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On resonant over-reflection of waves by jets

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It is well known that internal or Rossby waves propagating across a jet can be amplified, a phenomenon usually referred to as over-reflection. In some cases, over-reflection can be infinitely strong – physically, this means that the reflected and transmitted waves can exist without an incident one, i.e. they are spontaneously emitted by the mean flow. In this article, it is shown that infinitely strong over-reflection (resonant over-reflection) occurs for gravity-wave scattering by ageostrophic jets in a rotating barotropic ocean and Rossby-wave scattering by a two-jet configuration on the quasigeostrophic beta-plane. It is further demonstrated that, generally, a resonantly over-reflected wave is always marginal to instability, i.e. either an increase or a decrease of its wavenumber transforms it into an unstable eigenmode localised near the jet.

Keywords: Wave-flow interaction; Resonant over-reflection; Rossby waves; Internal waves

1. Introduction

It has been known for more than 30 years that, if an internal or Rossby wave propagates across a jet, the joint energy flux of the reflected and transmitted waves may exceed that of the incident one (Jones 1968, Dickinson 1970, McKenzie 1972, Lindzen 1974, Eltayeb and McKenzie 1975, Acheson 1976, Van Duin and Kelder 1982, Basovich and Tsimring 1984, Take-hiro and Hayashi 1992, Ollers *et al*. 2003). This effect, usually referred to as over-reflection, is caused by the interaction of the wave with the jet's critical levels, i.e. the lines where the velocity of the mean flow matches the corresponding component of the wave's phase speed.

It is intuitively clear that, since over-reflection transfers energy from jets to waves, it is conducive to the jet's instability (Lalas and Einaudi 1976, Acheson 1976, Lindzen and Tung 1974, Rosenthal and Lindzen 1983, Lindzen 1988). In particular, over-reflection indeed causes instability if the wave is reflected back towards the critical level by a rigid wall or a turning point beyond which the medium is not transparent (Lindzen and Rosenthal 1976, Davis and Peltier 1979, Balmforth 1999). A similar effect has been examined for acoustic waves in compressible fluids by Gill (1965), Blumen *et al*. (1975) and Broadbent and Moore (1979). Furthermore, radiational instability of vortices can

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also be interpreted in terms of over-reflection (Ford 1994, Ford *et al.* 2000, LeDizés and Billant 2009).

This article is concerned with the limiting case of over-reflection where the reflected and transmitted waves are *infinitely* strong – which can be interpreted as spontaneous radiation of waves by the flow. This effect is traditionally referred to as "resonant over-reflection". We shall, however, use a more succinct term, "hyper-reflection", which also emphasises that this effect is stronger than over-reflection – which is, in turn, stronger than the usual reflection.

Two reasons make hyper-reflection worth studying. First, a situation where a jet responds to a *small-amplitude* incident wave by an *infinitely strong* reflected one is fascinating for a theoretician. Second, hyper-reflection may be responsible, at least partially, for oceanic/atmospheric wave generation, as "bursts" of internal gravity waves were observed in numerical simulations of Viúdez and Dritschel (2006) at large distances from the jet. Such burst cannot be caused by either over-reflection (which is an effect of a "moderate" nature) or the usual unstable modes (localized near the jet).

Three cases of hyper-reflection have been reported in the literature:

(1) Lindzen (1974) observed hyper-reflection of internal waves propagating vertically in a stratified flow with a piece-wise constant velocity and a constant Väsälä frequency. McIntyre and Weissman (1978) clarified the energy budget of wave-flow interaction in this problem, and its weakly nonlinear extension was examined by Grimshaw (1976, 1979).
(2) Maslowe (1991) showed that a similar effect occurs for a Rossby wave and a jet on the β-plane, provided the critical level is located at the jet's maximum.
(3) Lott *et al.* (1992) demonstrated the existence of hyper-reflection for internal waves propagating vertically in a flow with a smooth "step-like" velocity profile and a variable Väsälä frequency (both determined by hyperbolic functions).

Note that settings 2 and 3 are described by Sturm–Liouville-kind problems, with coefficients involving second-order poles located at the critical level. Setting 1, in turn, is a limiting case of setting 3 (with the width of the velocity "step" and the variation of the Väsälä frequency both tending to zero) – thus, it effectively involves a second-order pole multiplied by a discontinuous coefficient.

In this article, we shall demonstrate that hyper-reflection can also occur for gravity waves on the f-plane (section 2) and Rossby waves on the β-plane (section 3), of which both involve *first*-order poles. We shall also examine hyper-reflection in a general formulation, concentrating on its connections with instability (section 4).

2. Gravity waves and ageostrophic jets on the f-plane

2.1. *The governing equations*

Consider a thin layer of an ideal fluid with a free upper boundary, on a sphere rotating with an angular velocity Ω. If the spatial scale of the flow is much smaller than the sphere's radius and we are interested in motions near a certain reference point located at a latitude θ, we can take advantage of the so-called f-plane approximation, replacing the sphere with a tangent plane. Since the layer is thin, we shall also use the shallow-water approximation.

Then the motion of the fluid can be characterized by the horizontal velocity (u_*, v_*) and depth h_*, which depend on the horizontal Cartesian coordinates (x_*, y_*) and time t_* (the asterisks indicate that the corresponding variables are dimensional). We shall also introduce the Coriolis parameter $f = 2\Omega \sin\theta$, the mean depth of the layer H_0 and the deformation radius

$$R_d = \sqrt{gH_0}/f, \tag{1}$$

where g is the acceleration due to gravity.

We shall use the following dimensionless variables:

$$x = \frac{x_*}{R_d}, \qquad y = \frac{y_*}{R_d}, \qquad t = ft_*, \tag{2a–c}$$

$$u = \frac{u_*}{fL_d}, \qquad v = \frac{v_*}{fL_d}, \qquad h = \frac{h_*}{H}, \tag{2d–f}$$

in terms of which the shallow-water equations governing the fluid are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial h}{\partial x} = v, \tag{3a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial h}{\partial y} = -u, \tag{3b}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0. \tag{3c}$$

Note that, in addition to homogeneous fluids, these equations can also describe a two-layer density-straified fluid under additional assumptions that the upper (lighter) layer is much thinner than the lower (heavier) one, and that the density difference $\Delta\rho$ is much smaller than the mean density ρ_0. In this case, g in (1) should be replaced with the reduced gravity $g' = g\Delta\rho/\rho_0$.

Equations (3a–c) admit a steady solution describing a parallel flow along the x-axis (a zonal flow),

$$u = U(y), \qquad v = 0, \qquad h = H(y),$$

where the depth and velocity are related by the geostrophy condition,

$$\frac{\mathrm{d}H}{\mathrm{d}y} = -U.$$

We shall assume that

$$H \to H_\pm, \qquad U \to 0 \qquad \text{as} \qquad y \to \pm\infty,$$

i.e. the depth $H(y)$ is a "step-like" function, whereas the flow $U(y)$ is a meridionally localized jet.

Consider a small-amplitude wave superposed on the jet (figure 1),

$$u = U(y) + \widetilde{u}, \qquad v = \widetilde{v}, \qquad h = H(y) + \widetilde{h},$$

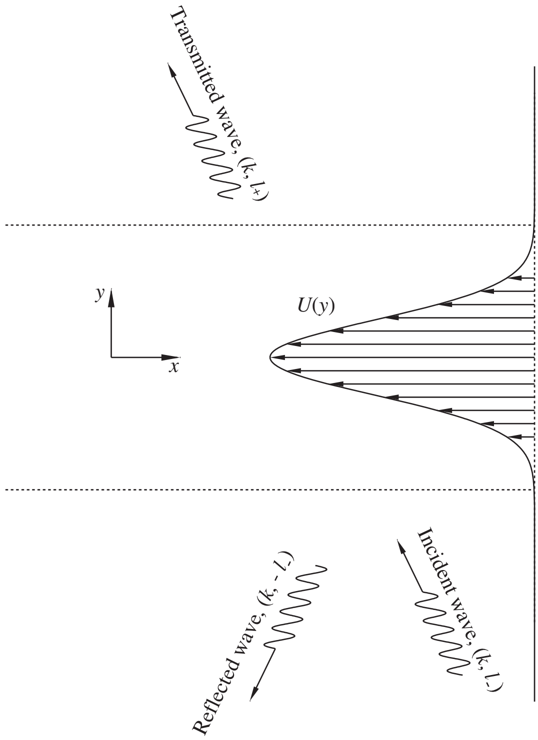

Figure 1. The setting: scattering of waves by a jet.

where the tilded variables describe the wave. Linearizing (3a–c), we obtain

$$\frac{\partial\widetilde{u}}{\partial t}+U\frac{\partial\widetilde{u}}{\partial x}+\widetilde{v}\frac{\mathrm{d}U}{\mathrm{d}y}+\frac{\partial\widetilde{h}}{\partial x}=\widetilde{v}, \tag{4a}$$

$$\frac{\partial\widetilde{v}}{\partial t}+U\frac{\partial\widetilde{v}}{\partial x}+\frac{\partial\widetilde{h}}{\partial y}=-\widetilde{u}, \tag{4b}$$

$$\frac{\partial\widetilde{h}}{\partial t}+\frac{\partial\left(U\widetilde{h}+\widetilde{u}H\right)}{\partial x}+\frac{\partial(\widetilde{v}H)}{\partial y}=0. \tag{4c}$$

We shall consider solutions with harmonic dependence on the zonal coordinate and time,

$$\widetilde{u}=\mathrm{Re}\left\{\hat{u}(y)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\right\},\qquad \widetilde{v}=\mathrm{Re}\left\{\hat{v}(y)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\right\},\qquad \widetilde{h}=\mathrm{Re}\left\{\hat{h}(y)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\right\}, \tag{5a–c}$$

where ω and k are the frequency and zonal wavenumber. Substitution of (5a–c) into (4a–c) yields (hats omitted)

$$\mathrm{i}(kU-\omega)u+(U'-1)v+\mathrm{i}kh=0, \tag{6a}$$

$$\mathrm{i}(kU-\omega)v+u+h'=0, \tag{6b}$$

$$\mathrm{i}(kU-\omega)h+\mathrm{i}kHu+(Hv)'=0, \tag{6c}$$

where the primes denote differentiation with respect to y. Equations (6a–c) can be reduced to a single equation for h,

$$(Fh')' + \left(\frac{kF'}{\omega - kU} - k^2 F + 1 \right) h = 0, \tag{7a}$$

where

$$F = \frac{H}{(\omega - kU)^2 - 1 + U'}. \tag{7b}$$

Observe that, generally, (7a,b) may involve singular points of two different types, which will be denoted by y_c and y_a (in both cases, the subscript represents the point's number, i.e. c, $a = 1, 2, \ldots$). y_c are the points where the denominator in (7a) vanishes,

$$\omega - kU(y_c) = 0, \tag{8a}$$

whereas, at $y = y_a$, the denominator of (7b) vanishes,

$$[\omega - kU(y_a)]^2 - 1 + U'(y_a) = 0. \tag{8b}$$

We assume that singular points of different types do not coincide,

$$y_c \neq y_a,$$

and that the singularities are simple poles, i.e.

$$U'(y_c) \neq 0, \qquad 2[\omega - kU(y_a)][-kU'(y_a)] + U''(y_a) \neq 0.$$

Note that, even though the coefficients of equation (7a,b) are singular at $y = y_a$, its general solution is, surprisingly, regular (Boyd 1976). Indeed, it can be readily deduced using the Frobenius method that

$$h = A_a \left[1 - \frac{k\xi}{\omega - kU(y_a)} + \mathrm{O}(\xi^2) \right] + B_a \left[\xi^2 + \mathrm{O}(\xi^3) \right] \qquad \text{as} \qquad y \to y_a, \tag{9}$$

where $\xi = y - y_a$ and A_a and B_a are constants of integration. Accordingly, the points y_a will be referred to as "apparent singularities".

The points y_c, in turn, are the critical levels, and the solution of (7a,b) near $y = y_c$ is

$$h = A_c \left\{ 1 - \frac{F'(y_c)}{F(y_c)\, U'(y_c)} \left[\eta \ln \eta + \tfrac{1}{2} - U'(y_c)\, \eta \right] + \mathrm{O}(\eta^2 \ln \eta) \right\} + B_c \left[\eta + \mathrm{O}(\eta^2) \right] \qquad \text{as} \qquad y \to y_c, \tag{10}$$

where $\eta = y - y_c$ and A_c and B_c are constants of integration. Since it is *a priori* unclear which branch of the logarithm in (10) should be chosen for $\eta < 0$, this singularity needs to be regularized.

Since Rayleigh (1883), equations describing disturbances in a mean flow are regularized by introducing infinitesimal friction. As the results of regularization do not depend on which model of friction is used (e.g. Case 1960, Dikey 1960, Maslowe 1986), we shall use the simplest one, assuming that the frequency ω has an infinitesimal positive imaginary part (which is sometimes referred to as the "Rayleigh viscosity"). Accordingly, (7a,b) will be replaced with

$$(Fh')' + \left(\frac{kF'}{\omega + \mathrm{i}0 - kU} - k^2 F + 1 \right) h = 0, \tag{11a}$$

$$F = \frac{H}{(\omega + \mathrm{i}0 - kU)^2 - 1 + U'}. \tag{11b}$$

In the next subsection, we shall introduce the boundary conditions describing wave scattering by a jet.

2.2. *The boundary conditions and scattering coefficients*

Assume that the incident wave has a unit amplitude and is coming from $y \to -\infty$ – accordingly, the reflected and transmitted waves propagate towards $y \to -\infty$ and $y \to +\infty$, respectively (see figure 1). The corresponding boundary conditions are

$$h \to \begin{cases} \mathrm{e}^{\mathrm{i}l_- y} + R\mathrm{e}^{-\mathrm{i}l_- y} & \text{as} \quad y \to -\infty, \\ T\mathrm{e}^{\mathrm{i}l_+ y} & \text{as} \quad y \to +\infty, \end{cases} \tag{12}$$

where R and T are the reflection and transmission coefficients, l_- and $-l_-$ are the meridional wavenumbers of the incident and reflected waves, and l_+ is the wavenumber of the transmitted wave. Substitution of (12) into (11a) yields

$$l_\pm = \sqrt{\frac{\omega^2 - 1}{H_\pm} - k^2}.$$

This formula implies that the incident wave is characterised by ω and k, with $l_\pm$ being "secondary" parameters. It is more convenient, however, to characterise the incident wave by its wavevector (k, l_-), in which case,

$$\omega = \sqrt{1 + H_-\left(k^2 + l_-^2\right)}, \qquad l_+ = \sqrt{\frac{H_- l_-^2 - (H_+ - H_-)k^2}{H_+}}. \tag{13a,b}$$

Note that, if l_- is sufficiently small, expression (13b) yields imaginary l_+. Then, subject to a proper choice of the sign of $\mathrm{Im}\{l_+\}$, expression (12) shows that the wave field decays towards $y \to +\infty$. In this case the transmission coefficient T can be assumed to be zero.

The scattering coefficients $R(k, l_-)$ and $T(k, l_-)$ are *a priori* unknown and, thus, to be determined together with the solution h from the boundary-value problem (11)–(13). Note, however, that R and T satisfy a *unitarity condition* (derived in appendix A),

$$l_+ H_+ |T|^2 + l_- H_- \left(|R|^2 - 1\right) = \pi k \left(\omega^2 - 1\right) \sum_c \left[\frac{Q' |h|^2}{Q^2 |kU'|} \right]_{y = y_c}, \tag{14a}$$

where

$$Q(y) = \left(1 - U'\right) / H \tag{14b}$$

is the jet's potential vorticity (PV). Observe that, in the absence of critical levels, the right-hand side of (14a) vanishes, and the unitarity condition reduces to a requirement that the energy fluxes of the reflected and transmitted waves add up to that of the incident wave. Equation (14a) can also be interpreted in terms of pseudo-energy, i.e. the difference between the energy of the mean flow and that perturbed by the wave (e.g. Hayashi and Young 1987). Note, however, that the scattering coefficients are defined for the wave field infinitely far from the jet, where the densities of energy and pseudo-energy coincide – so the two interpretations are equivalent.

If, however, critical levels *are* present, the wave energy (pseudo-energy) is *not* conserved. To understand whether it is generated or dissipated, observe that (13a) implies $\omega^2 - 1 > 0$. Then, the contributions of critical levels to the energy balance – as described by (14a) – depend only on the PV gradient at $y = y_c$:

- if $kQ'(y_c) > 0$, the corresponding critical level *amplifies* the wave (over-reflection),
- if $kQ'(y_c) < 0$, the critical level *absorbs* the wave (under-reflection).

Note that apparent singularities do not contribute to the unitarity condition (14a) – hence, wave energy is neither generated nor absorbed there.

The single most convenient characteristic of scattering in the problem at hand is the *non-unitarity coefficient* given by

$$S(k, l_-) = \frac{l_+ H_+}{l_- H_-} |T|^2 + |R|^2. \tag{15}$$

As follows from (14a), $S > 1$ corresponds to over-reflection, $S < 1$ corresponds to under-reflection and $S = 1$ implies that the wave energy (pseudo-energy) is conserved.

2.3. *Numerical results*

The boundary-value problem (11)–(13) was solved numerically using an algorithm described in appendix B. We shall present the results for the so-called Bickley jet,

$$H = 1 + \tfrac{1}{2} \Delta H \tanh(y/W), \tag{16}$$

where ΔH is the depth change across the jet and W is the jet's width. Note that, for positive ΔH and W, the jet flows westwards ($U < 0$).

When computing the non-unitarity coefficient S as a function of, say, l_-, it is convenient to keep the zonal phase speed ω/k constant (in which case the critical level does not move when l_- is changed).

The graphs of S versus l_- for various values of ω/k are shown in figure 2 for the jet (16) with

$$\Delta H = 1, \quad W = 0.25. \tag{17}$$

The following conclusions can been drawn:

- For sufficiently small values of the incident wavenumber l_-, the transmitted wave does not exist (becomes "non-propagating"). Indeed, as follows from (13b), if $l_- \to 0$, then l_+ becomes imaginary. The reflected wave still exists, however, and R can still be computed in such cases – but they are not

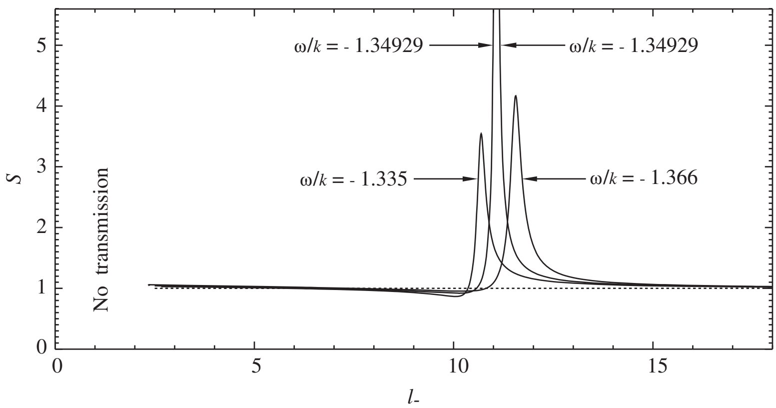

Figure 2. The dependence of the non-unitarity coefficient S [defined by (15)] on the meridional wavenumber l_- of the incident wave, for the Bickley jet (16), (17). Each curve is computed for a fixed value of the phase velocity ω/k (indicated on the graph). Waves for which $S > 1$ correspond to over-reflection.

particularly interesting and we just let the curves in figure 2 terminate in the small-l_- region.

- Figure 2 shows that, for $\omega/k \approx -1.34$ and $l_- \approx 11$, over-reflection is anomalously high. The corresponding value of the zonal wavenumber [which can be determined using (13a)] is $k \approx -6.9$.
- Further computations specifically targeting the region of anomalously large S suggest that, for certain values $k = k_0$ and $l_- = l_0$, the non-unitarity coefficient S is truly infinite – i.e. over-reflection turns into *hyper*-reflection.

  So far, this conclusion is based on numerical evidence only – but later it will be supported by analytical and qualitative arguments. We also emphasize that, even though figure 2 illustrates the results with moderate S, the full range of our computations reached $S \sim 1000$ (as our numerical method is fully reliable for up to $S \sim 700$, after which its accuracy slowly deteriorates).

  We have also computed the wavevector for which hyper-reflection occurs for the Bickley jet (16), (17),

$$k_0 \approx -6.8665, \quad l_0 \approx 11.0693.$$

- The asymptotics of S as $(k, l_-) \to (k_0, l_0)$ turned out to be difficult to compute. We can only state that the integral of S in the (k, l_-) plane over a region including the hyper-reflection point (k_0, l_0) diverges, i.e. the singularity of S is stronger than, or equivalent to, a second-order pole.

We have also computed the dependence of the hyper-reflection wavevector $(k_0,\ l_0)$ on the jet's width W. The results are shown in figure 3: one can see that, as the jet becomes narrower, the hyper-reflected wave becomes shorter.

The dependence of $(k_0,\ l_0)$ on the depth change across the jet, ΔH, is shown in figure 4. Observe that, as $\Delta H \to 2$, the meridional wavenumber l_0 of the incident wave tends to infinity. This is probably caused by the fact that, in this limit, the ocean's depth $H(y)$ vanishes as $y \to -\infty$.

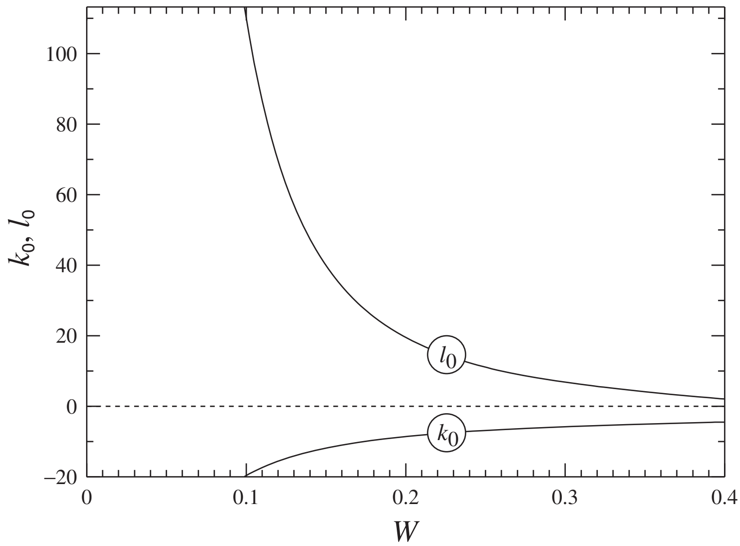

Figure 3. The wavevector (k_0, l_0) of the hyper-reflected wave versus the width W of the Bickley jet (16) with $\Delta H = 1$.

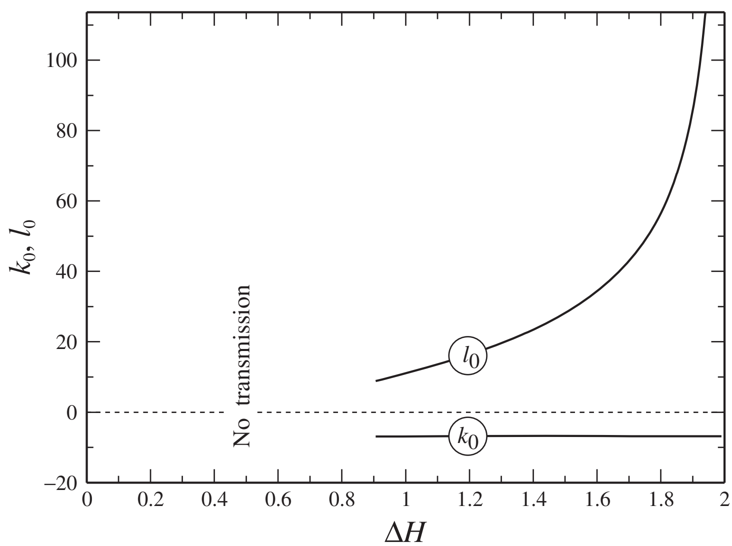

Figure 4. The wavevector (k_0, l_0) of the hyper-reflected wave versus the depth change ΔH of the Bickley jet (16) with $W = 0.25$. The curves terminate at $\Delta H \approx 0.906$ when l_+ becomes imaginary (i.e. the medium becomes non-transparent for waves as $y \to +\infty$).

For $\Delta H \lesssim 0.906$, the meridional wavenumber l_+ becomes imaginary, i.e. no transmitted wave exists – in which case, as before, the graph is terminated. Note, however, that hyper-reflection can still occur for this range of ΔH (as the reflection coefficient can still be infinite), and our computations show that $l_0 \to 0$ as $\Delta H \to 0$.

Finally, observe that the zonal wavenumber k_0 does not change much through the whole range of ΔH.

2.4. *Physical interpretation of hyper-reflection*

An important insight into the mechanism of hyper-reflection can be obtained by introducing

$$\phi = F^{1/2} h,$$

in which case (11a) becomes

$$-\phi'' + P\phi = l_-^2 \phi, \tag{18a}$$

where

$$P(y) = \frac{2FF'' - (F')^2}{4F^2} - \frac{1}{F} - \frac{kF'}{F(\omega + \mathrm{i}0 - kU)} + k^2 + l_-^2. \tag{18b}$$

The boundary conditions (12), in turn, become

$$h \to \begin{cases} \sqrt{k^2 + l_-^2}\left(\mathrm{e}^{\mathrm{i}l_- y} + R\mathrm{e}^{-\mathrm{i}l_- y}\right) & \text{as} \quad y \to -\infty, \\ \sqrt{k^2 + l_+^2}\ T\mathrm{e}^{\mathrm{i}l_+ y} & \text{as} \quad y \to +\infty. \end{cases} \tag{19}$$

Equation (18a) can be interpreted as the Schrödinger equation for a "quantum particle" with momentum l_- scattered by a "potential" $P(y)$ (e.g. Landau and Lifshitz 1981). Also note that the analogy between quantum particles and oceanic waves has been previously employed by LeDizés and Billant (2009).

A typical graph of $P(y)$ is shown in figure 5. Observe that it involves four singular points: two critical levels and two apparent singularities. Most importantly, the critical level located between the apparent singularities is an *amplifying* one ($Q' > 0$).

This circumstance suggests the following interpretation of hyper-reflection: imagine a wave "oscillating" to and fro between two apparent singularities acting as barriers. Then, each time the wave passes through the critical level, its amplitude grows.

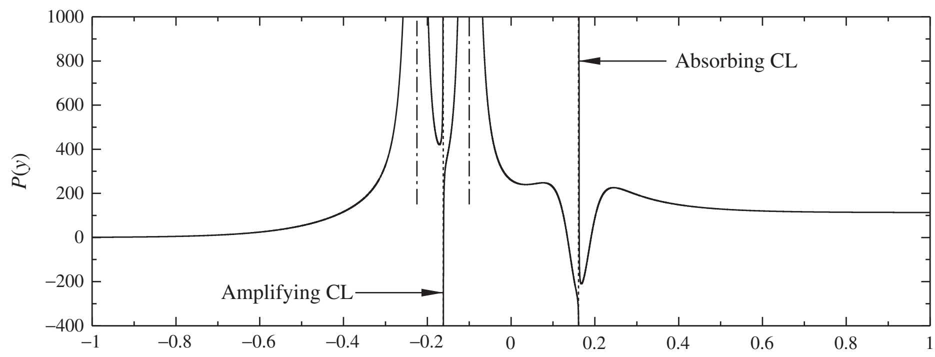

Figure 5. The potential $P(y)$, as determined by (18b) for the Bickley jet (16), (17) and the wave with $k = -6.8665$, $l_- = 11.069$. The radiating and absorbing critical levels are labelled and marked with dotted lines, the apparent singularities are marked with dashed-dotted lines.

Within the framework of this model, hyper-reflection occurs if the amplification of the wave by the critical level exceeds the loss of wave energy through the barriers.

Note, however, that a doubly-reflected wave strengthens the "original" one only if their phases coincide – i.e. the above interpretation neglects the wave interference. This aspect of the problem will be explored in the next section using a mathematically simpler example.

Also observe that, as $y \to \pm\infty$, $P(y)$ should tend to a constant sufficiently fast. To understand why, consider, for example,

$$P \to P_+ + \frac{P_1}{y} \qquad \text{as} \qquad y \to +\infty,$$

then (18a) yields

$$\phi \to \text{const} \times y^{-\mathrm{i}P_1/2l_+} \mathrm{e}^{\mathrm{i}l_+ y} \quad \text{as} \quad y \to +\infty,$$

which is inconsistent with the boundary conditions (19). To avoid this problem, we assume

$$P = P_\pm + \mathrm{O}(y^{-1-\gamma}) \qquad \text{as} \qquad y \to \pm\infty, \tag{20}$$

where $\gamma > 0$ is a constant. In terms of the "physical" variables, this restriction amounts to

$$H = H_\pm + \mathrm{O}(y^{-1-\gamma}) \qquad \text{as} \qquad y \to \pm\infty,$$

which is implied everywhere in this article.

3. Rossby waves and jets on the quasigeostrophic β-plane

3.1. *Formulation*

Consider again a thin layer of ideal fluid on a rotating sphere, but this time assume the elevation of the free surface to be small (which amounts to the quasigeostrophic approximation). In this case, the motion of the layer can be characterised by the non-dimensional streamfunction ψ, related to its dimensional counterpart by

$$\psi = \frac{\psi_*}{fR_d^2},$$

where R_d is the deformation radius [given by (1)] and f is the Coriolis parameter. Assuming also the β-plane approximation, we shall write the governing equation in the form

$$\frac{\partial\left(\nabla^2\psi - \psi\right)}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} + \beta\frac{\partial\psi}{\partial x} = 0,$$

where (x, y) and t are the "old" dimensionless coordinates and time (given by (2)), and

$$\beta = \frac{R_d}{R_E}\tan\theta,$$

R_E is the Earth's radius and θ is the latitude.

We shall seek a solution in the form

$$\psi = -\int U(y)\mathrm{d}y + \widetilde{\psi},$$

where $U(y)$ and $\widetilde{\psi}$ describe a zonal flow and a small-amplitude wave, respectively. Following the same routine as in the previous section, one can obtain the following equation equivalent of (11a):

$$-\psi'' + \left[k^2 + 1 - \frac{k(U'' - U - \beta)}{\omega + \mathrm{i}0 - kU}\right]\psi = 0. \tag{21}$$

Recall, however, that the coefficients of (11a) involve a "step-like" function, $H(y)$, whereas the coefficients of (21) have equal limits as $y \to \pm\infty$. As a result, the meridional wavenumber of the transmitted wave equals that of the incident wave in the present case. Thus omitting the subscripts $\pm$, we have

$$\psi \to \begin{cases} \mathrm{e}^{\mathrm{i}ly} + R\mathrm{e}^{-\mathrm{i}ly} & \text{as} \quad y \to -\infty, \\ \psi \to T\mathrm{e}^{\mathrm{i}ly} & \text{as} \quad y \to +\infty, \end{cases} \tag{22}$$

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1}. \tag{23}$$

Equation (21) can be re-written in the "general" form (18a) with $\phi = \psi$, $l_- = l$ and

$$P(y) = k^2 + l^2 + 1 - \frac{k(U'' - U - \beta)}{\omega + \mathrm{i}0 - kU}. \tag{24}$$

Also note that the solution of the boundary-value problem (21)–(23) satisfies a unitarity condition

$$|T|^2 + |R|^2 - 1 = \frac{\pi}{|l|}\sum_c \left[\frac{U'' - U - \beta}{|U'|}|\psi|^2\right]_{y=y_c},$$

which is similar to the one derived by Benilov *et al.* (1992) for non-divergent Rossby waves.

3.2. *A two-jet configuration*

Observe that, unlike its counterpart (18b), potential (24) does not involve apparent singularities – thus, it is unclear what can act as barriers trapping the wave and, thus, give rise to hyper-reflection (as suggested by our interpretation in section 2.4).

As an alternative to apparent singularities, we shall consider a two-jet configuration, so the wave can be trapped between the jets. Such a setting is also motivated physically, as two distinct well-defined jets have been observed in the Antarctic Circumpolar Current by Gille (1994), multiple jets also exist in the tropical part of the Earth's ocean, as well as on Jupiter and Saturn.

To simplify the problem, we assume that the jets' velocities $U_{1,2}(y)$ are functions with compact non-overlapping supports. In terms of the general equation (18a), this implies that the potential is given by

$$P(y) = P_1(y) + P_2(y),$$

where

$$P_1 = 0 \quad \text{for} \quad y \notin (y_{1a}, y_{1b}),$$

$$P_2 = 0 \quad \text{for} \quad y \notin (y_{2a}, y_{2b}),$$

and $y_{1a} < y_{1b} < y_{2a} < y_{2b}$ (see figure 6(a)).

For the first jet, we shall introduce "from-right-to-left" scattering coefficients R_1, T_1 (see figure 6(b)),

$$-\psi_1'' + P_1\psi_1 = l^2\psi_1,$$

$$\psi_1 \to \begin{cases} T_1 e^{ily} & \text{for} \quad y < y_{1a}, \\ e^{ily} + R_1 e^{-ily} & \text{for} \quad y > y_{1b}. \end{cases} \tag{25}$$

For the second jet, we introduce "from-left-to-right" coefficients R_2, T_2 (figure 6(c)),

$$-\psi_2'' + P_2\psi_2 = l^2\psi_2,$$

$$\psi_2 \to \begin{cases} e^{ily} + R_2 e^{-ily} & \text{for} \quad y < y_{2a}, \\ T_2 e^{ily} & \text{for} \quad y > y_{2b}. \end{cases} \tag{26}$$

We shall treat R_1, T_1, R_2 and T_2 as known characteristics of the individual jets and use them to express the global scattering coefficients R and T defined by (21)–(23). The calculations involved are given in appendix C, whereas here we shall only present

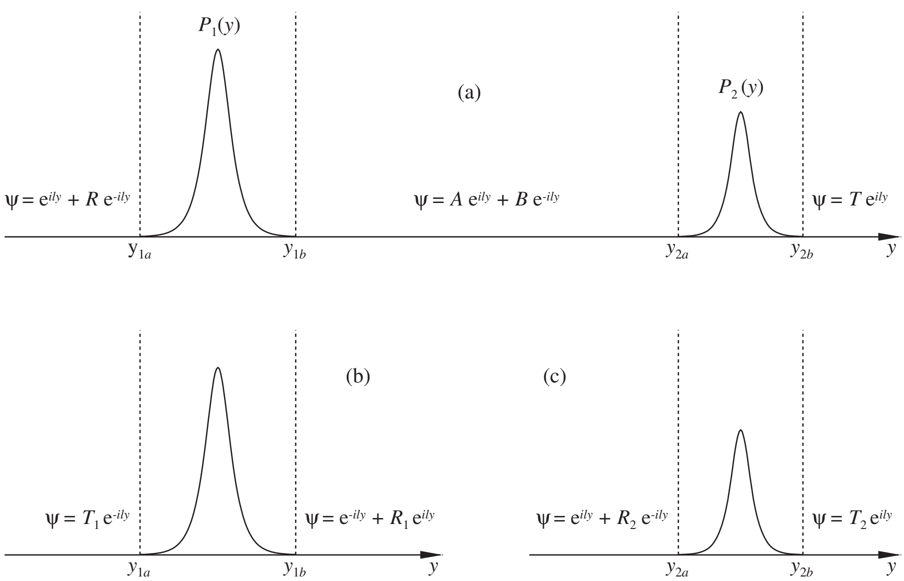

Figure 6. A schematic illustrating the scattering of waves by two potentials with compact, non-overlapping supports. Panels (b) and (c) illustrate wave scattering by the "individual" potentials P_1 and P_2, panel (a) corresponds to the "global" scattering problem.

the final result

$$T = \frac{T_2\left(1 - |R_1|^2\right)}{T_1^*(1 - R_1 R_2)}, \qquad R = \frac{T_1\left(R_2 - R_1^*\right)}{T_1^*(1 - R_1 R_2)}. \tag{27a,b}$$

Evidently, if

$$R_1 R_2 = 1, \tag{28}$$

the global scattering coefficients are both infinite, i.e. hyper-reflection occurs.

3.3. *Discussion*

Physically, condition (28) means that, after two successive reflections from P_1 and P_2, a wave trapped between the jets regains its original amplitude and phase. Given that the incident wave keep "pumping" energy into the space between the jets, it is clear that (28) should cause hyper-reflection.

It is less clear, however, why hyper-reflection does not occur when

$$R_1 R_2 = \text{real number greater than 1}. \tag{29}$$

This condition guarantees that, after two successive reflections, a wave trapped between the jets regains its original phase and a *larger* amplitude. Yet, formulae (27a,b) yield finite values of the scattering coefficients in this case!

One can only assume that, if (29) holds, the (steady) solution with finite R and T is physically meaningless, as it co-exists with *exponentially growing* solutions (see the next section). Thus, in a general solution, the steady component is, essentially, invisible against the background of rapidly growing unstable field.

We shall also point out that:

- Hyper-reflection by a two-jet configuration never occurs if the jets are mirror images of one another. In this case $R_1 = R_2$ – hence, condition (28) holds only if $R_1 = R_2 = \pm 1$. As a result, the zero denominators in expressions (27a,b) are cancelled out by zero numerators, and R and T remain finite.
- If the jets have identical shapes (i.e. can be obtained from one another by translation along the y axis), the transmission coefficient T remains finite even if condition (28) does hold. In this case, it can be shown that

$$R_2 = -\frac{R_1^* T_1}{T_1^*} \mathrm{e}^{\mathrm{i}lD},$$

  where D is the distance between the jets. Then, (28) implies that

$$|R_1|^2 = 1,$$

  and formulae (27a,b) show that T remains finite (but R can still be infinite).
- Mathematically, singularities associated with critical levels are not essential for hyper-reflection. Indeed, the two-jet example – or rather its "general" formulation through (18a) – shows that hyper-reflection can also occur if the potentials $P_{1,2}$ are analytical but *complex* functions. The latter property

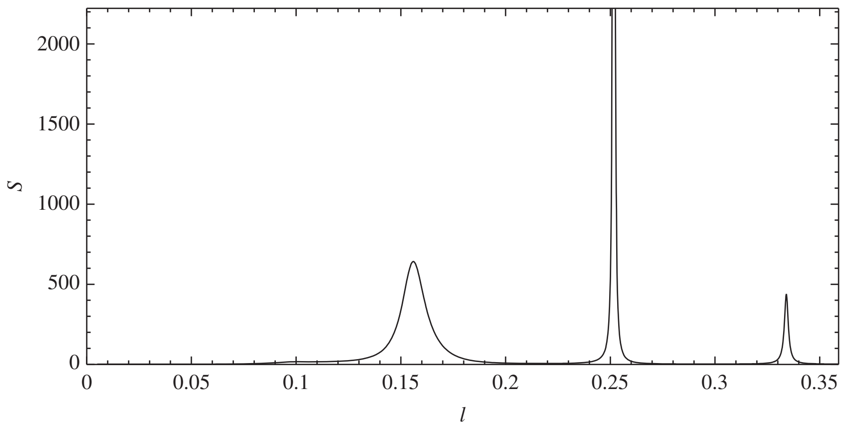

Figure 7. The dependence of the non-unitarity coefficient S (defined by (15)) on the meridional wavenumber l of the incident wave, for the double jet configuration (30a,b). The curve shown is computed for a fixed value of the phase velocity $\omega/k = 1.7713$.

guarantees that over-reflection by a single jet may still occur – hence, so can hyper-reflection by a two-jet configuration.

- In addition to hyper-reflection caused by trapping of waves between the jets, there can be instances of hyper-reflection of waves with their critical levels located at the jets' maxima (Maslowe 1991).
- There seems to be no specific reason why hyper-reflection cannot occur for a *single* jet and a wave with a critical level *not* located at the jet's maximum (which would be neither our setting, nor the one described by Maslowe (1991)). Still, we have been unable to find any examples of such.
- We have examined numerically the non-unitarity coefficient $S(l)$ for several examples of double jets. A typical example computed for

$$U = -0.7\,\mathrm{sech}^2[0.7(y+20)] - 2\,\mathrm{sech}^2[2(y-20)], \qquad \beta = 2, \qquad \text{(30a,b)}$$

is shown in figure 7. Out of the three peaks of $S(l)$ shown in the figure, the first and third correspond to over-reflection by the individual jets, with the middle one corresponding to hyper-reflection by the two jets as a system. Comparing figure 7 with figure 2, one can also deduce that, generally, over-reflection for quasigeostrophic jets on the β-plane (figure 7) is much stronger than that for ageostrophic ones (figure 2).

4. Hyper-reflected waves as marginally stable disturbances

In this section, we shall examine what happens with a hyper-reflected wave if its wavevector or the parameters of the jet are perturbed. In section 4.1, we shall keep our approach as general as possible, so it would be applicable to any hyper-reflecting potential. Then, in section 4.2, general results will be illustrated by the example of waves and jets on the β-plane.

4.1. *General results*

Assume for simplicity that

$$P(y) \to 0 \qquad \text{as} \qquad y \to \pm\infty.$$

This assumption holds for quasigeostrophic jets, but not for *a*geostrophic ones (where P is a "step-like" function, with different limits as $y \to \pm\infty$). Note, however, that all results obtained for the decaying potentials can be readily extended to step-like ones (but with more algebra required).

To describe a hyper-reflected wave, we shall use the general equation (18a) with $l_- = l$ and the following boundary conditions:

$$\phi \to \begin{cases} \bar{R}\mathrm{e}^{-\mathrm{i}ly} & \text{as} \quad y \to -\infty, \\ \phi \to \bar{T}\mathrm{e}^{\mathrm{i}ly} & \text{as} \quad y \to +\infty. \end{cases} \tag{31}$$

Comparing (31) with the standard boundary conditions (22), one can observe that (31) describes reflected/transmitted waves without an incident one – which is what hyper-reflection essentially is. However, (31) can also describe waves coming from infinity and absorbed by the jet. To eliminate the latter possibility, we shall introduce the meridional component of the waves' group velocity,

$$C_l = \frac{\partial\omega(k, l)}{\partial l}, \tag{32}$$

and require that, for a hyper-reflecting wave,

$$C(k_0, -l_0) < 0, \qquad C(k_0, l_0) > 0. \tag{33}$$

These conditions guarantee that the energy flux (which is proportional to the group velocity) is directed towards $\pm\infty$, i.e. away from the jet.

Note also that the coefficients $\bar{R}$ and $\bar{T}$ in conditions (31) are related to the original scattering coefficients R and T by

$$\bar{R} = \lim_{(k,l)\to(k_0,\, l_0)} \frac{R}{\sqrt{|R|^2+|T|^2}}, \qquad \bar{T} = \lim_{(k,l)\to(k_0,\, l_0)} \frac{T}{\sqrt{|R|^2+|T|^2}},$$

where (k_0, l_0) is the wavevector of the hyper-reflected wave.

Equation (18a) and the boundary conditions (31) form and eigenvalue problem, where ϕ is the eigenfunction and l is the eigenvalue. We shall distinguish two types of solutions: hyper-reflected waves ($\mathrm{Im}\{l\} = 0$) and captured waves ($\mathrm{Im}\{l\} > 0$). Solutions with $\mathrm{Im}\{l\} < 0$, in turn, grow as $y \to \pm\infty$ (see (31)) and, thus, will not be considered.

Now, let

$$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots,$$

where P_0 is the potential for which hyper-reflection occurs for the wavevector (k_0, l_0) and ε is a small parameter. A perturbation of P results in a perturbation of the solution, i.e.

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots, \qquad l = l_0 + \varepsilon l_1 + \varepsilon^2 l_2 + \cdots, \tag{34a,b}$$

$$\bar{R} = \bar{R}_0 + \varepsilon\bar{R}_1 + \varepsilon^2\bar{R}_2 + \cdots, \qquad \bar{T} = \bar{T}_0 + \varepsilon\bar{T}_1 + \varepsilon^2\bar{T}_2 + \cdots. \tag{34c,d}$$

In the next-to-leading order, (18a) yields

$$-\phi_1'' + P_1\phi_0 + P_0\phi_1 = l_0^2\phi_1 + 2l_0l_1\phi_0. \tag{35}$$

Recalling that

$$\phi_0 \to \begin{cases} \bar{R}_0 \mathrm{e}^{-\mathrm{i}l_0 y} & \text{as} \quad y \to -\infty, \\ \bar{T}_0 \mathrm{e}^{\mathrm{i}l_0 y} & \text{as} \quad y \to +\infty, \end{cases}$$

one can readily show that the term involving ϕ_0 on the right-hand side of (35) causes ϕ_1 to grow linearly as $y \to \pm\infty$. As a result, expansion (34a–d) is valid only for $|\,y| \ll \varepsilon^{-1}$ and should be treated as the *inner* solution of the problem.

The *outer* solution, in turn, is given by the boundary condition (31), which represents the asymptotics of ϕ in the region where $|P| \ll 1$, i.e. for $|y| \gg 1$. The outer and inner solutions "overlap" for $1 \ll |y| \ll \varepsilon^{-1}$ – thus, for matching, the outer limit of ϕ_1 should be equated to the inner limit of (31). This amounts to expanding (31) in ε, which yields

$$\phi_1 \to \begin{cases} \bar{R}_0 \mathrm{e}^{-\mathrm{i}l_0 y}(-\mathrm{i}l_1 y) + \bar{R}_1 \mathrm{e}^{-\mathrm{i}l_0 y} & \text{as} \quad y \to -\infty, \\ \bar{T}_0 \mathrm{e}^{\mathrm{i}l_0 y}(\mathrm{i}l_1 y) + \bar{T}_1 \mathrm{e}^{\mathrm{i}l_0 y} & \text{as} \quad y \to +\infty. \end{cases} \tag{36}$$

The boundary-value problem (35), (36) determines both ϕ_1 and l_1. The latter, however, is the more important characteristic, and it can be found without the former.

To find l_1, multiply (35) by ϕ_0 and integrate with respect to y over $(-Y, Y)$, where Y is an undetermined large number. Integrating the term involving ϕ_1'' by parts twice and taking into account that ϕ_0 satisfies

$$-\phi_0'' + P_0\phi_0 = l_0^2\phi_0,$$

we obtain

$$\left(\phi_0'\phi_1 - \phi_0\phi_1'\right)_{y=Y} - \left(\phi_0'\phi_1 - \phi_0\phi_1'\right)_{y=-Y} + \int_{-Y}^{Y} P_1\phi_0^2\,\mathrm{d}y = 2l_0l_1\int_{-Y}^{Y}\phi_0^2\,\mathrm{d}y. \tag{37}$$

Observe that, as $Y \to \infty$, the integral on the right-hand side of (37) diverges. Thus, at this stage, we let Y be large but not infinitely so, and re-arranging the first two terms in (37) using (36) yields

$$-\mathrm{i}l_1\left(\bar{T}_0^2 + \bar{R}_0^2\right)\mathrm{e}^{2\mathrm{i}l_0 Y} + \int_{-Y}^{Y} P_1\phi_0^2\,\mathrm{d}y \to 2l_0l_1\int_{-Y}^{Y}\phi_0^2\,\mathrm{d}y \qquad \text{as} \qquad Y \to \infty. \tag{38}$$

Next, introduce an auxiliary function

$$\hat{\phi}^2 = \begin{cases} \bar{R}_0^2 \mathrm{e}^{-2\mathrm{i}l_0 y} & \text{if} \quad y \le 0, \\ \bar{T}_0^2 \mathrm{e}^{2\mathrm{i}l_0 y} & \text{if} \quad y > 0, \end{cases} \tag{39}$$

in terms of which (38) can be re-written in the form

$$2l_0l_1\int_{-Y}^{Y}\hat{\phi}^2\,\mathrm{d}y - \mathrm{i}l_1\left(\bar{R}_0^2 + \bar{T}_0^2\right) + \int_{-Y}^{Y} P_1\phi_0^2\,\mathrm{d}y \to 2l_0l_1\int_{-Y}^{Y}\phi_0^2\,\mathrm{d}y \qquad \text{as} \qquad Y \to \infty.$$

Rearranging the integrals of $\hat{\phi}^2$ and ϕ_0^2 as a single integral and taking the limit $Y \to \infty$, we obtain

$$\int_{-\infty}^{\infty} P_1 \phi_0^2 \, \mathrm{d}y = l_1 \left[2l_0 \int_{-\infty}^{\infty} \left(\phi_0^2 - \hat{\phi}^2 \right) \mathrm{d}y + \mathrm{i} \left(\bar{R}_0^2 + \bar{T}_0^2 \right) \right]. \tag{40}$$

This equation is the final product of our derivation. It relates the perturbation l_1 of the eigenvalue to the perturbation P_1 of the potential. Observe that, if

$$P_1 \to 0 \qquad \text{as} \qquad y \to \pm\infty,$$

the integral on the left-hand side of (40) converges, and so does the integral on its right-hand side (subject to the general restriction (20) for P_0).

Most importantly, equation (40) is complex (as ϕ_0 is complex, and so are, generally, $\bar{R}_0$ and $\bar{T}_0$). Thus, for an arbitrary perturbation P_1, (40) yields either $\mathrm{Im}\{l_1\} > 0$ (captured wave), or $\mathrm{Im}\{l_1\} = 0$ (hyper-reflected wave), or $\mathrm{Im}\{l_1\} < 0$ (meaningless solution growing as $y \to \pm\infty$). We conclude that hyper-reflected waves are marginal to the captured ones.

Most importantly, these captured modes are unstable.

Indeed, since the wave's frequency ω is related to the wavenumber l by the dispersion relation, it follows that that the first-order correction to ω is

$$\omega_1 = C_l(k_0, l_0)\, l_1 + \delta\omega, \tag{41}$$

where C_l is given by (32) and $\delta\omega$ is the correction due to the perturbation of other parameters, such as the ocean's depth and the zonal wavenumber (see an example in the next subsection).

Note also that, since we deal with a conservative medium (a dissipative one would not support wave propagation without decay), ω and C_l are both real. $\delta\omega$ is real too (as it results from perturbations of real parameters), hence, (41) implies

$$\mathrm{Im}\{\omega_1\} = C_l(k_0, l_0)\, \mathrm{Im}\{l_1\}.$$

This equation shows that, since all captured modes correspond to $\mathrm{Im}\{l_1\} > 0$ and since $C_l(k_0, l_0)$ is positive (see (33)) – then $\mathrm{Im}\{\omega_1\} > 0$, i.e. the captured modes are indeed unstable.

It is worth mentioning that captured modes are unstable due to wave generation at the critical levels (see LeDiées and Billant 2009, and references therein) and exponentially decreasing as $y \to \pm\infty$ "tails" of these modes can be interpreted as waves emitted at an earlier time, when the wave field near the critical levels was exponentially weaker.

4.2. *An example: waves and jets on the β-plane*

To prove that a hyper-reflected wave is marginally unstable, it is sufficient to perturb the zonal wavenumber,

$$k = k_0 + \varepsilon k_1$$

and then verify that one of the two possible signs of k_1 gives rise to an unstable captured wave, whereas the other does not. The jet's shape $U(y)$ does not need to be perturbed.

Note also that, in this section, we do not necessarily imply that $U(y)$ represents a two-jet configuration. All we assume is that a hyper-reflected wave exists, and its wavevector is (k_0, l_0).

Perturbing the expressions for the frequency and potential, (23), (24), we obtain

$$\omega_1 = -\frac{\beta k_1}{k_0^2 + l_0^2 + 1} + \frac{2\beta k_0 (l_0 l_1 + k_0 k_1)}{\left(k_0^2 + l_0^2 + 1\right)^2}, \tag{42a}$$

$$P_1 = 2(l_0 l_1 + k_0 k_1)\left\{1 + \frac{\beta k_0^2 (U'' - U - \beta)}{\left[\beta k_0 - \mathrm{i}0 + k_0 U\left(k_0^2 + l_0^2 + 1\right)\right]^2}\right\}. \tag{42b}$$

Substitution of (42a,b) into the general formula (40) (where ϕ should be replaced with ψ) yields

$$l_1 = \frac{I k_0}{(J - I) l_0} k_1, \tag{43}$$

where

$$I = \int_{-\infty}^{\infty} \left\{1 + \frac{\beta k_0 (U'' - U - \beta)}{\left[\beta k_0 - \mathrm{i}0 + k_0 U\left(k_0^2 + l_0^2 + 1\right)\right]^2}\right\} \psi_0^2 \,\mathrm{d}y,$$

$$J = \int_{-\infty}^{\infty} \left(\psi_0^2 - \hat{\psi}^2\right) \mathrm{d}y + \bar{R}_0^2 - \bar{T}_0^2.$$

It follows from (43) that a solution with $\mathrm{Im}\{l_1\} > 0$ exists for either $k_1 > 0$ or $k_1 < 0$ – one way or another, captured waves do exist. Then from (42a), if $\mathrm{Im}\{l_1\} > 0$, then $\mathrm{Im}\{\omega_1\} > 0$ (instability).

Finally, observe that expression (42a) can be re-written in the form

$$\omega_1 = C_l(k_0, l_0)\, l_1 + C_k(k_0, l_0)\, k_1, \tag{44}$$

where the meridional component of the group velocity of Rossby waves, $C_l(k, l)$, is given by (32) and the zonal one is, similarly,

$$C_k(k, l) = \frac{\partial \omega(k, l)}{\partial k}.$$

Comparing (44) with the general expression (41), one can see that $\delta\omega$ in the latter corresponds to the second term of the former.

5. Summary and concluding remarks

We considered two examples of hyper-reflection of waves by jets. In both cases, the problem was reduced to a Schrödinger-type equation (18a) with potentials (18b) and (24).

(1) For a shallow-water jet on the *f*-plane, we argued that hyper-reflection occurs because the amplifying critical level is located between two apparent singularities (acting as barriers and reflecting waves back to the critical level).

(2) For the case of a two-jet configuration on the quasigeostrophic β-plane, the roles of barriers are played by the individual jets. It has been shown that, in this case, hyper-reflection occurs if a wave is trapped between the jets, after two successive reflections it regains its initial amplitude and phase.

In both cases hyper-reflection co-exists with instability due to disturbances localised near the jet(s), with the hyper-reflected wave playing the role of the marginally stable disturbance – i.e. it separates the spectral region where unstable eigenmodes exist from the region where no meaningful solution exists.

Furthermore, since the above conclusion was obtained through the general approach based on the Schrödinger equation (18a), it applies to all media with hyper-reflection. In particular, it agrees with the examples examined by Lindzen (1974), Maslowe (1991), Lott *et al.* (1992) – in all of which instability exists in a spectral region adjacent to a hyper-reflected wave.

Note, however, that the opposite to the above conclusion does *not* hold: if a steady state in a conservative medium is unstable and the spectral range of unstable disturbances is bounded by a certain wavenumber, the wave with this wavenumber is not necessarily a hyper-reflected one. This can be illustrated by any case where solutions exist on either side of the marginally stable wavenumber (unstable on one side and stable on the other) – whereas hyper-reflection implies unstable solutions on one side and non-existence of solutions on the other.

Finally, the connection between hyper-reflection and modal instability tells one about the latter just as much as it does about the former. Most importantly, it implies that the unstable modes that are spectrally close to the hyper-reflected wave have their eigenfunctions spread over large distances. This circumstance makes them capable of generating disturbances far from the unstable flow, and we believe that they are responsible for the "bursts" observed in numerical simulations of Viúdez and Dritschel (2006) at large distances from the jet – simply because the usual, localized modes cannot be.

The same should occur near all major oceanic currents, as all of them are, to some extent, unstable.

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Appendix A: The unitarity condition (14a)

To derive (14a), multiply (11a) by h^* (where the asterisk denotes complex conjugate), take the imaginary part, and integrate over the interval $-\infty < y < \infty$, which yields

$$\int_{-\infty}^{\infty} \mathrm{Im}\{(Fh')'h^*\}\mathrm{d}y + \int_{-\infty}^{\infty} \mathrm{Im}\left\{\frac{kF'}{\omega + \mathrm{i}0 - kU} - k^2F\right\}|h|^2\mathrm{d}y = 0.$$

Integrating the first term by parts and taking into account the boundary conditions (12), we obtain

$$\frac{l_+H_+|T|^2 + l_-H_-\left(|R|^2-1\right)}{\omega^2-1} - \int_{-\infty}^{\infty}\mathrm{Im}\{F\}\left|h'\right|^2\mathrm{d}y + \int_{-\infty}^{\infty}\mathrm{Im}\left\{\frac{kF'}{\omega+\mathrm{i}0-kU} - k^2F\right\}|h|^2\,\mathrm{d}y = 0. \tag{A.1}$$

In what follows, we shall use the formulae

$$\mathrm{Im}\left\{\frac{1}{f(y)\pm\mathrm{i}0}\right\} = \mp\pi\delta[f(y)] = \mp\pi\sum_n \frac{\delta(y-y_n)}{\left|f'(y_n)\right|},$$

$$\int_{-\infty}^{\infty}\delta'[f(y)]\,G(y)\,\mathrm{d}y = -\sum_n\frac{G'(y_n)}{\left|f'(y_n)\right|},$$

where δ is the Dirac delta-function and y_n are the roots of the equation $f(y)=0$. Applying the above formulae to (A.1), we obtain

$$\frac{l_+H_+|T|^2 + l_-H_-\left(|R|^2-1\right)}{\omega^2-1} = \pi k\sum_c\left(\frac{F'|h|^2}{|kU'|}\right)_{y=y_c} - \pi\sum_a\left[H\mathrm{sgn}(\omega-kU)\frac{|h'|^2 + \left(\frac{k|h|^2}{\omega-kU}\right)' + k^2|h|^2}{\left|2(\omega-kU)(-kU') + U''\right|}\right]_{y=y_a}. \tag{A.2}$$

Finally, using the asymptotics (9) of the solution as $y \to y_a$, one can show that

$$\left[\left|h'\right|^2 + \left(\frac{k|h|^2}{\omega-kU}\right)' + k^2|h|^2\right]_{y=y_a} = 0.$$

As a result, the second term on the right-hand side of (A.2) vanishes, and (A.2) reduces to (14a) as required.

Appendix B: The numerical method for problem (11)–(13)

If the coefficients of (11a) were regular, one could simply "shoot" the solution from $y = +\infty$ towards $y = -\infty$ using the following boundary condition:

$$h \to e^{il_+ y} \qquad \text{as} \qquad y \to +\infty, \tag{B.1a}$$

instead of the "correct" boundary conditions (12). For sufficiently large negative y, the coefficients of (11a) become close to constant – hence, the solution becomes

$$h \to b_1 e^{il_- y} + b_2 e^{-il_- y} \qquad \text{as} \qquad y \to -\infty. \tag{B.1b}$$

Comparing (B.1a,b) to the "correct" boundary conditions (12), one can use the linearity of the problem to show that

$$T = 1/b_1, \qquad R = b_2/b_1.$$

For the problem at hand, however, this simple approach needs to be modified, as the coefficients of (11a) can be singular. Typically, four singular points arise: two critical levels (determined by (8a)) and two apparent singularities (determined by (8b)).

To bypass this difficulty, one can extend (11a) and its solution into the plane of complex y and modify the path of integration in such a way that it "misses" the singular points. This approach was initially used by Boyd (1985) for a Chebyshev collocation method and by Benilov and Sakov (1999) for the Runge–Kutta method (as in this article). One would still have to keep the endpoints fixed, and also make sure that the modified path can be moved back to the real axis without touching any of the critical levels (the apparent singularities are unimportant in this case, as the solution is regular there). This would guarantee that the solution would arrive at its final destination with the correct value.

When implementing this plan, one should keep in mind that:

- The term i0 in the denominator in (11a) indicates that the critical-level singularity is located just *above* the real axis for $kU'(y_c) > 0$, and just *below* it for $kU'(y_c) < 0$. Therefore, the path of integration near $y = y_c$ can only be moved *downwards* and *upwards*, respectively.
- The coefficients of (11a) may have non-physical singularities at some complex values of y. The Bickley jet (16), for example, is singular at

$$y = \tfrac{1}{2} i\pi W, \ \ \pm\tfrac{3}{2} i\pi W, \ \ \pm\tfrac{5}{2} i\pi W \ldots.$$

When moving the path of integration into the complex plane, one should determine the locations of these non-physical singularities and make sure that the modified path can be moved back to the real axis without touching them.

Note also that the coordinates of non-physical singularities always have imaginary parts comparable to W, so one can miss them by simply keeping the path of integration sufficiently close to the real axis.

Appendix C: The scattering coefficients for a two-jet configuration

Observe that, in between the jets, the "global" solution can be represented in the form

$$\psi \to A\mathrm{e}^{\mathrm{i}ly} + B\mathrm{e}^{-\mathrm{i}ly} \qquad \text{for} \qquad y \in [y_{1a}, y_{2b}], \tag{C.1}$$

where A and B are undetermined constants (figure 6). Then, comparing (C.1) with (26), one can deduce

$$T/A = T_2, \qquad B/A = R_2. \tag{C.2}$$

Next, introduce

$$\phi = \psi - (\psi/R)^*,$$

where the asterisk denotes complex conjugate. Recalling that ψ satisfies (22), then one can show that

$$\phi = \begin{cases} [R - (1/R^*)]\mathrm{e}^{-\mathrm{i}ly} & \text{if} \quad y < y_{1a}, \\ [B - (A^*/R^*)]\mathrm{e}^{-\mathrm{i}ly} + [A - (B^*/R^*)]\mathrm{e}^{\mathrm{i}ly} & \text{if} \quad y_{1b} > y > y_{2a}. \end{cases}$$

Comparing this with (25), one can obtain

$$\frac{R - (1/R^*)}{B - (A^*/R^*)} = T_1, \qquad \frac{A - (B^*/R^*)}{B - (A^*/R^*)} = R_1. \tag{C.3}$$

Equations (C.2) and (C.3) form a set of equations for R, T, A and B. The required formulae, (27a,b), follow from this set.