Does Surface Tension Stabilize Liquid Films Inside a Rotating Horizontal Cylinder? Part 2: Multidimensional Disturbances

By E. S. Benilov

We examine the stability of a thin film of viscous fluid on the inside surface of a cylinder with horizontal axis, rotating about this axis. Depending on the parameters involved, the dynamics of the film can be described by several asymptotic models, one of which was examined by Benilov [*J. Fluid Mech.* 501:105–124 (2004)]. It turned out that the linearized stability problem for this model admits infinitely many neutrally stable eigenmodes, which form a complete set. Despite that, the film is unstable with respect to exploding disturbances, which grow infinitely in a finite time. The present paper examines the effect of surface tension on the stability of the film. Two cases are considered: short-scale disturbances (such that the axial wavelength λ is much smaller than the radius R of the cylinder) and long-scale disturbances (for which $\lambda \gtrsim R$). In the former case, surface tension is a stabilizing influence, because it regularizes the exploding solutions and makes all eigenmodes asymptotically (not just neutrally) stable. The latter case was previously examined by Acrivos and Jin [*J. Eng. Math.* 50:99–120 (2004)], who showed that surface tension destabilizes some of the eigenmodes. We argue, however, that the corresponding growth rate is much smaller than that of the so-called inertial instability.

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1. Introduction

While examining the two-dimensional stability of a viscous film inside a rotating horizontal cylinder, Benilov et al. [1] came across a linear equation with very unusual properties: it admits infinitely many stable solutions with harmonic dependence on time (eigenmodes)—which, however, are not representative of the stability properties of the system as a whole. It turned out that the eigenmodes coexist with exploding solutions, such that develop a singularity in a finite time (corresponding to a drop of fluid forming on the ceiling of the cylinder).

An even more exotic equation was derived in [2] for a three-dimensional generalization of the above problem: not only does it admit infinitely many stable eigenmodes, but the corresponding eigenfunctions form a complete set. Thus, an arbitrary initial condition can be represented by a series in the eigenmodes—but, even though all of these are stable, the series may still diverge giving rise to an explosion.

Note, however, that the equations derived in [1] and [2] do not take into account surface tension. In the former problem, it has been explored in [3]: it was shown that surface tension eliminates all exploding solutions, but *de*stabilizes some of the eigenmodes.

The present paper explores the effect of surface tension on the problem considered originally by [2]. In Section 2, we generalize the equation derived in [2] for surface tension and, in Sections 3–5, examine it. In Section 6, we compare our results to those of [4] who examined a similar problem.

2. The governing equations

Consider a thin film of liquid on the inside surface of an infinitely long cylinder of radius R (see Figure 1). Its axis is horizontal, and the cylinder is rotating about this axis with constant angular velocity Ω. The film is characterized by its density ρ, kinematic viscosity ν, and surface tension γ. We use cylindrical coordinates, thus the thickness $\hat{h}$ of the film depends on the azimuthal angle $\hat{\theta}$, axial coordinate $\hat{z}$, and time $\hat{t}$, where the hats mark the dimensional variables.

We use the following nondimensional variables:

$$t = \Omega \hat{t}, \qquad \theta = \hat{\theta}, \qquad z = \frac{\hat{z}}{R}, \qquad h = \frac{\hat{h}}{\varepsilon R} - \frac{\hat{h}^2}{2\varepsilon R^2}, \tag{1}$$

where g is the acceleration due to gravity and

$$\varepsilon = \left(\frac{\nu\Omega}{gR}\right)^{1/2}. \tag{2}$$

We note that such an unusual definition of h has been chosen because it simplifies the form of the governing equations.

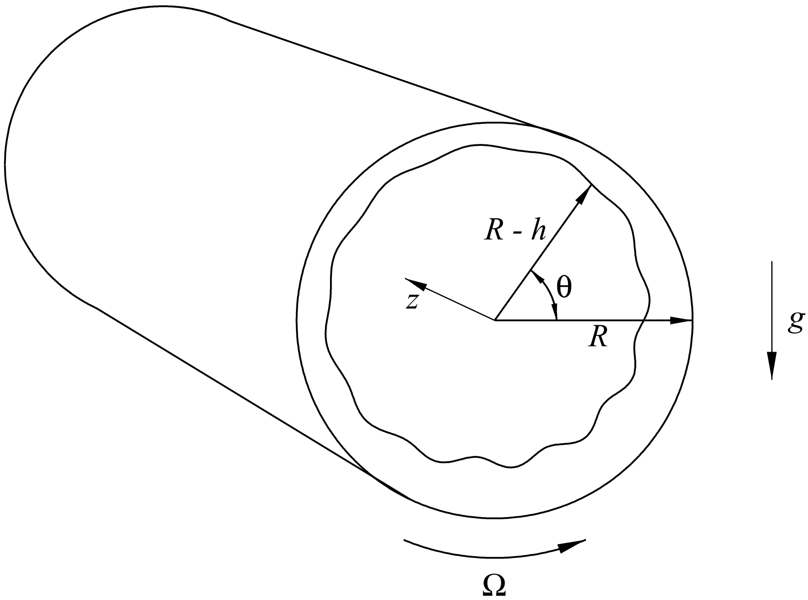

Figure 1. Formulation of the problem.

The evolution of the film is governed by a large number of parameters, depending on which various asymptotic equations can be derived through the lubrication theory (LT). Using the same approach as in [1, 2, 5], one can derive the following equation:

$$\begin{aligned}\frac{\partial h}{\partial t}+\frac{\partial h}{\partial \theta}-\frac{\partial}{\partial \theta}\left(\frac{1}{3}h^3\cos\theta\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{3}\varepsilon h^3\frac{\partial h}{\partial \theta}\sin\theta\right)\\ +\frac{\partial}{\partial z}\left(\frac{1}{3}\varepsilon h^3\frac{\partial h}{\partial z}\sin\theta\right)+\frac{\partial}{\partial \theta}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial \theta}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]\\ +\frac{\partial}{\partial z}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial z}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]=0,\end{aligned} \tag{3}$$

where

$$\beta=\frac{\gamma\varepsilon}{\rho g R^2} \tag{4}$$

is the nondimensional capillary coefficient. In Equation (3), the two terms following the time derivative describe viscous entrainment of the film by the rotating cylinder and the effect of gravity, and the terms involving ε and β describe the hydrostatic pressure effect and surface tension, respectively.

We shall not dwell on the applicability of (3) in detail, because it is no different from any other equation derived through the lubrication theory (for example, the limitations of LT have been discussed in [1, 2]). We only mention that the conditions of LT's validity include

$$\frac{\varepsilon \bar{h}^3}{\Delta\theta} \ll 1, \tag{5}$$

where $\bar{h}$ and $\Delta\theta$ are the characteristic amplitude and azimuthal scale of the nondimensional thickness $h(\theta, z, t)$. This condition makes the azimuthal gradient of hydrostatic pressure weaker than viscous entrainment.

Equation (3) is too complicated to be dealt with in its present form. To simplify it, assume

$$\Delta z \ll \Delta\theta, \tag{6}$$

where Δz is the axial scale of the solution. Then, the fifth term in Equation (3) is much larger than the fourth one—hence, the latter can be omitted. With regard to the surface-tension terms, we retain only the one that includes the highest-order derivative with respect to z. As a result, (3) reduces to

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} - \frac{\partial}{\partial \theta}\left(\frac{1}{3}h^3 \cos\theta\right) + \frac{\partial}{\partial z}\left(\frac{1}{3}\varepsilon h^3 \frac{\partial h}{\partial z}\sin\theta + \frac{1}{3}\beta h^3 \frac{\partial^3 h}{\partial z^3}\right) = 0. \tag{7}$$

This equation can be simplified further by assuming

$$\bar{h}^2 \ll 1, \tag{8}$$

in which case viscous entrainment is much stronger than gravity. Omitting the latter term, we can reduce (7) to

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + \frac{\partial}{\partial z}\left(\frac{1}{3}\varepsilon h^3 \frac{\partial h}{\partial z}\sin\theta + \frac{1}{3}\beta h^3 \frac{\partial^3 h}{\partial z^3}\right) = 0. \tag{9}$$

Approximation (8) has been previously used in [1, 2] (where it was termed "the small-flux limit") and has been shown to work for up to $\bar{h} \simeq 0.5$.

To illustrate restriction (8), we need to express $\bar{h}$ in terms of dimensional variables. Recalling how h is nondimensionalized [see (1)] and keeping in mind that, in the LT, the quadratic term in (1) is much smaller than the linear term, we have

$$\bar{h} \approx \frac{\hat{\bar{h}}}{\varepsilon R}. \tag{10}$$

Now, condition (8) can be rewritten in the form

$$\frac{\hat{\bar{h}}^2}{\varepsilon^2 R^2} \ll 1.$$

Thus, if ε is small, the small-flux approximation is stronger than the usual thin-film approximation (which requires $\hat{\bar{h}} \ll R$).

3. Eigenmodes

It turns out that the full short-scale model (7) is too complicated to be examined analytically, and we confine ourselves to studying its small-flux limit (9).

Equation (9) admits the following steady state:

$$h(\theta, z, t) = \bar{h},$$

where $\bar{h}$ is a constant (observe that, because the effect of gravity was assumed weak, the thickness of the film is, to leading order, uniform). In order to examine this steady state for stability, assume

$$h = \bar{h} + h'(\theta, z, t), \tag{11}$$

where h' represents the disturbance. Substituting (11) into (7) and omitting nonlinear terms, we obtain (primes dropped)

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + \frac{1}{3}\varepsilon\bar{h}^3\frac{\partial^2 h}{\partial z^2}\sin\theta + \frac{1}{3}\beta\bar{h}^3\frac{\partial^4 h}{\partial z^4} = 0. \tag{12}$$

Before we proceed, it is convenient to introduce

$$z_{\text{new}} = \sqrt{\frac{3}{\varepsilon\bar{h}^3}}\,z. \tag{13}$$

Rewriting (12) in terms of z_{new} and omitting the subscript $_{\text{new}}$, we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + \frac{\partial^2 h}{\partial z^2}\sin\theta + \mu\frac{\partial^4 h}{\partial z^4} = 0, \tag{14}$$

where

$$\mu = \frac{3\beta}{\varepsilon^2\bar{h}^3} \tag{15}$$

[recall that β, ε, and $\bar{h}$ are defined by (4), (2), and (10)]. Note that, except for the surface-tension term, (14) coincides with equation (2.13) of [2].

Now, seek a solution of (14) in the form

$$h(\theta, z, t) = \phi(\theta)\mathrm{e}^{i(kz-\omega t)}, \tag{16}$$

where ω and k are the frequency and axial wavenumber, respectively. Substitution of (16) into (14) yields a first-order ODE for $\phi(\theta)$; solving this ODE with the periodicity condition, we obtain an infinite set of solutions

$$\phi_n = \exp(in\theta - k^2\cos\theta), \tag{17}$$

$$\omega_n = n - i\mu k^4, \tag{18}$$

where n is an integer (the mode number). Observe that, because $\operatorname{Im}\omega_n < 0$, all modes are asymptotically stable, i.e., all harmonic disturbances decay.

Observe also that eigenfunctions (17) turned out to be independent of μ—in fact, they are exactly the same as those found in [2] for $\mu = 0$. Then, as shown in [2], they form a basis in the space of 2π-periodic functions with absolutely convergent Fourier series—hence, they can be used for solving the initial-value problem.

Let the initial condition for Equation (14) be

$$h = H(\theta, z) \qquad \text{at} \qquad t = 0. \tag{19}$$

According to the completeness theorem proved in [2], this initial condition can be represented by a Fourier integral with respect to z combined with a series in $\phi_n(\theta)$,

$$H(\theta, z) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} H_{kn} \phi_n \mathrm{e}^{ikz}\, \mathrm{d}k, \tag{20}$$

where the coefficients H_{kn} are

$$H_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} H(\theta, z)\, \exp(-in\theta + k^2 \cos\theta - ikz)\, \mathrm{d}z\, \mathrm{d}\theta \tag{21}$$

(for details, see [2]). Then, to solve the initial-value problem (14), (19), we need to multiply each term of series (20) by $\mathrm{e}^{-i\omega_n t}$:

$$h(\theta, z, t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} H_{kn} \exp[in\theta - k^2 \cos\theta + ikz - (in + \mu k^4)t]\, \mathrm{d}k, \tag{22}$$

where ω_n is given by (18).

Formulae (21)–(22) deliver the solution to the initial-value problem (14), (19).

4. Exploding solutions modified by surface tension

Let the initial condition be

$$H(\theta, z) = \exp\left(-\frac{z^2}{2W_0^2}\right), \tag{23}$$

which represents a ring-like disturbance encircling the cylinder from the inside. The z cross-section of the ring is Gaussian and is independent of θ (i.e., it is of constant width W_0 and unit amplitude). Substituting (23) into (21) and carrying out the integration, we obtain

$$H_{kn} = \frac{W_0}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} W_0^2 k^2\right) \mathrm{I}_n(k^2),$$

where I_n is the modified Bessel function. Then, substituting H_{kn} into (22) and using the identity

$$\sum_{n=-\infty}^{\infty} \exp[in(\theta - t)]\,\mathrm{I}_n(k^2) = \exp[k^2 \cos(\theta - t)] \tag{24}$$

(which follows from formula 5.8.4.4 of [6], volume 2), we obtain

$$h(\theta, z, t) = \frac{W_0}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left\{ - \left[\frac{1}{2} W_0^2 + \cos\theta - \cos(\theta - t)\right] k^2 + ikz - \mu k^4 t \right\} \mathrm{d}k. \tag{25}$$

4.1. The case of zero surface tension, $\mu = 0$ (brief review of [2])

If $\mu = 0$, and

$$\frac{1}{2} W_0^2 + \cos\theta - \cos(\theta - t) > 0, \tag{26}$$

the integral (25) converges and can be evaluated analytically,

$$h(\theta, z, t) = \frac{W_0}{\sqrt{W_0^2 + 2[\cos\theta - \cos(\theta - t)]}} \times \exp\left\{-\frac{z^2}{2W_0^2 + 4[\cos\theta - \cos(\theta - t)]}\right\}. \tag{27}$$

Observe that, for sufficiently wide initial conditions ($W_0 > 2$), this solution is periodic and smooth—whereas, for narrow ones ($W_0 \le 2$), it explodes (i.e., the denominators in (27) vanish at a finite t). The time and place of the explosion are

$$t_\mathrm{e} = 2 \arcsin\left(\frac{1}{4} W_0^2\right), \qquad \theta_\mathrm{e} = \frac{1}{2}\pi + \arcsin\left(\frac{1}{4} W_0^2\right), \qquad z_\mathrm{e} = 0.$$

To understand the reason of the explosion, one needs to clarify the physical meaning of the governing equation (14). For example, its second term describes propagation of disturbances due to rotation (to leading order, viscosity makes the film move together with the cylinder). The third term describes diffusion of disturbances due to hydrostatic pressure (it involves a diffusion operator with sign-variable diffusivity, $-\sin\theta$; observe that, in the upper half of the cylinder, $0 < \theta < \pi$, the diffusivity is negative). Finally, the fourth term describes surface tension (which is currently absent, $\mu = 0$).

Thus, if the initial disturbance is sufficiently narrow, antidiffusion causes a singularity somewhere in the region of negative diffusivity (physically, it corresponds to a drop forming on the cylinder's ceiling). If, however, the initial disturbance is wide, the rotation of the cylinder moves the developing

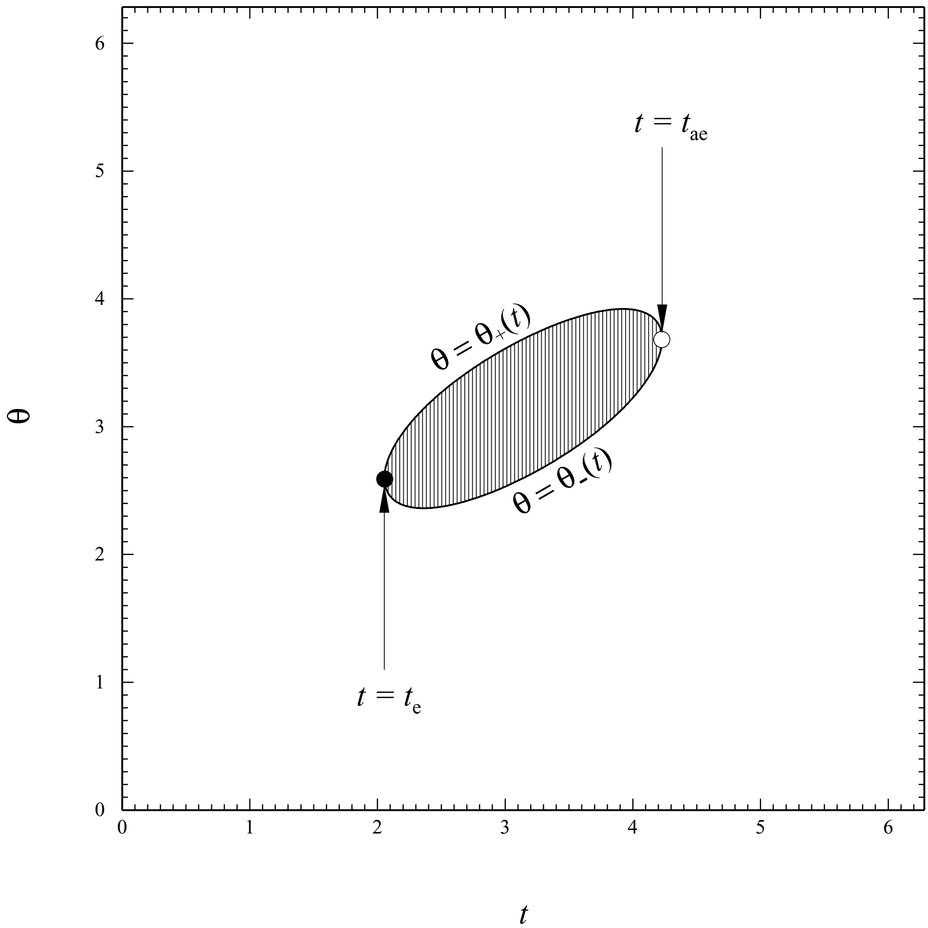

Figure 2. Convergence properties of integral (25) on the (θ, t)-plane for $W_0 = 1.85$, $\mu = 0$. In the shaded region, condition (26) does not hold, and (25) diverges. The place and time of the explosion (the first occurrence of the divergence) are indicated by a filled circle. Observe that, after the antiexplosion (marked by an empty circle), (25) converges again.

singularity outside the region of negative diffusivity *before* it explodes. Then, propagating through the region of positive diffusivity, the disturbance regains its original form.

Recall also that, in the case of explosion, solution (27) is valid only in a certain region of the (θ, t)-plane, defined by condition (26) (see Figure 2). On the (θ, z)-plane (see Figure 3), the validity region consists of two strips of variable width

$$0 < \theta < \theta_-(t), \quad -\infty < z < \infty,$$
$$\theta_+(t) < \theta < 2\pi, \quad -\infty < z < \infty,$$

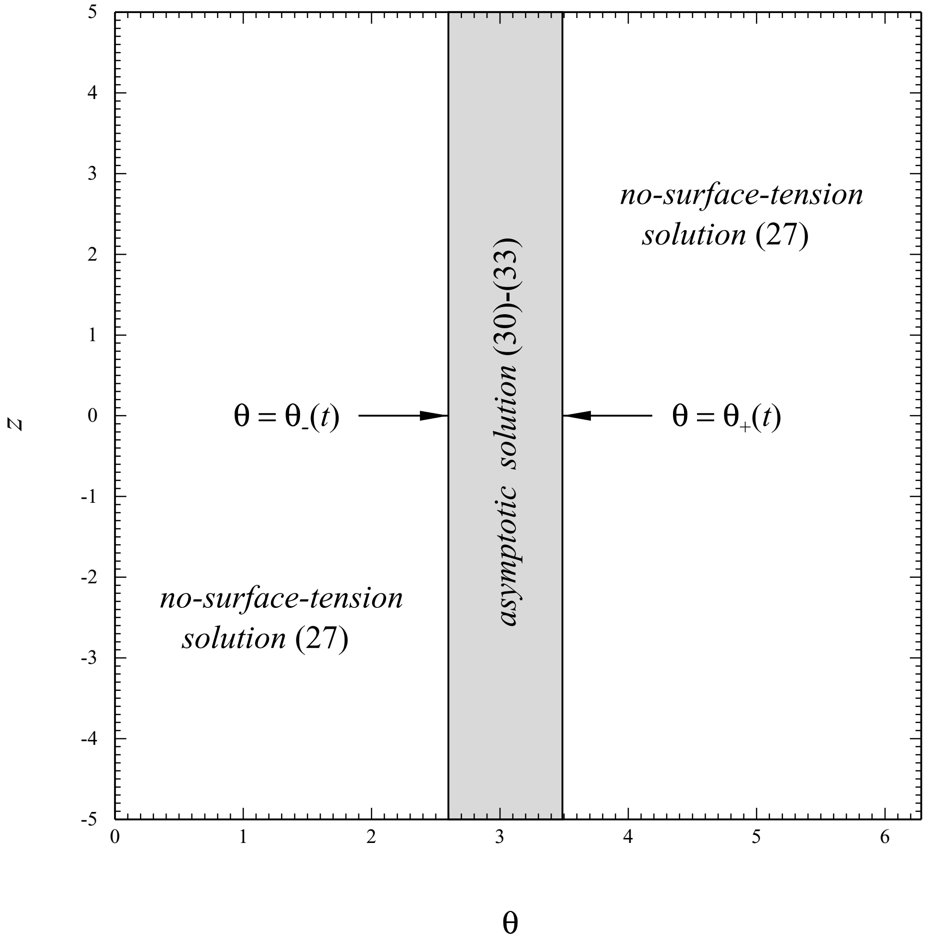

Figure 3. The regions of applicability of the no-surface-tension solution (27) and asymptotic solution (30)–(33) on the (θ, z)-plane for $W_0 = 1.85$, $\mu = 0.002$, $t = \pi$. The boundaries of the shaded strip are determined by condition (26); see also Figure 2.

with $\theta_{\pm}(t)$ determined by (26) and Figure 2. Note, however, that—for the case with no surface tension—the physical meaning of the $t > t_{\mathrm{e}}$ part of Figure 2 is unclear, as, after the explosion, solution (27) is unsafe.

4.2. The general case ($\mu \neq 0$): Asymptotic results

If $\mu \neq 0$, the integral in (25) cannot be evaluated exactly. Observe, however, that in many applications,

$$\mu \ll 1, \tag{28}$$

and we can evaluate (25) asymptotically. Note also that limit (28) is the most interesting one theoretically—as, for large μ, the solution is likely to quickly decay.

It turns out, however, that condition (28) is not sufficient—we actually need to assume

$$\mu t \ll 1. \tag{29}$$

Then, for those θ and t that satisfy condition (26), the solution is close to that with no surface tension (to leading order, the term involving μt can be simply ignored, and the resulting integral would still converge). If, however, (26) is *not* satisfied, the term involving μt must be taken into account, because it provides convergence of the integral as $k \to \infty$. Furthermore, if μt is small, the maximum of the integrand in (25) occurs at large k, which enables us to evaluate (25) using the method of steepest descent.

Examining (25) on a complex k-plane, one can see that the critical points satisfy

$$2\left[\cos(\theta - t) - \cos\theta - \frac{1}{2}W_0^2\right]k - 4\mu t k^3 + iz = 0. \tag{30}$$

This equation has two symmetrically located complex roots,

$$\begin{aligned} k_1 &= -\kappa_r + i\kappa_i, \\ k_2 &= \kappa_r + i\kappa_i, \end{aligned} \tag{31}$$

with their imaginary parts being of the same sign as that of z; and a purely imaginary root, with its imaginary part being of the opposite sign. Omitting technical details, we mention that the path of steepest descent passes through $k_{1,2}$, whereas k_3 turns out to be unimportant (see Figure 4). Then, integral (25) can be estimated as

$$h(\theta, z, t) \approx -\frac{W_0}{\sqrt{|\Phi''(k_1)|}}\mathrm{Im}\left\{\exp\left[\Phi(k_1) - \frac{i}{2}\arg\Phi''(k_1)\right]\right\}, \tag{32}$$

where

$$\Phi(k) = -\left[\cos(\theta - t) - \cos\theta - \frac{1}{2}W_0^2\right]k^2 + ikz - \mu k^4 t \tag{33}$$

and $\Phi'' = \mathrm{d}^2\Phi/\mathrm{d}k^2$.

The most important conclusion to be drawn from the asymptotic solution (30)–(33) is that even weak surface tension prevents the disturbance from reaching infinite amplitude [(32) shows that h remains finite for all θ, z, and t]. We conclude that surface tension transforms the explosive instability into a *transient* one, i.e., such that the disturbances initially grow, and sometimes quite significantly, but in the end always decay (see [7–9]). Note that examples

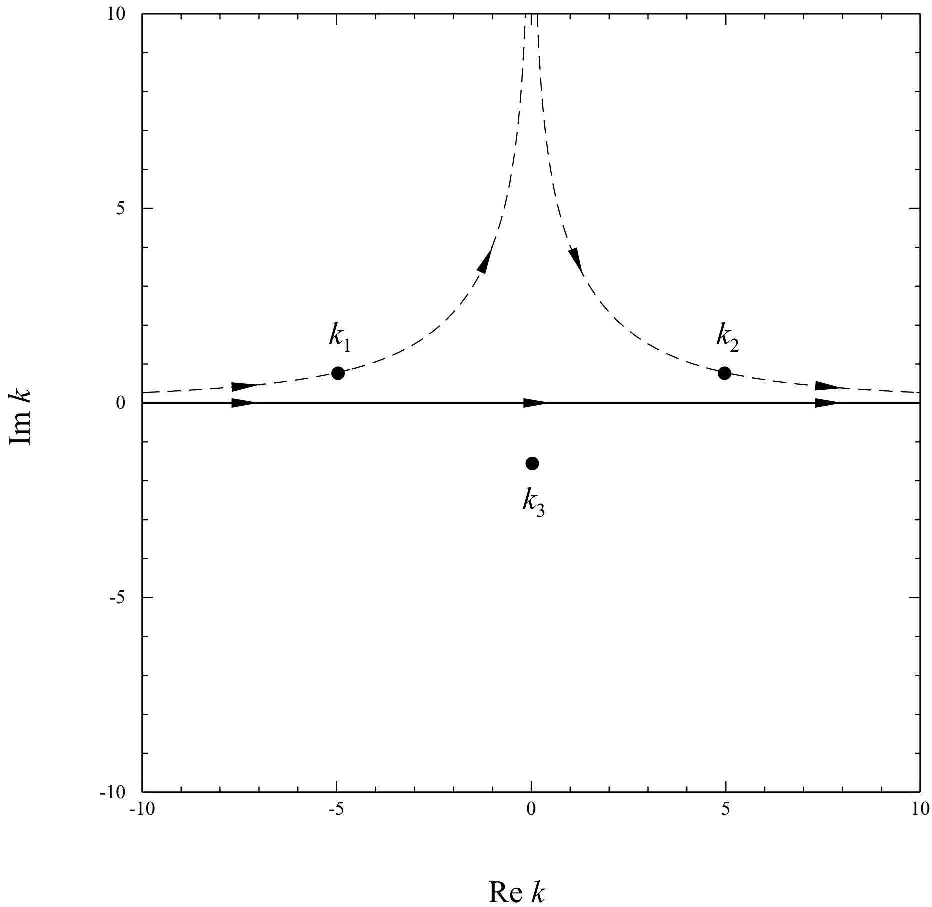

Figure 4. Integral (25) on the plane of complex k for $W_0 = 1.85$, $\mu = 0.002$, $\theta = \pi$, $z = 1$, and $t = \pi$. Black circles show critical points, solid line shows the original path of integration, and dotted line shows the path of steepest descent.

of transient instability have also been found in settings similar to the present one—for lubrication flows on a sloping surface (e.g., [10]) and for films with inverted density stratification (e.g., [11, 12]).

Next, observe that (30)–(33) describe a packet of short capillary waves radiated by the disturbance towards $z \to \pm\infty$. The axial wavenumber of the waves is $\mathrm{Re}\, k_1$, and the spatial decay rate (in the z-direction) is $\mathrm{Im}\, k_1$—both depend on (θ, z, t) through Equation (30). The temporal decay rate, in turn, equals $\mu \mathrm{Re}\, k_1^4$.

Note also that solution (30)–(33) is valid only if condition (26) does not hold—hence, it is confined to a strip of variable width

$$\theta_-(t) < \theta < \theta_+(t), \qquad -\infty < z < \infty,$$

with $\theta_{\pm}(t)$ defined by (26) (see Figures 2 and 3). Note also that, as can be seen from Figure 2, the strip exists only for $t \in (t_e, t_{ae})$—hence, the capillary waves radiated by the explosion at $t = t_e$ disappear without a trace at $t = t_{ae}$.

Finally, observe that integral (25) can be readily evaluated numerically (see the next section). At the same time, due to restriction (29), the asymptotic solution (30)–(33) does not describe the long-term evolution (unfortunately, this shortcoming cannot be remedied analytically, because, for $t \sim \mu^{-1}$, integral (25) involves neither small nor large parameters). Thus, (30)–(33) should be mostly used for interpretation and qualitative analysis of the solution, rather than for its calculation.

4.3. The general case ($\mu \neq 0$): Numerical results

Integral (25) was computed via truncation of its integrand at a sufficiently large k and the use of Simpson's Rule.[1] Typical results, for $W_0 = 1.85$, $\mu = 0.002$, are shown in Figures 5 and 6.

Figure 5 illustrates the short-term ($0 < t < 2\pi$) evolution of the disturbance: one can see how it grows, then bursts in capillary waves (confined to a strip stretched in the z-direction), then becomes smooth again. Another way to characterize the disturbance is to calculate the maximum amplitude at a given time

$$h_{\max}(t) = \max_{0<\theta<2\pi, -\infty<z<\infty} \{h(\theta, z, t)\} \tag{34}$$

(see Figure 6(a)). One can see that the maximum of $h_{\max}(t)$ is reached at approximately $t \approx 3.04$ and is about 108 times larger than the initial amplitude (which explains why the disturbance can hardly be seen in the first and last frames of Figure 5). Note also that, over the first period, the disturbance loses less than 0.5% of its initial amplitude (see Table 1).

Figure 6(b) illustrates the long-term ($0 < t < 20\pi$) evolution of the disturbance. Interestingly, the amplitudes of the solution's local maxima decrease much faster than those of the local minima (see Table 1). To illustrate the latter, observe that the loss of the initial amplitude over the first 10 periods is less than 3%, and that over the first 100 periods is less than 15%. Thus, after each period, the solution almost exactly restores its previous (minimum) amplitude—in this sense, the decay of the solution is very slow.

Thus, surface tension keeps the amplitude of the disturbance finite (prevents the explosion) and also causes decay.

[1]The truncation value of k and the Simpson's Rule step have been chosen to ensure that, if they are changed unfavorably by a factor of 2, the results would not change by more than 0.1%. Because the problem at hand involves just single integration and is not computationally challenging, such accuracy was not difficult to achieve.

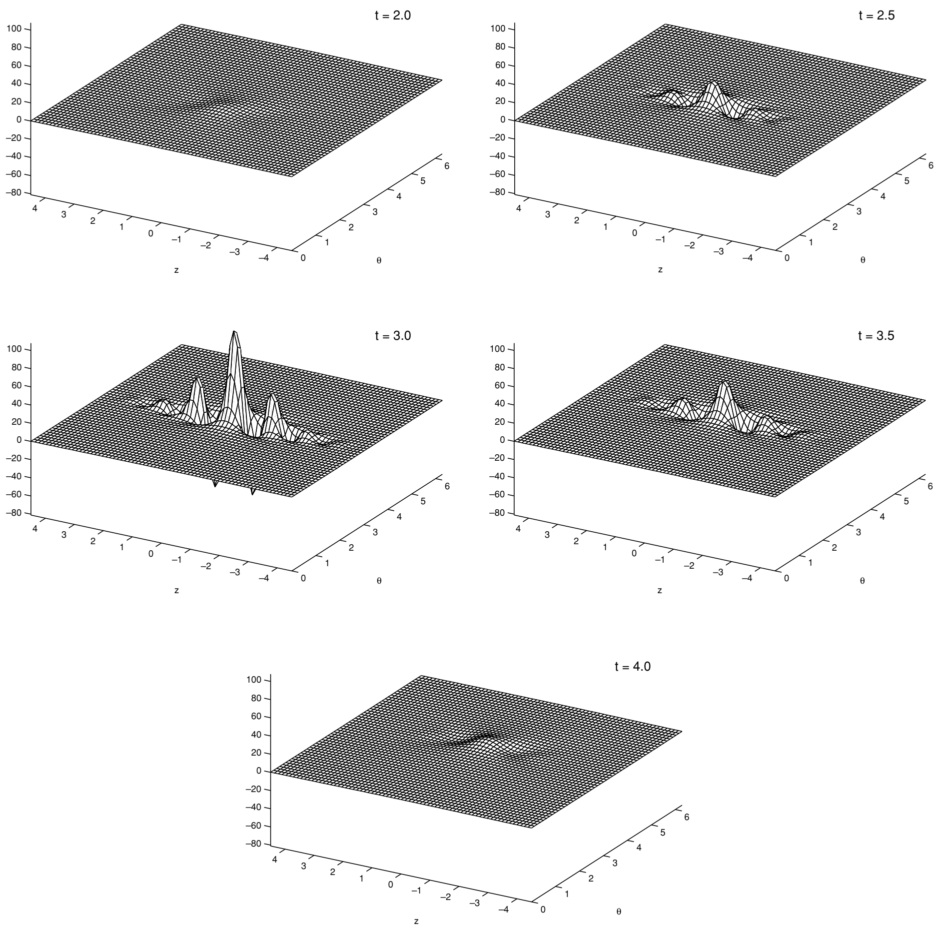

Figure 5. Snapshots of solution (25) with $W_0 = 1.85$, $\mu = 0.002$ for various values of t.

4.4. Applicability of Equation (14)

Observe that Equation (14) used for obtaining all of the above results, implies that the axial scale of the solution is much smaller than the azimuthal scale [see restriction (6)]. This, however, does not create a problem for the eigenmodes: if, for a particular mode, (6) holds initially—it will hold for all t (simply because the mode's wavenumbers n and k do not change).

For transient solutions, such as (25), the situation is different, as the spatial scales of those vary in time.

To validate (25), we should justify the omission of the θ-derivatives in Equation (3)—taking into account substitution (13), we have

$$\frac{1}{\varepsilon\bar{h}^3}\frac{\partial^2 h}{\partial z^2} \gg \frac{\partial^2 h}{\partial \theta^2}. \tag{35}$$

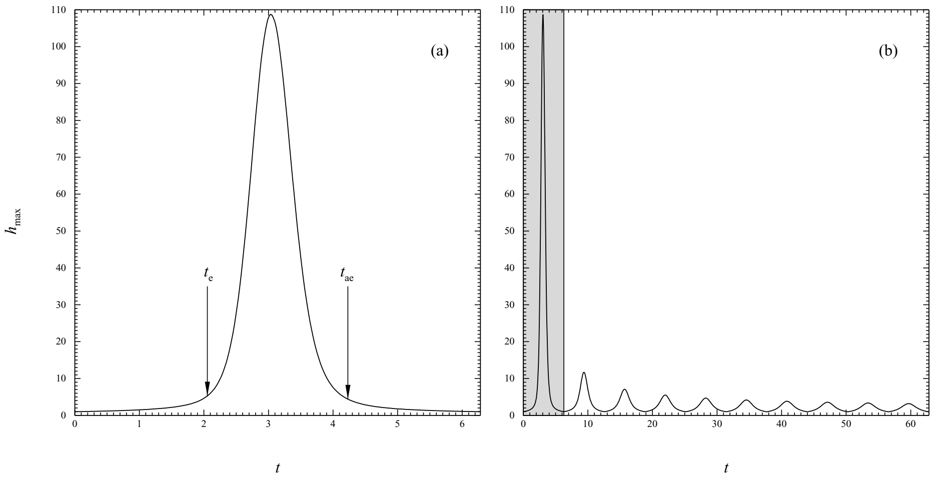

Figure 6. The maximum amplitude of solution (25) vs. time [$h_{max}(t)$ is determined by (34)]. (a) Short-term evolution (t_e and t_{ae} are the times of explosion and antiexplosion, both determined using the no-surface-tension approximation; see Figure 2). (b) Long-term evolution (the region corresponding to Figure 6(a) is shaded).

To estimate the derivatives of the solution, we use the asymptotic expression (32), according to which

$$\frac{\partial h}{\partial \theta} = \mathrm{O}\big(k_1^2\big), \qquad \frac{\partial h}{\partial z} = \mathrm{O}(k_1). \tag{36}$$

Table 1
The Long-Term Evolution of Solution (25) with $W_0 = 1.85$, $\mu = 0.02$ [$h_{max}(t)$ is Determined by (34)]

Time Period	Local Minima of $h_{max}(t)$	Local Maxima of $h_{max}(t)$
$0 < t \leq 2\pi$	0.9968	108.7372
$2\pi < t \leq 4\pi$	0.9938	11.6936
$4\pi < t \leq 6\pi$	0.9908	7.0846
$6\pi < t \leq 8\pi$	0.9880	5.5148
$8\pi < t \leq 10\pi$	0.9852	4.6994
$10\pi < t \leq 12\pi$	0.9825	4.1886
$12\pi < t \leq 14\pi$	0.9799	3.8328
$14\pi < t \leq 16\pi$	0.9773	3.5675
$16\pi < t \leq 18\pi$	0.9748	3.3603
$18\pi < t \leq 20\pi$	0.9724	3.1927
...	...	...
$198\pi < t \leq 200\pi$	0.8518	1.3872

Then, for $t = \mathrm{O}(1)$, (30) yields

$$k_1 = \mathrm{O}(\mu^{-1/2}), \tag{37}$$

and, substituting (36)–(37) into (35), we obtain

$$\mu \gg \varepsilon \bar{h}^3. \tag{38}$$

Thus, the transient solution of Equation (14) is applicable only if the effective capillary coefficient μ is not too small.

5. The general initial-value solution (21)–(22)

In the previous section, it was shown that the particular solution (25) remains finite for all t, and in this section, we prove the same for the general initial-value solution (21)–(22).

The latter is more convenient to analyze if it is rewritten in terms of the Fourier integral/series of the initial condition,

$$H(\theta, z) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} H_{pm} \exp(im\theta + ipz)\,\mathrm{d}p, \tag{39}$$

where

$$H_{pm}^{\mathrm{F}} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} H(\theta, z)\, \exp(-im\theta - ipz)\,\mathrm{d}\theta\,\mathrm{d}z \tag{40}$$

are the Fourier coefficients. The relationship between these and the original coefficients H_{kn} can be established by substituting (39) into (21). Changing the order of integration/summation where necessary (the validity of which will be discussed later), and using the formula

$$\int_{-\infty}^{\infty} \exp[i(p-k)z]\,\mathrm{d}z = 2\pi\delta(p-k),$$

where $\delta(p)$ is the Dirac delta function, we obtain

$$H_{kn} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} H_{km}^{\mathrm{F}} \exp[i(m-n)\theta + k^2 \cos\theta]\,\mathrm{d}\theta.$$

To evaluate the integral with respect to θ in this expression, change its limits to $(-\pi, \pi)$. Then, it becomes clear that its imaginary part is zero, and the real part can be calculated in terms of a modified Bessel function:

$$H_{kn} = \sum_{m=-\infty}^{\infty} H_{km}^{\mathrm{F}}\, \mathrm{I}_{m-n}(k^2). \tag{41}$$

Next, substituting (41) into (22), shifting the index of summation $n \to n + m$, replacing I_{-n} with I_n, and making use of identity (24), we obtain

$$h(\theta, z, t) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} H_{km}^{\mathrm{F}} \exp\{im(\theta - t) + ikz + k^2[\cos(\theta - t) - \cos\theta] - \beta k^4 t\} \, \mathrm{d}k. \quad (42)$$

To find our whether or not (42) can explode, examine its absolute convergence—i.e., consider

$$\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| H_{km}^{\mathrm{F}} \right| \exp\{k^2[\cos(\theta - t) - \cos\theta] - \mu k^4 t\} \, \mathrm{d}k.$$

For $t > 0$, the convergence of this integral/series can be guaranteed by the following sufficient conditions:

$$\mu > 0, \quad (43)$$

$$\sum_{m=-\infty}^{\infty} \left| H_{km}^{\mathrm{F}} \right| \leq A \mathrm{e}^{Bk^2} \qquad \text{for} \qquad k \geq k_0, \quad (44)$$

where A, B, and k_0 are constants. Note that, importantly, (43)–(44) also justify the changes of the order of integration/summation carried out while transforming the original solution (22) into (42) (it should be recalled here that the order of integration/summation may be changed only for absolutely convergent integrals/series).

The meaning of condition (43) is obvious (it simply requires surface tension to be present), thus we only need to illustrate the restrictions imposed by condition (44). To do so, consider an initial condition of the form

$$H(\theta, z) = P(\theta)\, Q(z).$$

Then, (44) yields

$$\sum_{m=-\infty}^{\infty} P_m^{\mathrm{F}} = A < \infty, \quad (45)$$

$$Q_k^{\mathrm{F}} \leq A \mathrm{e}^{Bk^2} \qquad \text{for} \qquad k \geq k_0, \quad (46)$$

where P_m^{F} and Q_k^{F} are the Fourier coefficients of the functions $P(\theta)$ and $Q(z)$, respectively. It is clear that (45) is satisfied by any continuous $P(\theta)$—whereas $Q(z)$ does not even have to be continuous: condition (46) is satisfied even for the Dirac delta function or any of its derivatives.

Thus, in the presence of surface tension, solution (40), (42) remains finite and smooth for a wide class of initial conditions, which effectively means that *capillary effects eliminate exploding solutions*.

6. Comparison with the results of [4]

In this paper, we have argued so far that surface tension is a stabilizing influence and, in particular, makes all eigenmodes asymptotically stable. Note, however, that this conclusion seems to be at odds with that of [4], who considered the same setting, but a slightly different asymptotic regime.

6.1. The large-scale model

Observe that, instead of (6), [4] assumed

$$\Delta z \gtrsim \Delta\theta. \tag{47}$$

Conditions (5) and (47) make both components of the hydrostatic pressure gradient [fourth and fifth terms of (3)] negligible, whereas the surface tension may or may not be of leading order. To be on the safe side, we retain the latter, after which Equation (3) reduces to

$$\frac{\partial h}{\partial t}+\frac{\partial h}{\partial \theta}-\frac{\partial}{\partial \theta}\left(\frac{1}{3}h^3\cos\theta\right)+\frac{\partial}{\partial \theta}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial \theta}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial z}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]=0. \tag{48}$$

In what follows, this equation will be referred to as the large-scale model, where the large applies to the axial scale Δz. The old equation, (7), will be referred to as the short-scale model.

As before, we restrict our analysis to the small-flux approximation. Then, taking into account (8), one can reduce (48) to

$$\frac{\partial h}{\partial t}+\frac{\partial h}{\partial \theta}+\frac{\partial}{\partial \theta}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial \theta}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\frac{1}{3}\beta h^3\frac{\partial}{\partial z}\left(h+\frac{\partial^2 h}{\partial \theta^2}+\frac{\partial^2 h}{\partial z^2}\right)\right]=0. \tag{49}$$

To examine the stability of this equation's steady-state solution, $h=\bar{h}$, assume

$$h(\theta,z,t)=\bar{h}+h'(\theta,z,t).$$

Linearizing (49), we obtain

$$\frac{\partial h'}{\partial t}+\frac{\partial h'}{\partial \theta}+\frac{1}{3}\beta\bar{h}^3\left[\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)^2\right]h'=0. \tag{50}$$

This equation has constant coefficients—hence, it admits harmonic solutions

$$h'=\mathrm{e}^{i(n\theta+kz-\omega t)}, \tag{51}$$

where ω is, again, the frequency, and n and k are the azimuthal and axial wavenumbers (to ensure 2π-periodicity in θ, n must be an integer). Substituting (51) into (50), we obtain

$$\omega = n + \frac{1}{3} i\beta\bar{h}^3 \left[n^2 + k^2 - (n^2 + k^2)^2\right]. \tag{52}$$

Then,

$$\operatorname{Im}\omega = \frac{1}{3}\beta\bar{h}^3 k^2(1 - k^2) > 0 \qquad \text{for} \qquad n = 0, \quad k \in (0, 1),$$

which corresponds to instability and is equivalent to the corresponding result [formula (85)] of [4]. The characteristics of the most unstable disturbance are

$$k_{\max} = \frac{1}{\sqrt{2}}, \qquad (\operatorname{Im}\omega)_{\max} = \frac{1}{12}\beta\bar{h}^3.$$

Dimensionally [see (1)–(2), (4)], these parameters correspond to

$$\hat{k}_{\max} = \frac{k_{\max}}{R} = \frac{1}{\sqrt{2}R}, \qquad (\operatorname{Im}\hat{\omega})_{\max} = \Omega(\operatorname{Im}\omega)_{\max} = \frac{\gamma}{12\rho R\nu}\left(\frac{\hat{\bar{h}}}{R}\right)^3, \tag{53}$$

where $\hat{\bar{h}}$ is the dimensional thickness; R and Ω are the radius and angular velocity of the cylinder; ρ, γ, and ν are the density, surface tension, and kinematic viscosity.

Thus, the axial wavelength of the most unstable disturbance is comparable to the radius of the cylinder—which explains why the instability is missed by the short-wave model (these scales are simply not included in it, because it implies $\Delta z \ll 1$). Observe also that, mathematically, instability is caused by the k^2 term in expression (52). Tracing this term back to Equation (50), we see that it appears due to the cylindricity of the problem (i.e., there would be no such term for a film on a plate). We conclude that the instability is caused by surface tension acting under the conditions of cylindrical geometry.

Note that, despite *de*stabilizing some of the harmonic solutions of the large-scale model, surface tension still stabilizes (eliminates) all of the exploding solutions. Indeed, any large-scale disturbance can explode only through a reduction of its spatial scale. Then, as a result of such reduction, it will enter the validity region of the short-scale model—where, as shown above, disturbances never collapse.

6.2. An example

To illustrate formulae (53), consider the following example: let the fluid under consideration be glycerin at 20°C, for which

$$\nu = 1.12 \times 10^{-3}\,\mathrm{m}^2/\mathrm{s}, \quad \gamma = 0.063\,\mathrm{N/m}, \quad \rho = 1.26 \times 10^3\,\mathrm{kg/m}^3. \tag{54}$$

Assume also that

$$\Omega = 1\,\text{rev/s}, \quad R = 5\,\text{cm}, \quad \hat{\bar{h}} = 2.5\,\text{mm}, \tag{55}$$

where $\hat{\bar{h}}$ is the dimensional mean thickness of the film. Then, for particular case (54)–(55), the e-folding time is

$$\hat{\tau} = \frac{1}{(\operatorname{Im}\hat{\omega})_{\max}} \approx 1790\,\text{min},$$

i.e., the instability is very weak. By comparison, the so-called inertial instability in this case is much faster: its e-folding time (estimated using the results of [5]) is $\hat{\tau} \approx 38\,\text{min}$.

Several other examples, similar to (54)–(55) have been considered and, in all cases, inertial instability turned out to be much stronger than the capillary one.

7. Concluding remarks

In this paper, we have explored the impact of surface tension on three-dimensional stability of a liquid film inside a rotating horizontal cylinder. For short disturbances (such that their wavelengths are much smaller than the radius R of the cylinder), surface tension turned out to be a stabilizing influence. Indeed, it makes all eigenmodes *asymptotically* stable (without surface tension, they would be *neutrally* stable) and regularizes exploding solutions that would exist without it.

For long disturbances, with wavelengths comparable to, or larger than R, surface tension is a *de*stabilizing influence, because it makes the eigenmodes unstable (which agrees with the conclusion of [4]). Physically, the instability is caused by the joint effect of surface tension and cylindrical geometry. It should be noted, however, that this (capillary) instability is much weaker than the so-called inertial one (for more information on the latter, see [5]).

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