-plane is governed by the following equation:

$$\left[\Delta\Psi - \left(\frac{1}{\rho_z}\Psi_z\right)_z\right]_t + J\left[\Psi, \Delta\Psi - \left(\frac{1}{\rho_z}\Psi_z\right)_z\right] + \beta\Psi_x = 0. \tag{1}$$

Here $\Delta \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, J is the Jacobian operator: $J(\Psi, \Phi) \equiv \Psi_x\Phi_y - \Psi_y\Phi_x$, and

$$\beta = (R_d/R_e)\cot\lambda,$$

where R_e is the earth's radius and λ is the latitude. Equation (1) should be supplemented by the no-flow conditions at the rigid boundaries of the layer:

$$\frac{1}{\rho_z}[\Psi_{zt} + J(\Psi, \Psi_z)] = 0, \quad \text{at} \quad z = 0, \tag{2}$$

$$\frac{1}{\rho_z}[\Psi_{zt} + J(\Psi, \Psi_z)] = 0, \quad \text{at} \quad z = -1. \tag{3}$$

As $\rho(z)$ has a discontinuity at $z = -h$:

$$\rho(-h+0) - \rho(-h-0) = -1, \tag{4}$$

Ψ must satisfy the following matching conditions (derived in Appendix A):

$$\left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h+0} - \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h-0} = 0, \tag{5a}$$

$$(\Psi)_{z=-h+0} - (\Psi)_{z=-h-0} = -\left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h}. \tag{5b}$$

We assume that the lower layer is homogeneous:

$$\rho(z) \equiv \text{const}, \quad \text{for} \quad z < -h \tag{6}$$

(stratification model (4), (6) is shown in Figure 1) and introduce $\Psi_{1,2}$:

$$\Psi = \begin{cases} \Psi_1, & \text{for} \quad z > -h, \\ \Psi_2, & \text{for} \quad z < -h. \end{cases}$$

In order to regularize the singularity in Equation (1) $(\rho_z)^{-1} \equiv \infty$ for $z < -h$), one should consider "almost" constant density: $|\rho_z| \ll 1$, and seek a solution in the form of a series in powers of ρ_z:

$$\Psi_2(t, x, y, z) = \Psi_2(t, x, y) + \Phi_2(t, x, y) \int_{-1}^{z} (1 + z')\rho_{z'}(z')dz' + O(\rho_z^2) \tag{7}$$

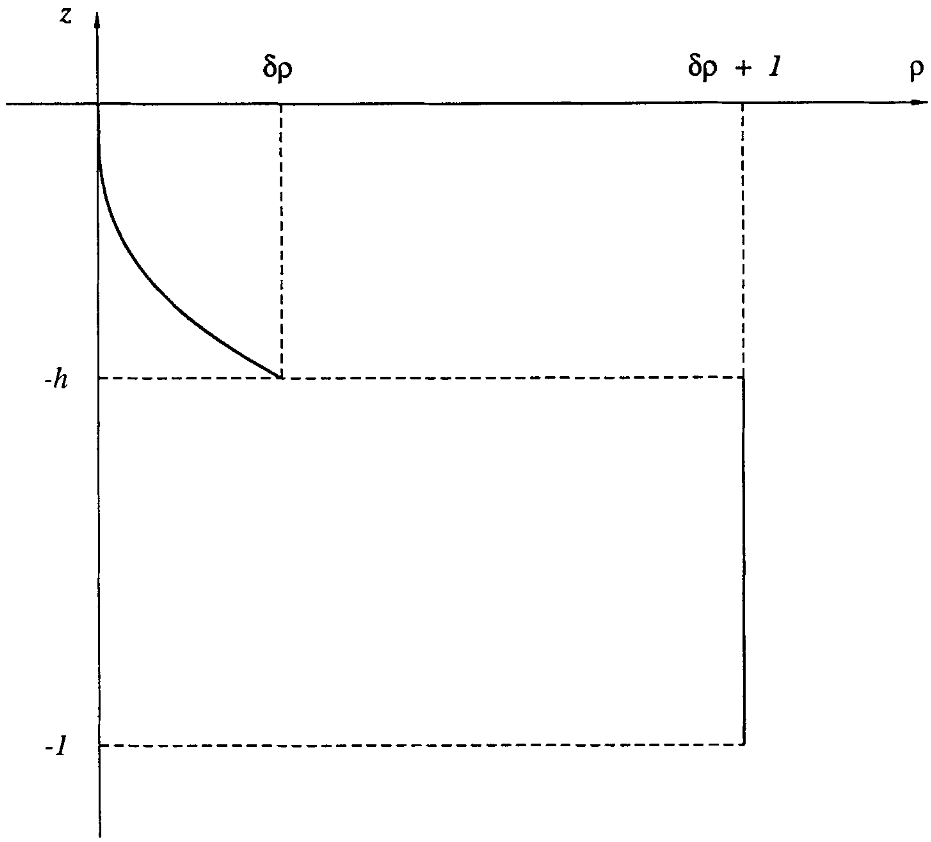

Figure 1 Non-dimensional model of stratification.

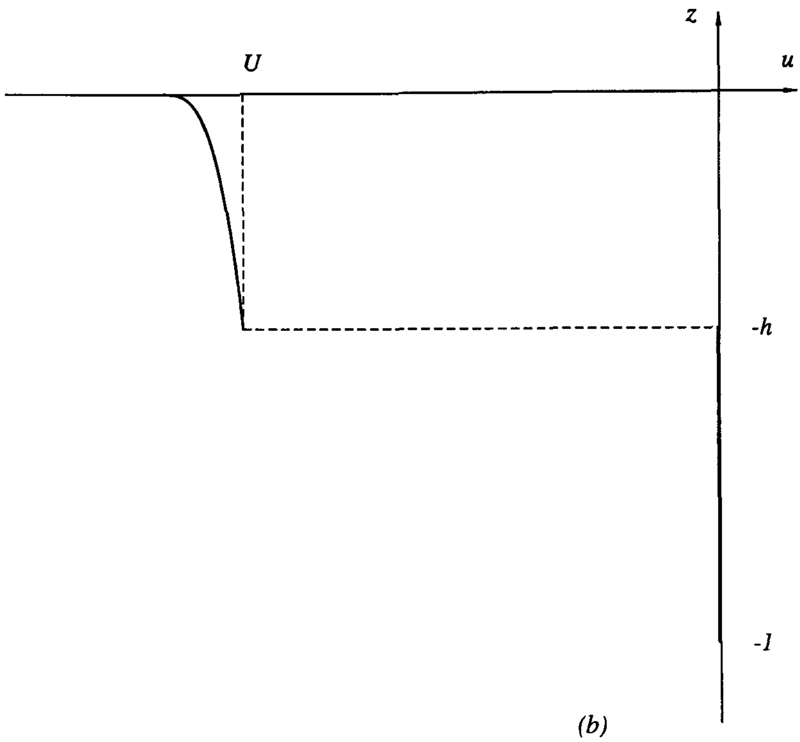

Figure 1 (*Continued*)

[this expansion automatically satisfies the bottom boundary condition (3)]. Substituting (7) into (1), (5) and taking the limit $\rho_z \to 0$, we obtain

$$\frac{\partial}{\partial t}(\Delta\Psi_2 - \Phi_2) + J(\Psi_2, \Delta\Psi_2 - \Phi_2) + \beta\frac{\partial \Psi_2}{\partial x} = 0; \tag{8}$$

$$\left.\begin{aligned} \left(\frac{1}{\rho_z}\frac{\partial \Psi_1}{\partial z}\right)_{z=-h} &= (1-h)\Phi_2, \\ (\Psi_1)_{z=-h} &= \Psi_2 - (1-h)\Phi_2. \end{aligned}\right\} \tag{9}$$

Boundary conditions (2) and (8)–(9) supplement (1), which is now to be solved in the interval $z \in (-h, 0)$.

In principle, (8) and (9) could have been derived from the primitive equations using stratification model (6) and the quasi-geostrophic approximation, which is how the layer models are usually derived (e.g. Pedlosky, 1987). This procedure would be much longer than our approach; on the other hand, it would clarify the physical meaning of Φ_2 (this quantity is proportional to the displacement of the interface).

Assuming (without loss of generality) that the lower layer is at rest, we describe zonal flow by

$$\Psi_1 = -y\,u_1(z), \qquad \Psi_2 = 0, \qquad \Phi_2 = -yS; \tag{10a}$$

where $u_1(z)$ is the velocity profile in the upper layer and S is the slope of the interface. Substitution of (10a) into (9) yields the following constraints:

$$S = \frac{1}{1-h}\left(\frac{1}{\rho_z}\frac{du_1}{dz}\right)_{z=-h}, \qquad H\left(u_1 + \frac{1}{\rho_z}\frac{du_1}{dz}\right)_{z=-h} = 0. \tag{10b}$$

Now we linearize the governing equations against the background of the steady solution (10), i.e. substitute

$$\Psi_1 = -y\,u_1(z) + \psi_1, \qquad \Psi_2 = +\psi_2, \qquad \Phi_2 = -yS + \Phi_2;$$

into (1–2), (8–9) and omit the nonlinear terms. Then we substitute the harmonic-wave solution:

$$[\Psi_1, \Psi_2, \Phi_2] = [\Psi_1(z), \Psi_2, \Phi_2]\exp[il(ct-x) - imy];$$

where c and (l, m) are the phase speed and the wavevector of the disturbance. Now ψ_2 and Φ_2 can be eliminated from the resulting equations, and we obtain

$$\left.\begin{aligned}
&(c-u_1)\left[k^2\psi_1 + \frac{d}{dz}\left(\frac{1}{\rho_z}\frac{d\psi_1}{dz}\right)\right] + \left[\frac{d}{dz}\left(\frac{1}{\rho_z}\frac{du_1}{dz}\right) + \beta\right]\psi_1 = 0,\\
&\frac{1}{\rho_z}\left[(c-u_1)\frac{d\psi_1}{dz} + \frac{du_1}{dz}\psi_1\right] = 0, \qquad \text{at} \quad z = 0,\\
&\frac{1}{\rho_z}\frac{d\psi_1}{dz} = -\frac{(1-h)(ck^2+\beta+S)}{c+(1-h)(ck^2+\beta+S)}\psi_1, \qquad \text{at} \quad z = -h;
\end{aligned}\right\} \tag{11}$$

where $k^2 = l^2 + m^2$. Finally we replace ψ_1 by ψ such that

$$\psi_1 = (c-u_1)\psi. \tag{12}$$

Substituting (12) into (11) and making use of (10b), we obtain (primes and indices omitted):

$$\left[\frac{1}{\rho_z}(c-u)^2\psi_z\right]_z + [k^2(c-u)^2 + \beta(c-u)]\psi = 0, \tag{13a}$$

$$\frac{1}{\rho_z}(c-u)^2\psi_z = 0, \quad \text{at} \quad z = 0, \tag{13b}$$

$$\frac{1}{\rho_z}(c-u)^2\psi_z = -\frac{c(c-u)(1-h)(ck^2+\beta)}{c-U+(1-h)(ck^2+\beta)}\psi, \qquad \text{at} \quad z = -h. \tag{13c}$$

Equations (13) form an eigenvalue problem. If the eigenvalue c is complex, the current with parameters $[\rho(z), u(z), h]$ is unstable.

3. ASYMPTOTIC SOLUTION TO THE EIGENVALUE PROBLEM (13)

Generally speaking, eigenvalue problem (13) cannot be solved analytically. If, however,

$$[(\delta\rho)h]^{-1} \gg k^2, \tag{14a}$$

$$[(\delta\rho)h]^{-1}|c-u| \gg \beta \tag{14b}$$

(where $\delta\rho = \rho(-h)$ is the relative density variation across the upper layer- see Figure 1), the last two terms in (13a) can be treated as a perturbation and the solution can be found asymptotically. Indeed, in this case $\psi(z)$ can be, to the leading order, approximated by

$$\psi_{(0)} \approx \text{const}_1 + \text{const}_2 \int (c-u)^{-2} dz. \tag{15}$$

Boundary condition (13b) "kills" the second term in (15), after which const_1 can be equated to 1:

$$\psi_{(0)} \approx 1. \tag{16}$$

Next we observe that (16) satisfies the lower boundary condition (13c) only if we can omit the right-hand side of (13c), i.e. when

$$(\delta\rho)^{-1}|c-u| \gg u \tag{17}$$

(it has been taken into account that $c \sim u$). Conditions (14) and (17) stipulate the validity of the zeroth-order solution (16).

The next term $\psi_{(1)}$ can be found using the straightforward perturbation technique, which would also yield the leading-order term for the dispersion relation $c(k)$. However, this procedure is associated with cumbersome calculations and will be bypassed. In order to derive the dispersion relation $c(k)$ without calculating $\psi_{(1)}$, we integrate (13a) with respect to z over $(-h, 0)$ and take into account (13b, c):

$$\frac{c(c-U)(1-h)(ck^2+\beta)}{\mathrm{c}-\mathrm{U}+(1-h)(ck^2+\beta)}\psi(-h) + \int_{-h}^{0} [k^2(c-u)^2 + \beta(c-u)]\psi\, dz = 0, \tag{18}$$

where

$$U = u(-h).$$

Since all terms in this equation are of the same order (the big term $\sim(\rho_z)^{-1}$ has been eliminated), we can substitute the zeroth-order solution (16)[2]:

$$\frac{c(c-U)(1-h)(ck^2+\beta)}{\mathrm{c}-\mathrm{U}+(1-h)(ck^2+\beta)} + \int_{-h}^{0} [k^2(c-u)^2 + \beta(c-u)]\, dz = 0. \tag{19}$$

[2] It should be emphasized that, in the vicinity of a critical level (if any), the solution, of course, cannot be approximated by (17). However, this region is narrow and its contribution to the integral term in (18) is small. Moreover, the integrand in this term vanishes at $u(z) = c$, which further downgrades the contribution of the critical level. Finally, we are interested in unstable perturbations, for which $\operatorname{Im} c \neq 0$ and $c - u \neq 0$.

This equation can be reduced to a cubic equation with respect to c and then solved analytically. However, due to the large number of parameters involved, the exact solution is very bulky and meaningless from a physical viewpoint. We shall analyse (18) using the assumption that the velocity variations in the upper layer are small:

$$u(z) = U + v(z), \quad |U| \gg |v|.$$

Substituting this into (19) and neglecting terms $O(v^2)$, we obtain

$$(c-U)\{h[k^2(c-U)+\beta][c-U+(1-h)(ck^2+\beta)] + c(1-h)(ck^2+\beta)\}$$
$$= h[2k^2(c-U)+\beta][c-U+(1-h)(ck^2+\beta)]\,\bar{v}, \tag{20}$$

where

$$\bar{v} = h^{-1}\int_{-h}^{0} v(z)\,dz = h^{-1}\int_{-h}^{0}[u(z)-U]dz$$

characterizes the mean variance of the velocity in the upper layer. First we consider (20) with

$$\bar{v} \to 0. \tag{21}$$

It should be emphasized that (21) is consistent with assumption (14b) only if $\delta\rho$ also tends to zero. In other words, we consider the following double limit

$$|v| \to 0, \quad \delta\rho \to 0 \quad \text{such that} \quad (\delta\rho)/v \to 0.$$

Substitution (21) breaks the cubic equation (20) into

$$h[k^2(c-U)+\beta][c-U+(1-h)(ck^2+\beta)] + c(1-h)(ck^2+\beta) = 0, \tag{22a}$$

$$c - U = 0. \tag{22b}$$

These equations can be easily solved:

$$C = -\tfrac{1}{2}(B-\sqrt{D})/A, \qquad c_2 = -\tfrac{1}{2}(B+\sqrt{D})/A, \qquad c_3 = U; \tag{23a, b, c}$$

where

$$A = h(1-h)k^4 + k^2, \qquad B = -h(1-h)Uk^4 - 2h[U-(1-h)\beta]k^2 + \beta,$$

$$D = U^2h^2(1-h^2)k^8 + 2h(1-h)U[(1-2h)\beta - 2U]k^4 + \beta^2.$$

Here $c_{1,2}$ are the two Rossby-wave modes; correspondingly, (22a) yields the standard dispersion relation of Phillips' instability (e.g. Pedlosky, 1987). c_3, in turn, describes the mode supported by the vertical shear of the flow.

Next we consider a perturbed ($\bar{v} \neq 0$) flow. From a physical viewpoint, the instability is likely to occur when two (or more) modes have a resonance:

$$c_1(h) = c_2(k), \quad \text{or} \quad c_2(k) = c_3(k), \quad \text{or} \quad c_1(k) = c_3(k). \qquad (24\text{a, b, c})$$

Resonance (24a) corresponds to Phillips' instability and is of no interest for us. The physical meaning of the other two resonances follows from the fact that the phase speeds of the resonant waves match the velocity in the upper layer [see (22b)]. In other words, these waves have critical levels. The mathematical meaning of the intermode resonance is discussed in Appendix B.

In order to find the values of parameters, for which resonances (24b, c) occur, (22b) should be substituted into (22a). We obtain

$$k^2 \text{ is arbitrary}, \qquad U = -\beta h, \qquad c = U; \qquad (25\text{a, b, c})$$

or

$$k^2 = -\beta/U, \qquad U < 0, \qquad c = U. \qquad (26\text{a, b, c})$$

In case (25), the resonance between the shear mode and one of the Rossby-wave modes occurs for all wavenumbers [substitute (25b) into (23b) and compare the latter to (23c)]. This case will be considered in Section 4.2. First we shall consider case (26), which makes sense only for flows with $U < 0$ (westward shear).

Now we return to the dispersion Equation (20) and formally expand it in the vicinity of the wave that satisfies conditions (26):

$$c = U + C, \qquad k^2 = -(\beta/U) + K; \qquad (27\text{a, b})$$

where

$$|C| \ll -U, \qquad |K| \ll -\beta/U. \qquad (28\text{a, b})$$

After substitution of (27) into (20), all linear in C, K or $\bar{v}$ terms cancel out, while the quadratic terms yield

$$[1 - 2h + h(1-h)(\beta/U)]C^2$$
$$- \{[h + (u/\beta)](1-h)UK - h[1 - (1-h)(\beta/U)]\bar{v}\}C + h(1-h)UK\bar{v} = 0. \qquad (29)$$

It can be readily demonstrated that C is real for all K (which means stability) if and only if

$$\left.\begin{array}{ll} -U \geqslant \beta h(1-h)/(1-2h), & \text{if } h < \frac{1}{2}, \\ -U \geqslant 0, & \text{if } h \geqslant \frac{1}{2}. \end{array}\right\} \qquad (30)$$

It should be emphasized that criterion (30) is applicable only to the critical-level instability, apart from which the modified model also describes the "classical" two-layer instability. In the case of westward flows, the latter does not take place if

$$0 \leqslant -U \leqslant \beta h \qquad (31)$$

(Pedlosky, 1987). Evidently, conditions (30) and (31) are inconsistent, which means that all flows with westward shear are unstable.

The critical-level instability does not affect flows with eastward shear, and the corresponding stability criterion of the modified model coincides with that of the unperturbed two-layer model: $0 \leqslant U \leqslant \beta(1-h)$.

3.1 Discussion

It is worth noting that mere existence of the continuous component in the velocity profile does not guarantee the instability: if the average velocity in the upper layer is equal to the velocity at the interface, $\bar{v}$ vanishes and the instability disappears. Indeed, using (29), we can find the maximum growth rate:

$$R = \max\{k \operatorname{Im} c\} \approx \sqrt{-\beta/U} \max\{\operatorname{Im} C\}$$

$$= \frac{h}{|h + (U/\beta)|\sqrt{h(1-h) + (1-2h)(U/\beta)}}\,\bar{v} \tag{32a}$$

and the corresponding wavenumber:

$$K_{\max} = \frac{h[1-(1-h)(\beta/U)] + (U/\beta) - (1-h)}{(1-h)[h+(U/\beta)]}\,\frac{\bar{v}}{U} \tag{32b}$$

(it should be understood that the absolute value of the wave number is given by $k_{\max} = \sqrt{-(\beta/U) + K_{\max}}$). Then, introducing the spectral boundaries of the instability:

$$\operatorname{Im} C \neq 0, \qquad \text{for} \quad K_1 < K < K_2,$$

we can find its spectral width:

$$\Delta = K_2 - K_1 = \frac{4h\sqrt{h(1-h) + (1-2h)(U/\beta)}}{(1-h)[h+(U/\beta)]^2}\left|\frac{\bar{v}}{U}\right|. \tag{32c}$$

Equalities (32) clearly demonstrate that, if $\bar{v} = 0$, both R and Δ vanish. The condition $\bar{v} \neq 0$, however, is not very restrictive, as velocity profiles of real oceanic currents are, in most cases, monotonic (see Figure 1b), which entails $\bar{v} \neq 0$. In the remainder of this paper we assume that $\bar{v}$ has the same sign as U, i.e. $\bar{v} < 0$.

A more severe constraint on $\bar{v}$ comes from conditions (14b) and (16) of applicability of our asymptotic solution:

$$|\bar{v}| \gg \delta\rho \max\{\beta h, U\}.$$

As it turns out, condition (14a) does not impose any additional restriction. Indeed, the critical level instability takes place in the vicinity of the wavenumber given by (26);

substituting (26) into (14a), we obtain

$$|U| \gg \beta h \delta\rho,$$

which is obviously a weaker constraint than $|U| \gg |\bar{v}|$.

The curve of marginal stability, corresponding to (29), is plotted on the $(k^2, U/\beta)$-plane in Figure 2.

4. SINGULAR POINTS OF THE INSTABILITY

It is worth noting that the critical-level instability and Phillips' instability coexist in the region

$$\left.\begin{array}{ll} \beta h < -U < \beta h(1-h)/(1-2h), & \text{if} \quad h < \frac{1}{2}, \\ \beta h < -U, & \text{if} \quad h \geqslant \frac{1}{2}; \end{array}\right\} \tag{33}$$

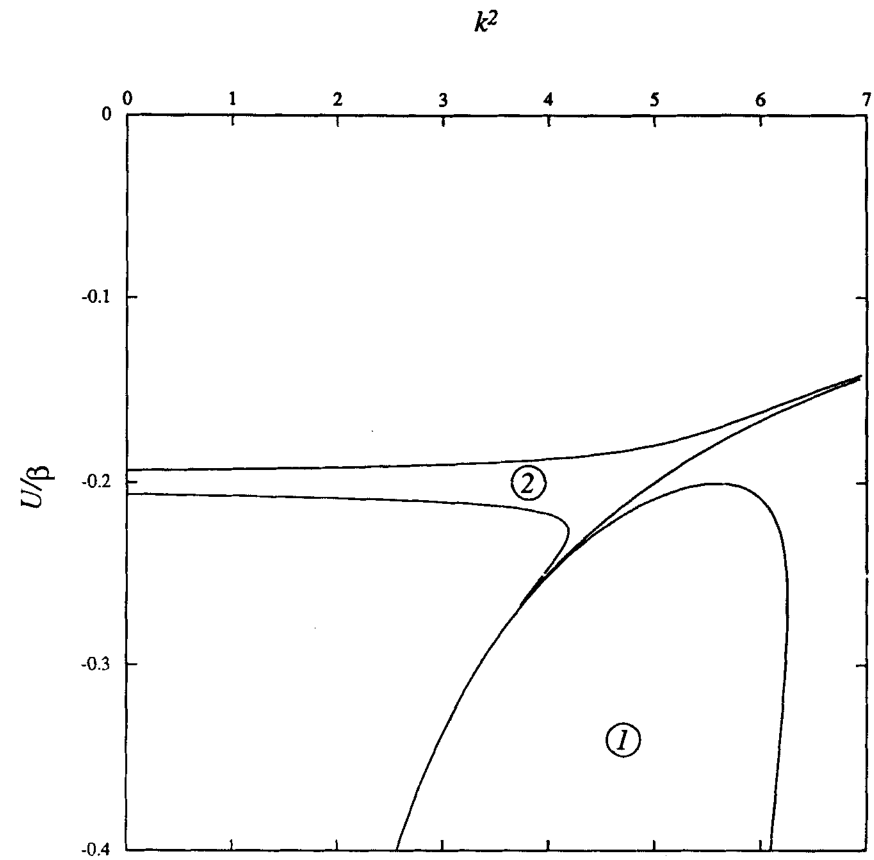

Figure 2 The curve of marginal stability on the $(k^2, U/\beta)$-plane. $h = 0.2$, $\bar{v} = 0.001\ U$. (1) Phillips' instability; (2) the critical-level instability.

however, the spectral regions of the two instabilities do not overlap (see Figure 2). Formulae (32) also demonstrate that the marginal stability curve is singular at the boundaries of the "coexistence interval". Assuming for simplicity that $h < \frac{1}{2}$, we get

$$R \to \infty, \quad \Delta \to \infty \quad \text{as} \quad -U \to \beta h;$$

$$R \to \infty, \quad \Delta \to 0 \quad \text{as} \quad -U \to \beta h(1-h)/(1-2h).$$

Since conditions (28) in these cases are violated, our expansion of (20) is not applicable and has to be modified.

In the above derivation of (29) from general dispersion equation (20), it was implicitly assumed that

$$C \sim K \sim \bar{v},$$

which is obviously incorrect in the vicinity of the singular points, where C grows. However, the new order of C's magnitude can be easily "guessed" for both of the points.

In what follows we shall use the following notation:

$$U' = -\beta h(1-h)/(1-2h), \quad U'' = -\beta h.$$

We shall demonstrate that, regardless of the singularities at $U = U', U''$, the qualitative side of the instability has been described correctly: the growth rate of the critical-level instability has maxima at $U = U', U''$.

4.1 $U \to U'$

From a physical viewpoint, this singularity corresponds to a triple resonance of the shear mode and both two-layer Rossby waves [(20) with $\bar{v} = 0$ has a triple root]. Accordingly, the coefficient of C^2 in (29) vanishes, and we should take into account terms $O(C^3)$. It can also be derived from (29) that the spectral boundaries of instability converge to

$$K_1 \to K_2 \to K_0 = \frac{(1-2h)^2\,\bar{v}}{h^3(1-h)\beta}, \quad \text{as} \quad U \to U'.$$

In order to describe the most general regime, we should keep in the dispersion equation as many terms as possible, which yields the following orders of the variables:

$$C \sim \bar{v}^{2/3}, \tag{34a}$$

$$|K - K_0| \sim \bar{v}^{4/3}, \quad |U + \beta h(1-h)/(1-2h)| \lesssim \bar{v}^{2/3}. \tag{34b}$$

Expanding (20) in accordance with (34), we obtain

$$-2h^{-1}(1-2h)C^3 + [1 - 2h + h(1-h)(\beta/U)]C^2$$
$$-\beta\frac{h^3(1-h)^2}{(1-2h)^2}(K - K_0)C - \frac{(1-h)(1-2h)}{h}\bar{v}^2 = 0.$$

If $h \neq \frac{1}{2}$ [3], the coefficient of the highest-order term in this equation does not vanish, and C is finite for all K. At the same time, equality (34a) demonstrates that in the vicinity of the singular point the instability is stronger than it is at regular points.

4.2 $U \rightarrow U''$

In order to understand why the maximum growth rate is unbounded in this case, we note that the margins of the instability region go to infinity:

$$K_{\max} \rightarrow -\infty, \quad \text{as} \quad U \rightarrow U''$$

[see (32b) and Figure 2]. Accordingly, condition (28b) is violated, and the dispersion Equation (29) fails (for one thing, it ignores the fact that the smallest allowable value of $K_{\max}$ is

$$K_{\max} = -h$$

– see (27b) with $U = -\beta h$). The correct dispersion equation can be derived using the assumption that

$$C \sim \bar{v}^{1/2}, \quad k^2 \sim 1, \quad |U + \beta h| \sim \bar{v}^{1/2}. \tag{35}$$

Substituting (35) into (20), we get

$$[1 - h^2(1-h)k^4]C^2 + (1-h)(1-hk^2)(U + \beta h)C - \beta h(1-h)(1-hk^2)\bar{v} = 0. \tag{36}$$

The region of instability, corresponding to (36), is a little wider than that described by (less accurate) equation (29) (see Figure 3), but the modified maximum growth rate is finite. Equation (36) also describes another region of instability (see region (3) in Figure 3) which has no analogue for (29) (see Figure 2), as region (3) is located far away from the curve $k^2 = -\beta/U$. This new region of critical-level instability overlaps with the region of Phillips' instability, and the two instabilities are, in this case, undistinguishable.

It is also worth noting, that the second region of the critical-level instability has a singular point at

$$k^2 = h^{-1}(1-h)^{-1/2},$$

where the coefficient of C^2 in (36) vanishes. We shall not discuss this singularity in detail, but note only that it corresponds to the triple resonance (similar to the case considered in Section 4.1) and can be regularized by the term $O(C^3)$.

5. CONCLUSIONS

Thus, the answer to the question formulated in the Introduction has been obtained: the continuous component of the velocity profile does destabilize all flows with westward shear. The instability occurs in the spectral region of disturbances with critical levels, but the unstable disturbances [given by (16)] do not have the logarithmic instability,

[3] For $h = \frac{1}{2}$ the singular point $U = U'$ does not exist at all.

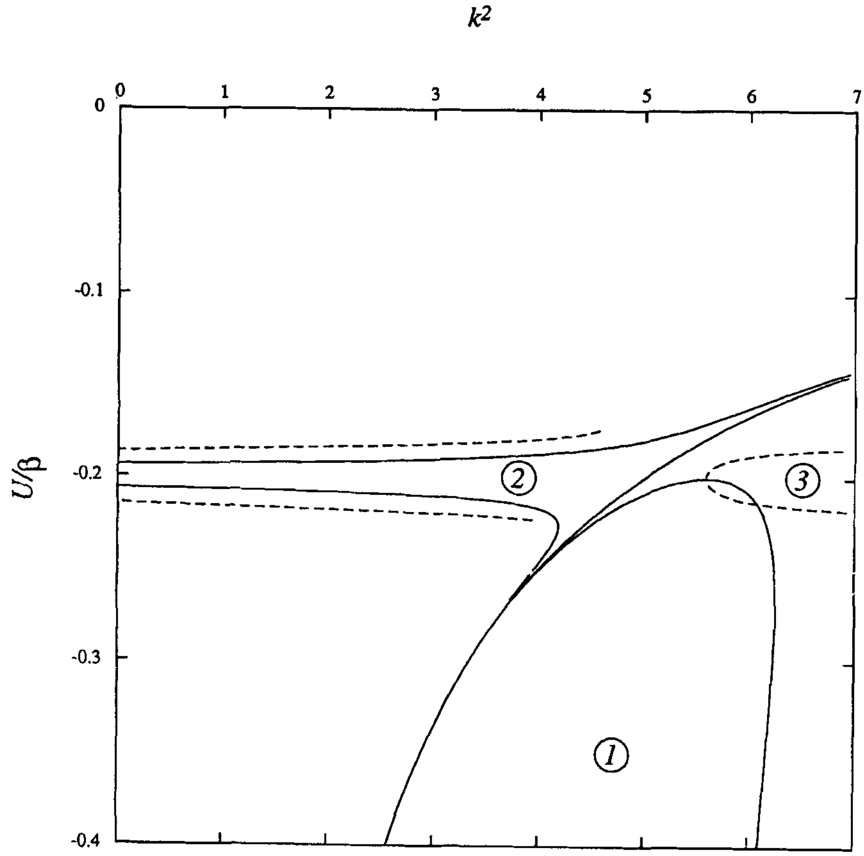

Figure 3 The curve of marginal stability on the $(k^2, U/\beta)$-plane corrected for $U \approx -\beta h$ by equation (36) (dashed line). $h = 0.2$, $\bar{v} = 0.001\,U$. (1) Phillip's instability; (2, 3) the critical-level instability.

which is commonly believed to destabilize the flow through rapid variation of heat flux (e.g. Pedlosky, 1987, p. 536). Instead of the interpretation based on the singularity, we interpret the instability discovered as a result of a resonance of Rossby waves and the "shear mode" supported by the vertical shear of the mean flow.

The critical-level instability can "coexist" with the standard Phillips' instability [the coexistence interval given by (33)], and at the endpoints of this interval has peaks. It is also worth noting that the instability of weak ($U \to 0$) flows takes place at short wavelengths (see formulae (32b) and Figure 2).

The fact that the instability occurs even for infinitesimal perturbation of the two-layer velocity profile, indicates that the two-layer model is structurally unstable and casts certain doubts on its suitability for description of realistic (continuously-stratified) flows. The results obtained suggest that those are more unstable than how they must be according to the two-layer. It should be emphasized, however, that this conclusion is not applicable to the two-layer model for Rossby waves in still water, as we have no evidence that it is structurally unstable or underrates any physically important quantity.

We note that continuous variations of the velocity and density in the lower layer should be expected to destabilize flows with positive U (eastward shear). This follows from the invariance of the unperturbed two-layer model with respect to simultaneous replacement of

$$h \text{ by } (1-h), \quad (1-h) \text{ by } h, \quad U \text{ by } -U;$$

and can be verified by straightforward calculations similar to those in this paper.

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APPENDIX A: DERIVATION OF MATCHING CONDITIONS (5)

Introducing the following variables:

$$A(t,x,y) = \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-1}, \tag{A.1a}$$

$$B(t,x,y,t) = \left(\frac{1}{\rho_z}\Psi_z\right)_{z=0}, \tag{A.1b}$$

$$\Phi(t,x,y,z) = \left(\frac{1}{\rho_z}\Psi_z\right)_z; \tag{A.1c}$$

we can rewrite system (1)–(3) in the form

$$(\Delta\Psi - \Phi)_t + J(\Psi, \Delta\Psi - \Phi) + \beta\Psi_x = 0, \tag{A.2a}$$

$$A_t + J(\Psi, A) = 0, \quad \text{at} \quad z = -1, \tag{A.2b}$$

$$B_t + J(\Psi, B) = 0, \quad \text{at} \quad z = 0. \tag{A.2c}$$

Integrating (A.1c) and taking into account (A.1b), we get

$$\frac{1}{\rho_z(z)}\Psi_z(t,x,y,z) = -\int_z^0 \Phi(t,x,y,z')dz' + B(t,x,y). \tag{A.3}$$

Multiplying (A.3) by $\rho_z(z)$ and integrating again, we obtain

$$\Psi(t,x,y,z) = \int_z^0 [\rho(z') - \rho(z)]\Phi(t,x,y,z')dz' + B(t,x,y)\rho(z) + \eta(t,x,y), \tag{A.4}$$

where $\eta(t, x, y)$ is an undetermined "constant" of integration, which should be treated as a new unknown function. Finally, putting in (A.3) $z = -1$ and taking into account (A.1a), we get

$$\int_{-1}^{0} \Phi(t, x, y, z)\, dz = B(t, x, y) - A(t, x, y). \tag{A.5}$$

Equations (A.2), (A.4) and (A.5) form a closed system, which is in some cases more convenient than the original Equations (1)–(3). In particular, if $\rho(z)$ has a discontinuity:

$$\rho(-h-0) = \rho(-h+0) + 1, \tag{A.6}$$

condition (A.3) clearly demonstrates that $[1/\rho_z(z)]\Psi_z(t, x, y, z)$ is a continuous function, i.e.

$$\left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h-0} = \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h+0}, \tag{A.7}$$

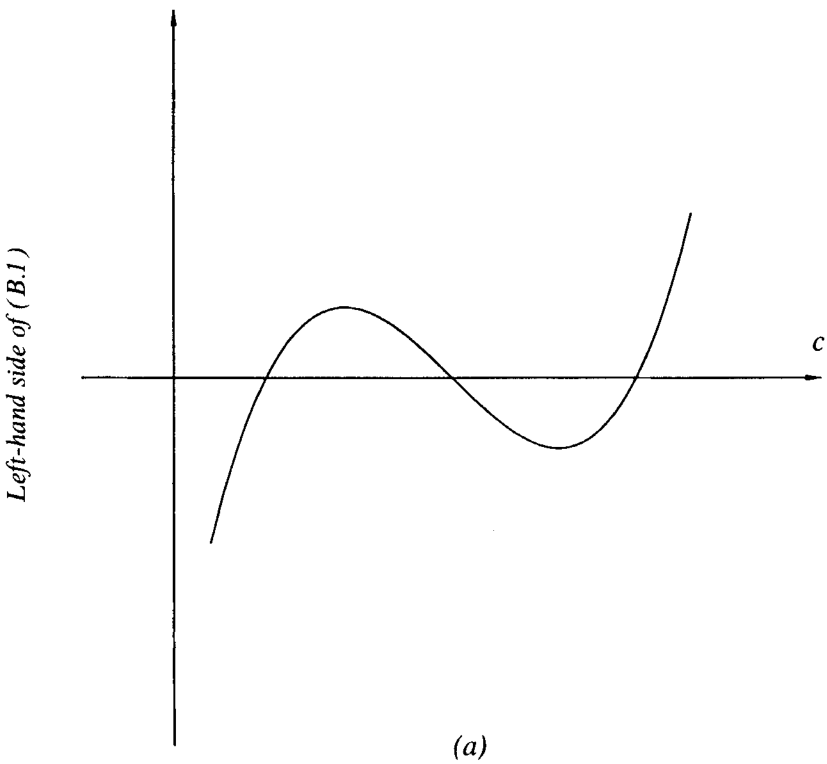

Figure 4 Graph of the left-hand side of equation (B.1) for $\bar{v} = 0$: (a) distinct roots; (b) one distinct and one double roots.

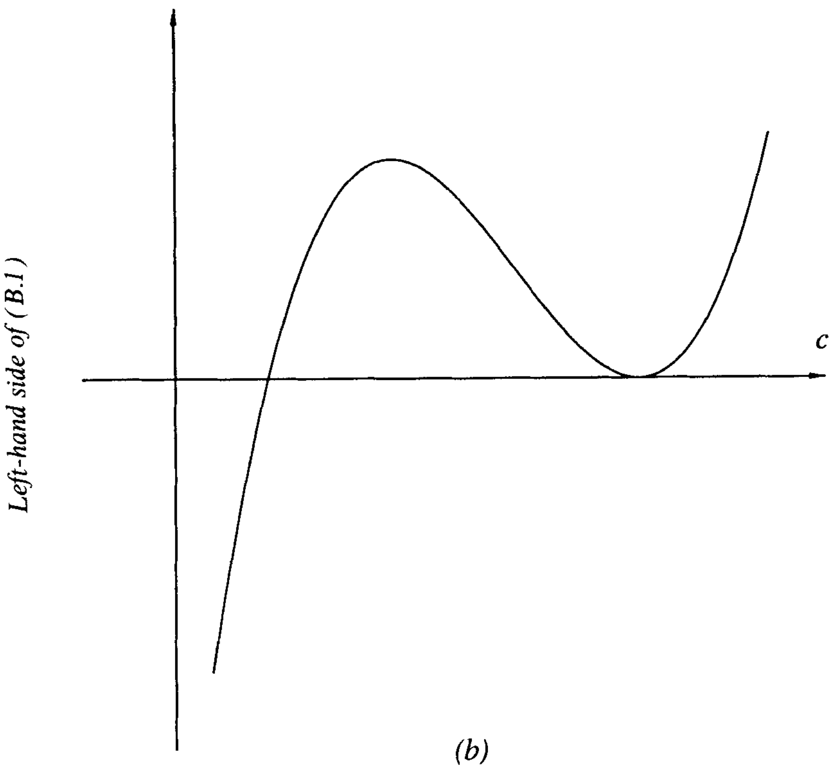

Figure 4 (*Continued.*)

which is equivalent to (5a). In order to derive (5b), we rewrite (A.4) as follows:

$$\Psi(t,x,y,z)=\int_z^0 \rho(z')\,\Phi(t,x,y,z')+\rho(z)\frac{1}{\rho_z(z)}\Psi_z(t,x,y,z)+\eta(t,x,y). \tag{A.8}$$

The first and the third terms on the right-hand side of (A.8) are continuous. Evaluating the discontinuity of the second term via (A.6) and taking into account (A.7), we obtain

$$(\Psi)_{z=-h-0}=(\Psi)_{z=-h+0}+\left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h},$$

which coincides with (5b):

It should be noted that this simple derivation implies that Φ is finite and

$$\int_{-h-0}^{-h+0}\Phi(t,x,y,z)\,dz=0.$$

This can be justified physically: Φ is the z-derivative of the vertical (Lagrangian) displacement of fluid particles and therefore can be assumed finite.

APPENDIX B: MATHEMATICAL INTERPRETATION OF THE INTER-MODE RESONANCE

We shall write Equation (20) in the general form:

$$a_0(k,\bar{v}) + a_1(k,\bar{v})c + a_2(k,\bar{v})c^2 + a_3(k,\bar{v})c^3 = 0. \tag{B.1}$$

If $\bar{v} = 0$, all three roots of Equation (B.1) are real:

$$c = c_1(k,0); \quad c = c_2(k,0); \quad c = c_3(k,0).$$

If $c_{1,2,3}$ are distinct roots (the non-resonant case), the graph of the left-hand side of (B.1) is shown in Figure 4a. The (resonant) case, where two modes have equal speeds (double root), is shown in Figure 4b.

If we now perturb Equation (B.1) by $\bar{v} \neq 0$ ($\bar{v} \ll 1$), distinct roots remain real (they can only slightly change their values). As for a double root (if any), it has a 50% chance of becoming complex:

a) If the local minimum of the right-hand side of (B.1) moves downwards, the double root splits into two distinct real roots (see Figure 5a);

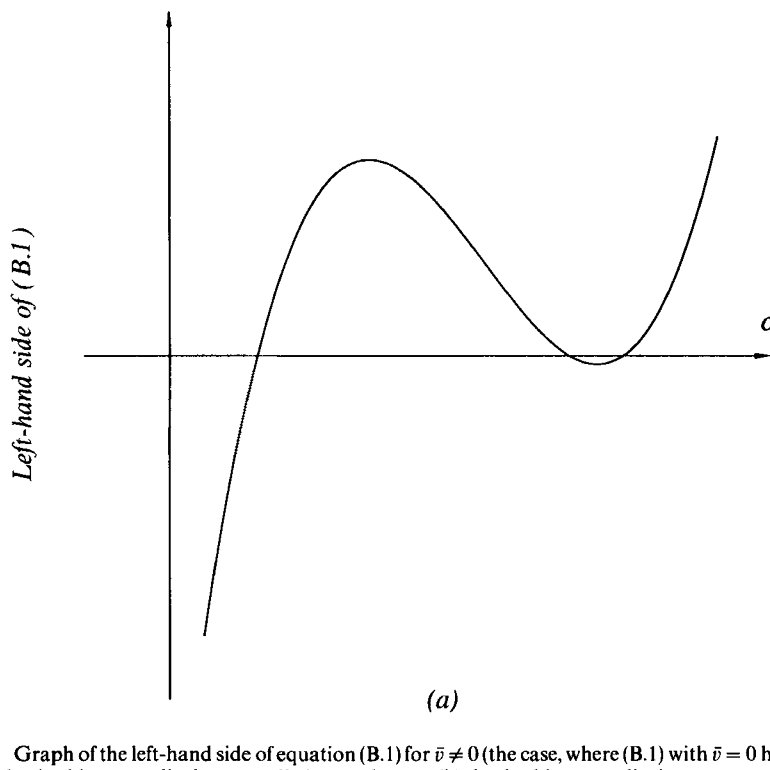

Figure 5 Graph of the left-hand side of equation (B.1) for $\bar{v} \neq 0$ (the case, where (B.1) with $\bar{v} = 0$ has a double root): (a) the double root splits into two distinct real roots; (b) the double root splits into two complex roots.

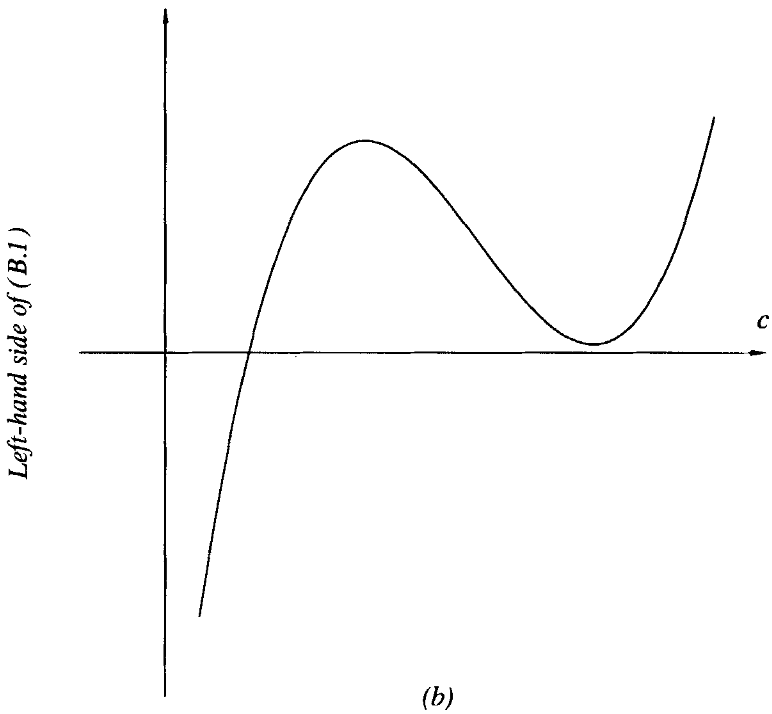

Figure 5 (*Continued.*)

b) if the local minimum of the right-hand side of (B.1) moves upwards, the double root splits into two complex roots (see Figure 5b).

This indicates that an instability can occur only if the unperturbed ($\bar{v} = 0$) equation (B.1) has two equal roots (resonance).