On the Surface Waves in a Shallow Channel with an Uneven Bottom

By Eugene S. Benilov

This paper examines surface gravity waves in a shallow channel with periodic or random bottom irregularities. Three types of bottom topography are distinguished, allowing a simplification of the basic shallow-water-wave equations. For two of them, asymptotic equations of the Korteveg–de Vries type are derived (the third type has been considered earlier by other authors).

1. Introduction

The nondimensional potential Ψ of long weakly nonlinear surface waves in a narrow channel with an uneven bottom satisfies the following equation:

$$\Psi_{tt} - (H\Psi_x)_x + \varepsilon^2\left(2\Psi_x\Psi_{xt} + \Psi_t\Psi_{xx} - \tfrac{1}{3}H^3\Psi_{xxxx}\right) + \;= 0, \tag{1}$$

where t and x are the time and horizontal distance variables, $H(x)$ is the nondimensional channel depth, and $\varepsilon \ll 1$ is the small parameter characterizing weakness of the effects of nonlinearity and dispersion:

$$\varepsilon^2 \sim \frac{A}{H_o} \sim \left(\frac{H_o}{\lambda}\right)^2.$$

(Here A and λ are the wave amplitude and the wavelength, respectively; H_o is the mean value of the depth of the channel.) Equation (1) was derived in [11] for the case of *smooth* bottom irregularities and, clearly, is still valid in all cases where $H_x = o(1)$ (for example, for *small* irregularities).

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One can see that (1) is a nonlinear equation with variable coefficients and, in spite of the existence of a small parameter, its analysis is a rather complicated problem. Nevertheless, three particular types of bottom irregularities exist for which (1) can be simplified:

$$H = 1 + \varepsilon h(x), \tag{2.1}$$

$$H = H(\varepsilon x), \tag{2.2}$$

$$H = H(\varepsilon^2 x). \tag{2.3}$$

There is a strong distinction between the cases (2.1 and 2.2) and (2.3), which can be understood if we introduce a characteristic spatial scale of nonlinearity and dispersion X_{nd}, i.e., the distance that a wave disturbance has to travel before being influenced by these effects. Evidently, since the nonlinear and the dispersive terms in Equation (1) are of the order of ε^2, $X_{nd} \sim \varepsilon^{-2}$. In addition to X_{nd}, the system contains one more spatial scales, namely, the characteristic length of a typical bottom obstacle: ${}^1X_i \sim 1$, ${}^2X_i \sim \varepsilon^{-1}$, ${}^3X_i \sim \varepsilon^{-2}$; where the superscript corresponds to the enumeration of the types of bottom irregularities (2). Hence, in the cases (2.1 and 2.2), there are additional small parameters characterizing the weakness of the dispersion-nonlinearity-inhomogeneity interaction: $X_{nd}/{}^1X_i \sim \varepsilon^2$, $X_{nd}/{}^2X_i \sim \varepsilon$. Accordingly, if the wave interacts with a *solitary* bottom obstacle, the effects of nonlinearity and dispersion have not enough time to influence the wave significantly. A solitary obstacle of the type (2.1) (*small* irregularity) causes a weak reflected wave, whereas that of the type (2.2) (*smooth* irregularity) gradually transforms the incident wave so that, having passed the obstacle, it regains its initial form. That is why only *spatially homogeneous* (periodic or random) bottom irregularities (2.1 and 2.2) can affect the wave strongly. The influence of such inhomogeneities is continuous and the characteristics of each elementary "act" of scattering are not important.

Most of the papers dealing with nonlinear surface waves in shallow channels with corrugated boundaries were devoted to the case (2.3) (compare [10, 14] and references therein). The only two papers [7, 15], dealing with the small (random) irregularities (2.1) were criticized in [1, 2, 6], and the question of the validity of their results has remained unclear until the present. Smooth irregularities (2.2) in the context of *nonlinear* waves have not been studied at all.

In Section 2 of the present paper, the *periodic* bottom irregularities (2.1 and 2.2) are considered. These irregularities are the most advantageous with regard to the application of rigorous asymptotic methods. With the help of a multiple-scale technique, two equations of the Korteveg-de Vries type are derived, describing the unidirectional waves traveling over the bottom irregularities (2.1 and 2.2). In Section 3, the results obtained are generalized for the case of *random* bottom topography, and in the instance of *small* irregularities are compared to those obtained in [7, 15]. In Section 4, the elementary analysis of the equations derived is presented.

2. Periodic bottom irregularities

Since the speed of long water waves depends upon the depth of the fluid (it is proportional to $H^{1/2}$), the phase (location) of a wave, traveling over the bottom irregularities, also fluctuates. In some cases, these fluctuations can hamper the application of direct asymptotic methods (compare [1, 2]) and, accordingly, should be "filtered out" by the following change of the spatial variable:

$$y = \int_{x_o}^{x} H^{-1/2}(x')\,dx'. \tag{3.1}$$

Then, the stationary reference system (y, t) should be changed for the moving one (y, ϑ):

$$\vartheta = y - t. \tag{3.2}$$

The transformations (3) have clear physical meaning: (3.1) makes the wave speed *uniformly* equal to unity, whereas (3.2) turns it into zero.

Substituting (3) into (1), we obtain

$$2\Psi_{\vartheta y} + \Psi_{yy} + \tfrac{1}{2}H^{-1}H_y(\Psi_y + \Psi_\vartheta)$$
$$+ \varepsilon^2\left[2H^{-1}(\hat{D}\Psi)(\hat{D}\Psi)_\vartheta + H^{-1}\Psi_\vartheta(\hat{D}^2\Psi) + \tfrac{1}{3}H(\hat{D}^4\Psi) + \cdots\right] = 0, \tag{4}$$

where $\hat{D} = \partial/\partial y + \partial/\partial\vartheta$ (it should be recalled that terms $\sim \varepsilon^2 H_y$ can be dropped—they are small for all types of bottom irregularities (2): $H_y = H^{1/2}H_x = o(1)$).

As seen earlier, the problem contains two characteristic spatial scales; correspondingly, we shall use an asymptotic method of multiple scales. Small bottom irregularities (2.1) will be considered first.

2.1. Small bottom irregularities

Along with the "fast" spatial variable y, we shall introduce a hierarchy of "slow" ones: $Y = \varepsilon^2 y$; $Y = \varepsilon^{2+m}y$, $m = 1,2,3\ldots$. Accordingly, spatial derivatives in (4) should be transformed as follows:

$$\frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \varepsilon^2\frac{\partial}{\partial Y} + \sum_{m=1}^{\infty} \varepsilon^{2+m}\frac{\partial}{\partial Y_m}. \tag{5}$$

Rewriting (2.1) as $H = 1 + \varepsilon h(y)$ and substituting it and (5) into (4), we shall seek the solution in the form of a series

$$\Psi(\vartheta, y, Y) = \Psi^{(o)}(\vartheta, Y) + \varepsilon\Psi^{(1)}(\vartheta, y, Y) + \varepsilon^2\Psi^{(2)}(\varepsilon, y, Y) + \cdots,$$

where Y_m are omitted from the formal lists of arguments. The leading-order term of this expansion describes a wave disturbance going to the right at near-unity speed. Equating the terms $\sim \varepsilon$, we obtain

$$2\Psi^{(1)}_{\vartheta y} + \Psi^{(1)}_{yy} = -\tfrac{1}{2} h_y \Psi^{(o)}_{\vartheta}. \tag{6.1}$$

Equation (6.1) should be supplemented with the conditions of the absence of the back-scattered wave at $y \to \infty$ (in front of the incident wave):

$$\Psi^{(1)}\Big|_{\substack{2y-\vartheta=\mathrm{const}\\ y\to\infty}} = 0 \tag{6.2}$$

and the absence of the forward-scattered wave at $y \to -\infty$:

$$\Psi^{(1)}\Big|_{\substack{\vartheta=\mathrm{const}\\ y\to -\infty}} = 0. \tag{6.3}$$

(Here the so-called *adiabatic turning on* of the depth variations is implied, i.e., the Goursat problem (6) is to be solved after replacing $h(y) \to h(y)e^{\nu y}$, $0 \le \nu \ll 1$. When the solution is obtained, ν can be turned into zero.) The solution to (6)

$$\Psi^{(1)} = \tfrac{1}{2}\int_0^{\infty} \Psi^{(o)}_{\vartheta}(\vartheta + 2\tau, Y)\, h(y+\tau)\, d\tau \tag{7}$$

describes a small-amplitude reflected wave going to the left.

The next ($\sim \varepsilon^2$) order of the perturbation theory yields an equation that, in terms of $\chi = 2\Psi^{(2)}_{\vartheta} + \Psi^{(2)}_{y}$, has the following form:

$$\chi_y = F(\vartheta, y, Y), \tag{8.1}$$

$$F = \tfrac{1}{2} h h_y \Psi^{(o)}_{\vartheta} - \tfrac{1}{2} h_y \left[\Psi^{(1)}_y + \Psi^{(1)}_{\vartheta}\right] - 2\Psi^{(o)}_{\vartheta Y} - 3\Psi^{(o)}_{\vartheta}\Psi^{(o)}_{\vartheta\vartheta} - \tfrac{1}{3}\Psi^{(o)}_{\vartheta\vartheta\vartheta\vartheta}. \tag{8.2}$$

Evidently, the periodicity of $h(x)$ entails the periodicity of $h(y)$ and, consequently, $F(y)$. As a result, χ can be bounded at $y \to \pm\infty$ only if the averaged-over-one-period value of F is equal to zero:

$$\langle F(y) \rangle \equiv (2p)^{-1} \int_{-p}^{p} F(y, Y, \vartheta)\, dy = 0, \tag{9}$$

where $2p$ is the period of $h(y)$. Substitution of (7) and (8.2) into (9) yields an asymptotic equation governing the slow evolution of $\Psi^{(o)}(\vartheta, Y)$ (the index $^{(o)}$ is omitted):

$$2\Psi_{\vartheta Y} + 3\Psi_{\vartheta}\Psi_{\vartheta\vartheta} + \frac{1}{3}\Psi_{\vartheta\vartheta\vartheta\vartheta} = \frac{1}{4}\zeta^2\Psi_{\vartheta\vartheta} + \frac{1}{2}\int_0^{\infty}\Psi_{\vartheta\vartheta\vartheta\vartheta}(\vartheta + 2\tau, Y)W(\tau)\,d\tau, \quad (10)$$

where

$$W(\tau) = \langle h(y)h(y+\tau)\rangle \quad (11)$$

and $\zeta^2 = W(0) \equiv \langle h^2(y)\rangle$. In obtaining (10) we used the equalities

$$\langle h_y(y)h(y+\tau)\rangle = -W_{\tau}(\tau), \qquad \langle h_y(y)h_y(y+\tau)\rangle = -W_{\tau\tau}(\tau)$$

and the condition $\langle h(y)\rangle = 0$ (the latter verifies the condition: $W(\tau) \to 0$ when $\tau \to \infty$).

The last issue to be discussed here is the relationship between $h(y)$ and $h(x)$. To clarify it, (2.1) should be substituted into (3.1) and then expanded in a series:

$$y = (x - x_o) + \varepsilon\int_{x_o}^{x} h(x')\,dx' + \cdots. \quad (12)$$

Clearly, if the period of h is not too large—for example, $p = O(1)$; (12) can be reduced to $y = (x - x_o) + o(1)$ for $(x - x_o) \le 2p$. Thus, in the equality (11), $h(y)$ can be replaced with $h(x)$.

2.2. *Smooth irregularities*

In the case of smooth irregularities (2.2), Equation (4) should be rewritten in terms of the natural spatial variable $y' = \varepsilon y$ (primes are omitted)

$$2\Psi_{\vartheta y} + \frac{1}{2}H^{-1}H_y\Psi_{\vartheta} + \varepsilon\left(\Psi_{yy} + \frac{1}{2}H^{-1}H_y\Psi_y + 3H^{-1}\Psi_{\vartheta}\Psi_{\vartheta\vartheta} + \frac{1}{3}H\Psi_{\vartheta\vartheta\vartheta\vartheta}\right) = o(\varepsilon). \quad (13)$$

Note that if we omitted in (13) *all* small terms (what corresponds to the geometrical-optical approximation), we would obtain the following relationship between the wave amplitude and the channel depth:

$$\Psi_{\vartheta} \sim H^{-1/4}.$$

As earlier, we should introduce the slow spatial variables $Y = \varepsilon y$; $Y_m = \varepsilon^{m+1} y$, $m = 1, 2, \ldots$:

$$\Psi = H^{-1/4}(y)\left[\Psi^{(o)}(\vartheta, Y) + \varepsilon Y^{(1)}(\vartheta, y, Y) + \cdots\right].$$

The subsequent calculations are rather straightforward and can be omitted. The condition of $\Psi^{(1)}$ being bounded at $y \to \pm\infty$ yields the equation governing the slow evolution of $\Psi^{(o)}$ (index $^{(o)}$ is dropped):

$$2\Psi_{Y\vartheta} + 3\alpha\Psi_{\vartheta}\Psi_{\vartheta\vartheta} + \tfrac{1}{3}\beta\Psi_{\vartheta\vartheta\vartheta\vartheta} = \gamma\Psi, \tag{14}$$

where

$$\alpha = \left\langle H^{-5/4}(y)\right\rangle, \qquad \beta = \left\langle H(y)\right\rangle, \qquad \gamma = 1/1\sigma\left\langle H^{-2}(y)\left[H_y(y)\right]^2\right\rangle.$$

The dependence of α, β and γ on $H(x)$ can be easily found with the help of the formula $\langle f(y)\rangle = \langle H^{-1/2}(x)\rangle \cdot \langle f(x) H^{-1/2}(x)\rangle$ (the first factor here is equal to the ratio of the period of $f(y)$ to the period of $f(x)$).

It is remarkable that (14) coincides with the Ostrovskiy equation [12] (compare also [3–5, 8]), describing long internal waves in a rotating basin with *even* boundaries.

3. Random bottom irregularities

From the mathematical point of view, the periodic case can be reduced to the case of random irregularities if the period of h tends to ∞. Hence, Equations (10) and (14) remain valid, provided the averaging $\langle\cdot\rangle$ is understood as the conventional spatial averaging, for example:

$$W(\tau) = \lim_{p\to\infty} (2p)^{-1} \int_{-p}^{p} h(y) h(y+\tau)\, dy.$$

If the correlation radius of $h(x)$ is not too big, for example $R = O(1)$, $h(y)$ can be replaced in this formula with $h(x)$ (as has been done in the periodic case).

As mentioned previously, small random irregularities were considered earlier in [7, 15]—*and Equation* (10) *differs significantly from the corresponding equations derived in those papers.*

To understand the reason for this discrepancy, Equation (10) should be "rederived" without making use of the transformation (3.1) (this approach would be equivalent to the asymptotical methods used in [7, 15]). As a result, the modified expression for the scattered wave field will contain one supplementary term (compare the following equality with (7)):

$$\Psi^{(1)} = \tfrac{1}{2}\int_{o}^{\infty} \Psi_{\vartheta}^{(o)}(\vartheta + 2\tau, X)h(x+\tau)\,d\tau$$

$$- \tfrac{1}{2}\Psi_{\vartheta}^{(o)}(\vartheta, X)\int_{-\infty}^{o} h(x+\tau)\,d\tau, \tag{15}$$

where $X = \varepsilon^2 x$ and the convergence of the improper integral in the last term is provided by the adiabatic turning on of the inhomogeneities. Then, instead of (10), we obtain the following equation governing the slow evolution of $\Psi^{(o)}$ (index $^{(o)}$ is omitted):

$$2\Psi_{\vartheta X} + 3\Psi_{\vartheta}\Psi_{\vartheta\vartheta} + \tfrac{1}{3}\Psi_{\vartheta\vartheta\vartheta\vartheta}$$

$$= \zeta^2\Psi_{\vartheta\vartheta} + \tfrac{1}{2}\int_{o}^{\infty} \Psi_{\vartheta\vartheta\vartheta}(\vartheta + 2\tau, X)W(\tau)\,d\tau + \tfrac{1}{2}\Psi_{\vartheta\vartheta\vartheta}\int_{-\infty}^{o} W(\tau)\,d\tau. \tag{16}$$

This equation *is* asymptotically equivalent to that derived in [7, 15], but it has one extra [in comparison to (10)] term in its right-hand side.[1] This difference is important, since the supplementary term has a diffusive structure ($\Psi_X \sim \Psi_{\vartheta\vartheta}$) and can cause strong wave damping. It should be emphasized that in the case of periodic bottom irregularities, the coefficient in front of this term is equal to zero[2]—and Equations (11) and (16) coincide.

To find out which equation—(10) or (16)—is valid for the case of random irregularities, we should calculate the mean value of the scattered-wave intensity

$$I = \left\langle [\Psi^{(1)}]^2 \right\rangle \tag{17}$$

for the case of "unfiltered" phase fluctuations. After the substitution of (15) into (17), the most "dangerous" contribution is given by the second term of (15):

$$I = \tfrac{1}{4}\Psi^2(\vartheta, X)\int_{-\infty}^{o}\int_{-\infty}^{o} \left\langle h(x+\tau_1)h(x+\tau_2)\right\rangle d\tau_1\,d\tau_2 + \cdots$$

$$= \tfrac{1}{4}\Psi^2(\vartheta, X)\int_{-\infty}^{o}\int_{-\infty}^{o} W(\tau_1 - \tau_2)\,d\tau_1\,d\tau_2 + \cdots$$

[1] The coefficients in front of the first terms in the right-hand sides of (10) and (16) also do not coincide, which can be explained by the difference between the reference systems (Y, ϑ) and (X, ϑ) (the former is moving at a nonzero mean speed).

[2] In this case, $W(\tau)$ is a periodic function with zero mean value.

and the improper integral here diverges for any $W(\tau)$ except $\int_{-\infty}^{\infty} W(\tau) = 0$. Thus, the perturbation theory, yielding Equation (16), contains a divergency. It is remarkable that the divergency is absent in the only case when (16) coincides with the correct equation (10).

To understand the nature of the divergency, we should substitute (2.1) into (3.1) and then calculate the phase fluctuation of a wave coming from minus infinity $x_o = -\infty$:

$$\delta\vartheta = \delta\left(\int_{-\infty}^{x} \left[1+\varepsilon h(x')\right]^{-1/2} dx' - t\right) = -\tfrac{1}{2}\varepsilon \int_{-\infty}^{x} h(x')\,dx'$$

$$= -\tfrac{1}{2}\varepsilon \int_{-\infty}^{o} h(x+\tau)\,d\tau.$$

Now the physical meaning of the second term in the expression for the scattered field (15) is clear: it describes phase fluctuations of the incident wave:

$$\Psi = \Psi^{(o)}(\vartheta+\delta\vartheta, X) + \Psi_{\text{bsc}} \approx \Psi^{(o)}(\vartheta, X) + \left[\Psi_{\text{bsc}} + \delta\vartheta\cdot\Psi_{\vartheta}^{(o)}(\vartheta, X)\right],$$

where Ψ_{bsc} stands for the back-scattered-wave field [compare this expression with (15)]. Evidently, owing to the divergency of wave phase fluctuations, any correct asymptotic theory has to "filter" them out with the help of some transformation of variables [apparently, the transformation (3.1) is not unique]. We shall not dwell on this question in detail but note only that more information on this matter can be found in the papers [1, 2] dealing with similar problems.

Thus, Equations (10) and (14) are uniformly applicable for the cases of periodic and random bottom topography.

4. Elementary properties of the equations derived

It is more convenient to rewrite Equations (10) and (14) in terms of the velocity of the fluid $u = \Psi_{\vartheta}$. Then, the first term in the right-hand side of (10) and all the coefficients in both equations can be eliminated:

$$u_Y = uu_{\vartheta} + u_{\vartheta\vartheta\vartheta} = \int_{o}^{\infty} u_{\vartheta\vartheta}(\vartheta+2\tau, Y)\,W(\tau)\,d\tau, \tag{18}$$

$$u_Y + uu_{\vartheta} + u_{\vartheta\vartheta\vartheta} = -\int_{\vartheta}^{\infty} u(\vartheta', Y)\,d\vartheta' \tag{19}$$

In obtaining (19), we have taken into account that for the right-going waves,

$$u|_{\vartheta\to\infty} = 0.$$

Either equation should be supplemented with the initial condition

$$u|_{Y=O} = u(\vartheta, O).$$

Equation (18) will be discussed first.

4.1. *Equation (18)*

As it will be seen later, the solutions to (18) strongly depend on the function $W(\tau)$. Accordingly, as an example of periodic irregularities, we shall consider sinusoidal bottom topography and then discuss the case of random irregularities with $W(\tau)$ monotonically decreasing at $(0, \infty)$.

4.1.1 Let $h = \sqrt{2}\,\zeta\cos(qx)$, where $q = \pi/p$. Correspondingly, $W = \zeta^2\cos(q\tau)$ and, after the introduction of an auxiliary function $v(\vartheta, \mathrm{Y})$:

$$4v_{\vartheta\vartheta} + q^2 v = -2u_{\vartheta}, \qquad (v, v_{\vartheta})|_{\vartheta\to\infty} = 0, \tag{20.1}$$

the integral term in (18) can be transformed into a differential one:

$$u_Y = uu_{\vartheta} + u_{\vartheta\vartheta\vartheta} = \zeta^2 v_{\vartheta\vartheta}. \tag{20.2}$$

(This trick also works when $h(x)$ can be represented in the form of a finite sum of sines and cosines). One of the main characteristics of a wave system is its dispersion relationship for small-amplitude waves. To determine it for the system (20), we should neglect the nonlinear term in (20.2) and set

$$(u, v) = (u_o, v_o)\exp[i\omega Y - i(k - i0)\vartheta], \tag{21}$$

where the term $i0$ makes the (right-going) wave (21) vanish at $\vartheta\to\infty$. Substitution of (21) into (20) yields

$$\omega = k^3 + \tfrac{1}{4}\zeta^2 k^2\left(\frac{1}{k + \frac{1}{2}q - i0} - \frac{1}{k - \frac{1}{2}q - i0}\right), \tag{22}$$

where

$$\frac{1}{k \pm \frac{1}{2}q - i0} \equiv \hat{R}\left(\frac{1}{k \pm \frac{1}{2}q}\right) + i\pi\,\delta\left(k \pm \tfrac{1}{2}q\right),$$

$\delta(k)$ is the Dirac delta-function and the operator $\hat{R}(1/k)$ is defined by the equality $\hat{R}(1/k)f(k) \equiv \text{v.p.} \int_{-\infty}^{\infty} k^{-1} f(k)\,dk$. The dispersion relationship (22) contains two δ-functions with pure imaginary coefficients, corresponding to the *instantaneous* vanishing of the wave with wave number equal to $\frac{1}{2}q$. This property of Equation (20) has a clear physical reason: the reflection of the wave with $k = \frac{1}{2}q$ is resonant (compare [9]) and, therefore, very fast (in comparison with the characteristic time scale of the evolution of other waves). The energy loss caused by backward scattering makes the "undirectional" Equation (20) nonconservative.

Knowledge of the dispersion relationship of linear waves permits us to derive some conclusions concerning the evolution of nonlinear solutions to (20). In particular, all *periodic* waves with wavelengths

$$\lambda = \frac{2\pi}{\frac{1}{2}q} m \equiv 4mp, \qquad m = 1,2,3\cdots \tag{23}$$

damp (although not instantly), whereas other periodic solutions tend to become stationary. This statement can be ascertained with the help of the Fourier transformation of the solution:

$$\tilde{u}(k) = (2\pi)^{-1}\int_{-\infty}^{\infty} u(x)\, e^{ikx}\, dx = \sum_{m=-\infty}^{\infty} \tilde{u}_m \delta(k - 2\pi m/\lambda). \tag{24}$$

Evidently, if $\lambda \neq 4mp$, the "damping" harmonic with $k = \frac{1}{2}q$ in the series (24) is skipped. If the condition (23) holds, this harmonic is present and (owing to the nonlinear wave interaction) begins to "pump out" energy from other Fourier-components of the wave field. The behavior of solitons is quite similar; in this case the solution has a continuous Fourier spectrum and the resonant harmonic cannot be skipped.

4.1.2 In the case of random bottom irregularities, the dispersion relationship of Equation (18)

$$\omega = -k^3 + ik^2 \int_0^{\infty} W(\tau)\, e^{-2ik\tau}\, d\tau$$

has a definitely dissipative character ($\operatorname{Im}\omega > 0$ for all $W(\tau)$, decreasing at $(0,\infty)$). Furthermore, if the correlation radius of bottom irregularities is small in comparison with the characteristic wave length (the case of short-scale irregularities), we can approximate $W(\tau)$ by a δ-function: $W(\tau) = \zeta^2\delta(\tau)$, and (18) turns into the well-known Korteveg-de Vries-Burgers equation

$$u_Y + uu_\vartheta + u_{\vartheta\vartheta\vartheta} = \zeta^2 u_{\vartheta\vartheta}.$$

In the opposite limiting case (large-scale, or *smooth* irregularities), the integral term in (18) can be expanded in a series. Then, truncating the series, we obtain

$$u_Y + uu_\vartheta + u_{\vartheta\vartheta\vartheta} = -\tfrac{1}{2}\zeta^2 u_\vartheta + \tfrac{1}{8} W_{\vartheta\vartheta}(0) \int_\vartheta^{\infty} u(\vartheta', Y)\, d\vartheta'.$$

Naturally, this equation can be easily reduced to Equation (19).

4.2. Equation (19)

Equation (19) conserves the energy invariant

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\left[3u^3 - (u_\vartheta)^2 + \Psi^2\right] d\vartheta$$

(where $\Psi = -\int_\vartheta^\infty u(\vartheta', Y)\, d\vartheta'$) and therefore can be written in the Hamiltonian form

$$u_Y = -\frac{\partial}{\partial\vartheta}\frac{\delta E}{\delta u}.$$

Indeed, in the case of *smooth* inhomogeneities of wave medium, the amplitude of the reflected wave is exponentially small and the energy losses due to scattering "disappear" within the framework of power expansion of the wave field in all orders of the perturbation theory.

Another rare feature of Equation (19) is associated with the mass invariant: one can see that a solution to this equation does not grow at $\vartheta \to -\infty$, only if its mass is equal to zero:

$$\int_{-\infty}^{\infty} u\, d\vartheta = 0. \tag{25.1}$$

(This condition has been discussed in [3–5, 8].) The equality (25.1) should be understood as a constraint of the allowed initial conditions $u(\vartheta, 0)$, and it is not the only constraint. As a matter of fact, there is an infinite series of analogous equalities severely limiting the ensemble of localized solutions to Equation (19). In particular, the next constraint of this series can be obtained via integration of Equation (19) over the interval $(-\infty, \infty)$:

$$\frac{\partial}{\partial Y}\int_{-\infty}^{\infty} u(\vartheta, Y)\, d\vartheta = -\int_{\infty}^{\infty}\int_{\vartheta}^{\infty} u(\vartheta', Y)\, d\vartheta' d\vartheta.$$

Taking into account (25.1) and integrating by parts, we have

$$\int_{-\infty}^{\infty} \vartheta u\, d\vartheta = 0 \tag{25.2}$$

($u(\vartheta, 0)$ should be assumed to decrease at $\vartheta \to \pm\infty$ sufficiently fast). The third constraint has a more complicated form:

$$\int_{-\infty}^{\infty}\left[\tfrac{1}{2}u^2 + \vartheta^2 u\right] d\vartheta = 0, \tag{25.3}$$

and so on. If the initial condition for Equation (19) $u(\vartheta, 0)$ is not consistent with *all* these constraints, a "tail", growing at $\vartheta \to -\infty$, will arise instantaneously behind the wave.

Now we can contribute to the explanation of the results obtained in [8], where the two-dimensional generalization of Equation (19)

$$u_Y + uu_\vartheta + u_{\vartheta\vartheta\vartheta} = \int_\vartheta^\infty \left[u(\vartheta', Y) - u_{zz}(\vartheta', Y) \right] d\vartheta' \tag{26}$$

was solved numerically with the initial conditions that were, or were not, consistent with the corresponding two-dimensional generalization of the constraint (25.1):

$$\int_{-\infty}^{\infty} (u - u_{zz})\, d\vartheta = 0. \tag{27}$$

The numerical results of [8] for initial conditions of "consistent" and "inconsistent" types "...turn out to be surprisingly similar and indicate that...the constraint (27) is not crucial." Now, the findings of the present paper clearly demonstrate that the constraint (27), being only a particular example of an *infinite* series of analogous equalities, cannot indeed be important. A significant difference can be presumably observed only between the initial conditions that are or are not consistent with the first *10 or more* constraints of the series.

In the end, we shall discuss the existence of stationary solutions to Equation (19). In the most simple, but yet most interesting, long-wave limiting case:

$$u_Y + uu_\vartheta = -\int_\vartheta^\infty u(\vartheta', Y)\, d\vartheta' \tag{28}$$

a wide class of periodic solutions was found in [12]. The ensemble of amplitudes of these waves is limited, and the wave with limiting value of the amplitude has pointed crests. It was also demonstrated in [12], that Equation (28) does not have any regular soliton-like solution. With regards to the more accurate Equation (19), we can suppose that the term $u_{\vartheta\vartheta\vartheta}$ would "smooth" the pointed crests, but the question of the existence of solitons is still unclear and can be clarified only by a numerical experiment.

5. Concluding remarks

This section outlines some questions resulting from the answers obtained in the present work.

(1) Apparently, the asymptotic technique developed for the description of wave motion in channels with an uneven bottom is uniformly applicable to channels with an uneven bottom and *corrugated walls*.

(2) At the same time, the consideration of *two-dimensional* bottom topography $H = H(\mathbf{x})$, $\mathbf{x} = (x, y)$ proves to be a much more complicated problem (at least, for the case of random irregularities). Unfortunately, the transformation of variables (3.1) has no trivial two-dimensional generalization and the problem seems to have a solution only in the case when this transformation is not important—i.e., for the periodic bottom irregularities (for more information, see [2]).

(3) Several questions are associated with the Equations (19) and (26), of which the most interesting one is this: what will be the "true" evolution of the initial conditions, which are inconsistent with some of the corresponding constraints? (Indeed, it can hardly be believed that the solution can instantaneously become unlocalized). Presumably, within the framework of the original inviscid water-wave equations, these solutions undergo a fast transformation ("adjustment") to the whole set of the constraints, whereas extra mass, momentum, and energy can be carried away or balanced by "fast" waves of some other nature [which were "lost" during the derivation of (19) or (26)]. In the instance of a *rotating* channel with an *even* bottom, those could be Poincaré or Kelvin waves (see [4]); but in our case, the original Equation (1) describes the ordinary surface waves only. The complete solution of this problem should also comprise the calculation of the adjustment of an arbitrary initial condition within the framework of Equation (1), which provides the initial condition for the "slow-time" Equations (19) or (26).

(4) If the influence of the uneven bottom is weak, (18) and (19) can be treated as perturbed Korteveg-de Vries equations, which could be analyzed by means of any suitable asymptotic technique (see [10, 14] and references therein). Unfortunately, it is impossible in the most interesting case of smooth irregularities, for neither the Korteveg-de Vries soliton nor the cnoidal wave are consistent with the constraints (25). It seems reasonable in this case to start the asymptotic analysis from the original Equation (1). This approach can be much easier than the consideration of an arbitrary nonstationary solution, outlined in the previous item.

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