ON WATER-WAVE PROPAGATION IN A LONG CHANNEL WITH CORRUGATED BOUNDARIES

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The scattering of surface water-waves in a channel with corrugated bottom and walls is investigated. We study two cases: (1) ordinary wave reflection; (2) scattering into specific "mode" waves, i.e. into oscillations propagating along the channel's axis and having a standing-wave structure in its cross-section. An analytical expression for the damping-due-to-scattering coefficient is obtained. The second type of scattering proved to be more effective.

1. Introduction

Surface-wave scattering in basins with a periodically uneven bottom was thoroughly investigated during recent years, e.g. Refs. [1–6]. In particular, Bragg scattering (occuring when the wavelength of the incident wave is approximately equal to the half-period of the bottom irregularities) was studied in Ref. [4]. An attempt of straightforward generalization of these results was made in Ref. [7], where channels with corrugated walls and bottom were considered. A wave, penetrating into such a channel, is damped due to scattering, its energy being transferred to reflected waves. The practical interest of the author of [7] was concerned with a design of harbour resonators.

We note, that in the case of a *channel* (not "open" basin) there is an additional type of surface-wave oscillations. These oscillations are formed by a pair of ordinary surface waves with wave vectors being equiangular to the channel's axis. The phases of the waves are related in such a way that a standing wave arises in the cross-section of the channel. Such oscillations (we shall call them mode waves) have a dispersion relation which is essentially different from that of ordinary surface waves. As a consequence, resonant (Bragg) scattering into mode oscillations essentially differs from the case which was considered in Ref. [7]. In particular, the scattering of an ordinary surface wave into a first mode appears to be much more effective. This can also be important in view of practical applications of the theory.

In the present paper we investigate scattering of surface waves into surface and first mode in channels with corrugated boundaries. In Section 2 we derive an equation governing Bragg wave scattering of the waves in the channels by means of a multiple-scale method. In Section 3 some solutions of this equation are obtained and analysed.

2. Statement of the problem and the governing equations

We consider surface waves in a channel with an ideal fluid (cf. Fig. 1). The corrugated walls and bottom of the channel are defined by the equalities

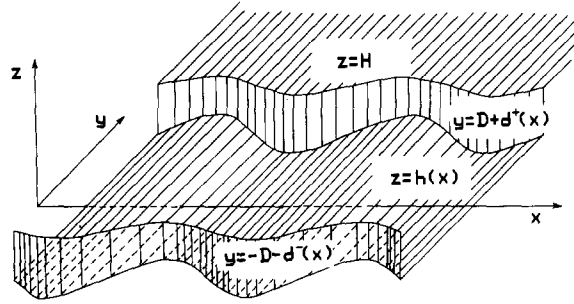$$z = h(x), \qquad y = D + d^+(x), \qquad y = -D - d^-(x)$$

Fig. 1. Statement of the problem.

(the x-axis is directed along the channel's axis, the z-axis is in the upward direction). The mean width of the channel is equal to $2D$, the upper boundary of the fluid is free and situated (in the absense of disturbances) at $z = H$ (H is the mean depth of the channel). The amplitudes of the undulations are small compared with a typical wavelength $\lambda = 2\pi/k$:

$$|kh| \ll 1, \qquad |kd^{\pm}| \ll 1. \tag{1a}$$

Surface waves in the channel are assumed to be small

$$|k\eta| \ll 1, \tag{1b}$$

where η is the elevation of the free boundary. We consider potential motion of the fluid:

$$\Delta\Phi = 0. \tag{2}$$

The inequalities given by eq. (1) enable us to use linearized boundary conditions, which are taken at undisturbed positions of the boundaries of the fluid (both rigid and free). Substituting $h \to \varepsilon h$, $d^{\pm} \to \varepsilon d^{\pm}$ (where $0 \le \varepsilon \le 1$ is a small parameter), we can write down boundary conditions in the form:

$$\Phi_z = \varepsilon[(\Phi_x h)_x + \Phi_{yy}h] \qquad \text{at } z = 0,$$

$$\Phi_{tt} + g\Phi_z = 0 \qquad \text{at } z = H, \tag{3}$$

$$\Phi_y = \pm\varepsilon[(\Phi_x d^{\pm})_x + \Phi_{zz}d^{\pm}] \quad \text{at } y = \pm D;$$

where g is the acceleration due to gravity.

First we shall consider free waves in the channel with even boundaries. In this case $\varepsilon = 0$, and the boundary value problem defined by eqs. (2) and (3) has the following solutions:

$$\Phi = \varphi_{n,k}(y, z) \exp(i\omega t - ikx); \tag{4a}$$

$$\varphi_{n,k} = \frac{\cosh(Kz)}{\cosh(KH)} f_n(y) = \frac{\cosh(Kz)}{\cosh(KH)} \times \begin{cases} 1 & \text{if } n = 0, \\ \sqrt{2}\sin(\varkappa_n y) & \text{if } n \text{ is odd}, \\ \sqrt{2}\cos(\varkappa_n y) & \text{if } n \text{ is even}; \end{cases} \tag{4b}$$

where the factor $\sqrt{2}$ is inserted for convenience. The solution (4a, b) can be fixed by the wavenumber k and the modenumber n, the rest parameters being determined by the formulas:

$$\varkappa_n = \frac{\pi n}{2D}, \qquad K = (k^2 + \varkappa_n^2)^{1/2}, \qquad \omega^2 = gK \tanh(KH). \tag{4c}$$

Figure 2 shows the dispersion curves of these free waves. The zeroth mode ($n = 0$) corresponds to the ordinary surface waves.
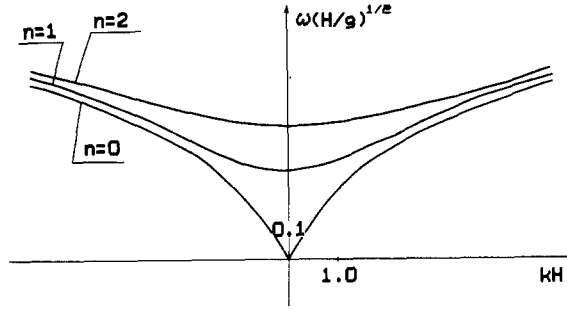

Fig. 2. Dispersion curves for surface waves in a channel, for the particular case $D = H$.

To carry out an asymptotic analysis of the boundary value problem given by eqs. (2) and (3) for the case $\varepsilon \neq 0$ we shall make use of a multiple-scale method (see, e.g. Ref [8]). Accordingly let us, along with the "fast" variables (x, t), introduce a hierarchy of "slow" variables

$$X = \varepsilon x, \qquad T = \varepsilon t, \qquad X_j = \varepsilon^{1+j} x, \qquad T_j = \varepsilon^{1+j} t, \quad j = 1, 2, 3, \ldots$$

The derivatives in (2) and (3) should be changed according to the formulas

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X} + \cdots, \qquad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}.$$

We shall look for the solution in the form of an asymptotic series

$$\Phi = \Phi^{(0)} + \varepsilon \Phi^{(1)} + \cdots.$$

To describe "mutual" scattering of the waves (k_1, n_1) and (k_2, n_2), the zeroth approximation to the potential should be chosen in the form

$$\Phi^{(0)}(x, y, z, t, X, T) = A_1(X, T) \varphi_{n_1, k_1}(y, z) \exp(i\omega_{n_1, k_1} t - ik_1 x)$$

$$+ A_2(X, T) \varphi_{n_2 k_2}(y, z) \exp(i\omega_{n_2, k_2} - ik_2 x). \tag{5}$$

(Here and hereafter X_j and T_j are omitted from formal lists of arguments). We note also that wave amplitudes A_1, A_2 in (5) are slowly varying in space and time.

Apparently, wave scattering in media with stationary perturbations does not change the frequency of a wave:

$$\omega_{n_1, k_1} = \omega_{n_2, k_2} = \omega. \tag{6}$$

Consequently, $K_{n_1, k_1} = K_{n_2, k_2} = K$, and if the wavenumbers of the incident wave k_1 and the mode numbers n_1, n_2 are given, eq. (6) determines the wavenumber of the reflected wave.

It is well-known that a scattering wave "feels" only those Fourier-components of the inhomogeneities which satisfy Bragg resonant conditions (e.g. [4]):

$$p = k_1 - k_2 + \varepsilon(\Delta p), \qquad q^{\pm} = k_1 - k_2 + \varepsilon(\Delta q^{\pm}), \tag{7a}$$

where p, $q^\pm$ are the wavenumbers resonant components due to boundary undulations, and Δp, $\Delta q^\pm$ are the corresponding (small) detuning wavenumbers. Thus, we can assume the undulations to be not only periodic but sinusoidal as well:

$$h = h_0 \cos(px), \qquad d^\pm = d_0^\pm \cos(q^\pm x + \alpha^\pm). \tag{7b}$$

The phase of $h(x)$ can be assumed to be zero without any loss of generality. Equation 7(a) yields

$$px = (k_1 - k_2)x + (\Delta p)X, \qquad q^\pm x = (k_1 - k_2)x + (\Delta q^\pm)X. \tag{7c}$$

We should note that the resonant condition adduced in Refs. [4, 7]:

$$k_2 = -k_1, \qquad p \simeq q^\pm \simeq 2k_1$$

is satisfied only if $n_1 = n_2 = 0$. In the general case

$$k_2 = [(k_1)^2 + (\varkappa_{n_1})^2 - (\varkappa_{n_2})^2]^{1/2}. \tag{8}$$

Equalities (5), (6) and (8) define the zeroth approximation of our perturbation theory. Note that the dependence of $A_{1,2}$ on the slow variables remains undetermined.

In the next order of the perturbation theory we seek $\Phi^{(1)}$ in the form

$$\Phi^{(1)}(x, y, z, t, X, T) = e^{i\omega t}[\Psi(y, z, X, T) e^{-ik_1 x} + \chi(y, z, X, T) e^{-ik_2 x}$$

$$+ \Psi'(X, T, y, z) e^{-i(2k_1 - k_2)} + \chi'(y, z, X, T) e^{-i(2k_2 - k_1)}]. \tag{9}$$

Substituting (5), (7) and (9) into eqs. (2) and (3), and equating coefficients in front of the exponents with equal indices, we obtain four boundary value problems determining Ψ, χ, Ψ' and χ' as functions of y and z. For example, Ψ satisfies Helmholtz equation with a right-hand side:

$$\Psi_{yy} + \Psi_{zz} - (k_1)^2 \Psi = 2ik_i \frac{\partial A_1}{\partial X} \varphi_1 \tag{10a}$$

and the following boundary conditions

$$\Psi_z = -\tfrac{1}{2}h_0[(\varkappa_{n_2})^2 + k_1 k_2]A_2\varphi_2 \, e^{-i(\Delta p)X} \qquad \text{at } z = 0$$

$$g\Psi_z - \omega^2\Psi = -2i\omega \frac{\partial A_1}{\partial T} \varphi_1 \qquad \text{at } z = H \tag{10b}$$

$$\Psi_y = \pm\tfrac{1}{2}d_0^\pm(K^2 - k_1 k_2)A_2\varphi_2 \, e^{-i(\Delta q^\pm)X} \qquad \text{at } y = \pm D$$

where $\varphi_j = \varphi_{n_j, k_j}$; $j = 1, 2$. Since the parameters (k_1, ω) are the eigenvalues of the homogeneous boundary value problem corresponding to eq. (10), the system (10) can be solved only under certain conditions. In order to determine these conditions, we multiply (10a) by $\varphi_1(y, z)$ and integrate over the domain $(-D < y < D, 0 < z < H)$. Using Green's formula and the boundary conditions (10b) (or simply integrating by parts) we obtain an equation for A_1:

$$i\left(\frac{\partial A_1}{\partial T} + c_1 \frac{\partial A_1}{\partial X}\right) = A_2\{Uh_0 \exp[-i(\Delta p)X]$$

$$+ V^+ d_0^+ \exp[-i(\Delta q^+)X - i\alpha^+] + V^- d_0^- \exp[-(\Delta q^-)X - i\alpha^-]\}. \tag{11a}$$

The analogous procedure for χ yields:

$$i\left(\frac{\partial A_2}{\partial T} + c_2 \frac{\partial A_2}{\partial X}\right) = A_1\{Uh_0 \exp[i(\Delta p)X]$$

$$+ V^+ d_0^+ \exp[i(\Delta q^+)X + i\alpha^+] + V^- d_0^- \exp[i(\Delta q^-)X + i\alpha^-]\}. \tag{11b}$$

Here $c_{1,2}$ is the group velocity of surface waves: $c_{n,k} = \mathrm{d}\omega_{n,k}/\mathrm{d}k$. The scattering coefficients U and V have the forms

$$U = \frac{g}{4\omega} \frac{(\varkappa^2 + k_1 k_2)}{\cosh^2(KH)} \delta_{n_1, n_2},$$ (12a)

where δ_{n_1, n_2} is the Kronecker delta, so that $\varkappa_{n_1} = \varkappa_{n_2} = \varkappa$, and

$$V^{\pm} = \left(\frac{c_1 c_2}{k_1 k_2}\right)^{1/2} \frac{K^2 - k_1 k_2}{8D} (f_{n_1} f_{n_2})\bigg|_{y=\pm D}$$ (12b)

Here $f_n(y)$, $\varkappa_n$ and K are defined by the relations (4b, c). Terms proportional to U correspond to wave scattering due to bottom irregularities, and terms proportional to $V^{\pm}$ describe wave scattering due to boundary undulations. We remark that the expressions (12) for the scattering coefficients in the particular case $n_1 = n_2 = 0$ coincide with the corresponding results of Refs. [4, 7]. Equations (11) form the desired system governing the slow evolution of the amplitudes of the incident and reflected waves.

3. Stationary scattering of surface waves

Let us consider a gravity wave whose front, which is normal to the channel's axis, arrives at the channel's entrance from the left. If the transitional diffraction effects are negligible the further evolution of the wave is determined by the system (11). One can see that scattering of the incident wave by the undulations of the channel's boundaries causes the appearance of a reflected wave and, consequently, damping of the incident wave. We consider a stationary case, when (11) can be reduced to

$$ic_1 \frac{\partial A_1}{\partial X} = WA_2, \qquad ic_2 \frac{\partial A_2}{\partial X} = W^* A_1;$$ (13)

where an asterisk denotes the complex conjugate. We are interested in the case of "maximum" reflection corresponding, apparently, to the equality $\Delta p = \Delta q^{\pm} = 0$. Let us also set $n_1 = 0$ (the incident wave is an ordinary surface wave). We remark that the presence of the Kronecker delta in eq. (12a) shows that the bottom undulations do not participate in intermode wave scattering (this fact is quite natural from the physical point of view). Since we are interested just in the intermode interaction, we assume $h_0 = 0$ as well as (for simplicity) $d_0^+ = d_0^- = d_0$. All these simplifications yield

$$W = \frac{d_0}{8D} \left(\frac{c_1 c_2}{k_1 k_2}\right)^{1/2} (K^2 - k_1 k_2)(e^{-i\alpha^+} f_{n_2}|_{y=D} + e^{-i\alpha^-} f_{n_2}|_{y=-D}).$$

The solution of eq. (13) has the form $A_{1,2} \sim e^{-\gamma X}$, where

$$\gamma = |W|(-c_1 c_2)^{-1/2}$$ (14)

is the desired damping coefficient of the incident wave. One can see that γ is a real number because the directions of the group velocities of the incident and the reflected waves are opposite ($c_1 c_2 < 0$). The final expressions for γ are

$$\gamma = \begin{cases} \dfrac{d_0}{2D} k_1 & \text{if } n_2 = 0 \\[2mm] \dfrac{d_0}{2D} k_1 \dfrac{(k_1 - k_2)}{(-2k_1 k_2)^{1/2}} & \text{if } n_2 \neq 0 \end{cases}$$ (15a)

$$k_2 = \left[k_1^2 - \left(\frac{\pi n_2}{2D}\right)^2\right]^{1/2}.$$ (15b)

These results give the maximum values of γ, corresponding to the following values of $\alpha^{\pm}$: $\alpha^{\pm} = \pm\pi/2$ if n_2 is an odd number and $\alpha^{\pm} = 0$ if n_2 is an even number.

Let us now discuss the applicability of these results. One can see that the condition $|\Phi^{(0)}| \gg \varepsilon|\Phi^{(1)}|$ after substitution of $\Phi^{(0)}$ and $\Phi^{(1)}$ entails the inequality $\varepsilon(kD) \ll 1$ (we remind that $\varepsilon \sim kd$). Thus, the expressions (15) become incorrect in the short-wave limit (when $k \gtrsim (dD)^{-1/2}$). This restriction remained unnoticed in Ref. [7].

We should also remark that if $k_1 \to \varkappa_{n_2}(k_2 \to 0)$ the damping coefficient tends to infinity $\gamma \to \infty$ (cf. (15b)). This follows from the vanishing of the group velocity of the wave mode when $k_2 \to 0$ (cf. (14) and Fig. 2). Accordingly, we have to take into account a next-order term in (11b), proportional to $\partial^2 A_2/\partial X^2$. The proportionality coefficient can be easily found by analogy with similar equations governing evolution of wave envelopes (e.g. [9–11]) as $\Omega = \frac{1}{2}\mathrm{d}^2\omega_{n,k}/\mathrm{d}k^2$. As a result a more accurate variant of the stationary equations (11) for the case $|k_2| \ll |k_1|$ has the form

$$\mathrm{i}c_1\frac{\partial A_1}{\partial X} = WA_2, \qquad \mathrm{i}c_2\frac{\partial A_2}{\partial X} + \Omega\frac{\partial^2 A_2}{\partial X^2} = W^*A_1. \tag{16a}$$

In order to avoid unnecessary accuracy, we should also substitute $k_1 = K = \varkappa_{n_2}$, $k_2 = 0$ in all the terms, except the ones that are proportional to c_2:

$$\Omega = \frac{g}{4\omega\varkappa_{n_2}}\left[\frac{\varkappa_{n_2}H}{\cosh^2(\varkappa_{n_2}H)} + \tanh(\varkappa_{n_2}H)\right]$$

$$W = \frac{d_0}{8D}\left(\frac{2c_1\Omega}{\varkappa_{n_2}}\right)^{1/2}(\varkappa_{n_2})^2(\mathrm{e}^{-\mathrm{i}\alpha^+}f_{n_2}|_{y=D} + \mathrm{e}^{-\mathrm{i}\alpha^-}f_{n_2}|_{y=-D}) \tag{16b}$$

(a rigorous asymptotic derivation of eq. (16) is quite analogous to that of eq. (13), differing only by substitution $k_2 \to \varepsilon k_2$). Accordingly, for γ we have:

$$\mathrm{i}c_1c_2\lambda^2 - \Omega c_1\lambda^3 = -\mathrm{i}|w|^2, \qquad \gamma = \mathrm{Re}\,\lambda; \tag{17}$$

where the appropriate root should be fixed by the inequality $\mathrm{Re}\,\lambda > 0$. Equation (17) complements the expression (15) at $k_1 \simeq \varkappa_{n_2}$. The dependence of γ on the wavenumber of the incident wave k_1 is shown in Fig. 3. One can see that *scattering into mode waves* ($n_2 \neq 0$) *is more effective than that into an ordinary surface wave* ($n_2 = 0$) (of course, if the former is not forbidden – for $k_1 \gtrsim \varkappa_{n_2}$). This is the main result of the present paper.
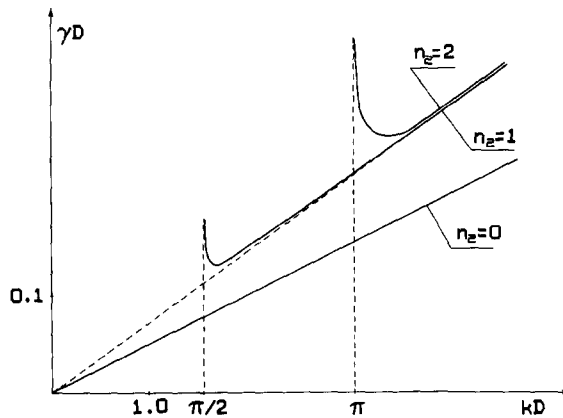

Fig. 3. Damping coefficient versus wave number of the incident surface wave $n_1 = 0$, $d/D = 0.1$.

4. Conclusions

We have considered surface-wave propagation in a channel with corrugated boundaries. It was shown that the waves fade (due to backward Bragg reflection) and the damping coefficient attains its maximum value at scattering into mode waves, i.e. into oscillations propagating along the channel's axis and having a standing-wave structure in the cross-section of the channel. The damping coefficient is determined by the formulas (15) (for $k_1 > \varkappa_{n_2}$) and by eq. (17) (at $k_1 \simeq \varkappa_{n_2}$).

In the present paper we have considered only the particular case when the three types of inhomogeneities of the channel (corrugated walls and bottom) were all "tuned" to the only resonance (6). it is clear that calculations of three or more resonances do not differ principally from the case considered, but in this situation the channel becomes "opaque" in a wide range of the incident waves' wavenumbers. Four or more resonances assume boundary irregularities to comprise more than one Fourier component. This fact can be taken into account in practical applications of the theory.

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