Propagation of nonlinear waves in a fluctuating medium

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1. The scattering of nonlinear waves in a weakly dispersive medium with random inhomogeneities is an important problem in the general theory of waves. Powerful general methods have been developed for the linear case (see Ref. 1 and the literature cited there), but in the nonlinear case it is nontrivial to take into account inhomogeneities and far fewer results have been obtained in this case. There are three different approaches to the problem of nonlienar waves in a fluctuating medium: the Born approximation, perturbation theory for systems which are almost integrable, and the mean-field method. The Born approximation[2,3] is applicable only to small space–time intervals, while use of the second appraoch[4] assumes that the "homogeneous" form of the problem is integrable by means of the methods used to solve inverse-scattering problems. The mean-field method[5] has been widely used in different physical problems. Although it assumes that the nonlinearity and inhomogeneity of the problem are additive, this assumption was later shown to be incorrect.[6]

The present paper is devoted to a modification of the mean-field method, which allows a rigorous asymptotic justification. The modification is based on a transformation to a reference frame which moves with the fluctuating velocity of the wave. In this frame the phase fluctuations can be filtered out and the evolution of the wave profile can be described. We will show that because of this transformation to a new reference frame, secular terms are absent in all orders of perturbation theory.

2. We consider the standard equation describing the evolution of sound waves of small, but finite, amplitude in a one-dimensional medium with small fluctuations in the speed of sound:

$$u_{tt} - (1 + \epsilon\alpha)^2 u_{xx} = \epsilon^2 (u^2)_{xx}, \qquad (1)$$

where ε is a small positive number, and $\alpha(x, t)$ is a stationary random function with a zero mean. We first consider purely temporal fluctuations: $\alpha = \alpha(t)$. We transform to a reference frame moving with the fluctuating velocity (which is to be determined:

$$x' = x - \int c(t)dt, \quad t' = t. \qquad (2)$$

In terms of these new variables Eq. (1) takes the form (the primes are omitted)

$$u_{tt} - 2cu_{tx} - c_t u_x + [c^2 - (1 + \epsilon\alpha)^2]u_{xx} = 0. \qquad (3)$$

In analyzing (3) we shall use the asymptotic method of multiple time scales; i.e., in addition to the "fast" time t, we introduce a hierarchy of "slow" times: $T = \varepsilon^2 t$, $T_1 = \varepsilon^3 t$, and so on. The time derivatives in (3) can then be transformed to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T} + \epsilon^3 \frac{\partial}{\partial T_1} + \ldots,$$

The solution of Eq. (3) is written in the form of asymptotic series

$$u(x, t, T, \ldots) = u^{(0)}(x, T, \ldots) + \epsilon u^{(1)}(x, t, T, \ldots) + \ldots, \qquad (4)$$
$$c(t, T, \ldots) = 1 + \epsilon c^{(1)}(t, T, \ldots) + \ldots$$

The lowest-order terms in (4) correspond to waves travelling to the right with a velocity close to the speed of sound. To zero order in ε the perturbation-theory equations are satisfied automatically. To first order we obtain an equation for the scattered field $u^{(1)}$:

$$u^{(1)}_{tt} - 2u^{(1)}_{tx} = c^{(1)}_t u^{(0)}_x + 2(\alpha - c^{(1)})u^{(0)}_{xx}. \qquad (5.1)$$

Equation (5.1) should be supplemented with the condition that there be no reflected waves in the limit $t \to -\infty$:

$$u^{(1)}\Big|_{\substack{x = \text{const} \\ t \to -\infty}} = u^{(1)}\Big|_{\substack{x + 2t = \text{const} \\ t \to -\infty}} = 0. \qquad (5.2)$$

Here it is implicitly assumed that the fluctuation is switched on adiabatically; i.e., Eq. (5) should be solved by replacing $\alpha(t) \to \alpha(t)\, e^{\nu t}$, $0 < \nu \ll 1$ and then by taking the limit $\nu = 0$. This device ensures that the improper integrals arising below converge. The problem (5) can be trivially integrated:

$$u^{(1)} = \int_0^\infty u_x(x + 2\tau, T)\alpha(t - \tau)d\tau - u^{(0)}_x(x, T) \int_{-\infty}^{t} [\alpha(\tau) - c^{(1)}(\tau)]d\tau. \qquad (6)$$

Because of the arbitrariness in the choice of $c(t)$, we assume

$$c^{(1)} = \alpha(t), \qquad (7)$$

which will be justified below.

To second order in ε we have an inhomogeneous linear equation for $u^{(2)}$

$$\frac{\partial}{\partial t}(u^{(2)}_t - 2u^{(2)}_x) = F(x, t, T),$$
$$F = 2u^{(0)}_{Tx} + [(u^{(0)})^2]_{xx} + 2\alpha u^{(1)}_{tx} + \alpha_t u^{(1)}_x + 2c^{(2)}u^{(0)}_{xx} + c^{(2)}_t u^{(0)}_x. \qquad (8)$$

Because of the stationary nature of $\alpha(t)$, the finiteness of $u^{(2)}$ can be ensured by requiring that

$$\langle F \rangle \equiv \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} F(x, t, T)\, dt = 0, \qquad (9)$$

and, taking the special case $\langle c^{(2)} \rangle = 0$, we obtain with the help of (6)-(8) an equation describing the slow evolution fo the wave [the index (0) is omitted]:

$$u_T + \frac{\sigma^2}{2} u_x + uu_x + \int_0^\infty u_{xx}(x + 2\tau, T) W(\tau)\, d\tau = 0, \qquad (10)$$

where $W(\tau) = \langle \alpha(t)\alpha(t + \tau) \rangle$ is the correlation function of the speed of sound fluctuations,[1)] $\sigma^2 = W(0)$. Equation (10) involves only determined coefficients, which is an advantage over the original equation (1). Since the field is averaged in a reference frame moving with the local speed of sound and "following" the phase fluctuations of the wave, $u(x, T)$ describes the mean wave form and not the mean field. We note also that the fast-time average in our method arises as a natural condition for the absence of secular terms and in this sense our method differs in an essential way from the ensemble average technique ordinarly used in problems of this kind. Our averaging technique better corresponds to the experimental situation, where one typically averages over time.

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