DOES SURFACE TENSION STABILIZE LIQUID FILMS INSIDE A ROTATING HORIZONTAL CYLINDER?

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Summary

We examine the stability of a thin film of viscous fluid inside a cylinder with horizontal axis, rotating about this axis. Depending on the parameters involved, the dynamics of the film can be described by several asymptotic equations, one of which was examined by Benilov, O'Brien, and Sazonov (*J. Fluid Mech.* 2003 **497**, 201–224). It turned out that the linear stability problem for this equation admits infinitely many harmonic eigenmodes, which are all neutrally stable. Despite that, the film is unstable with respect to 'exploding' (non-harmonic) disturbances, which grow infinitely in finite time.

The present paper examines the effect of surface tension on the stability of the film. Given the generally stabilizing nature of surface tension, it comes as no surprise that it eliminates the exploding solutions and makes most eigenmodes asymptotically (not just neutrally) stable. For a certain parameter range, however, some of the eigenmodes, paradoxically, become *un*stable.

1. Introduction

Harmonic solutions (eigenmodes) play an important role in stability analysis, as even a single growing mode can destabilize an otherwise stable system. If, on the other hand, all modes are bounded in time and the corresponding eigenfunctions form a complete set, the system is normally regarded as stable. In this case, an arbitrary initial condition can be represented as a series of eigenmodes; and since all of these are stable, so too should be the solution to the initial-value problem.

The above argument, however, is not water-tight: examples are known in which each term of the series is bounded but the series as a whole diverges and the solution 'explodes', that is, develops a singularity in a finite time. A partial differential equation with such properties has been found in (**1**), in a model describing thin viscous films inside a rotating horizontal cylinder (the exploding solutions in this case correspond to a drop of fluid forming on the 'ceiling' of the cylinder). Similar results, for both eigenmodes and exploding solutions, were obtained in (**2**) for a related problem.

Note, however, that neither (**1**) nor (**2**) took into account surface tension which—given the stabilizing nature of this effect (**3** to **7**), its influence on the film's stability in general, and the exploding solutions in particular, can be important.

The present paper aims to explore how the problem examined originally in (**1**) is influenced by capillary effects. In section 2, we introduce an equation which extends the (**1**) model to include surface tension. We shall then examine the eigenmodes of this equation (section 3) and investigate the effect of surface tension on the exploding solutions (section 4).

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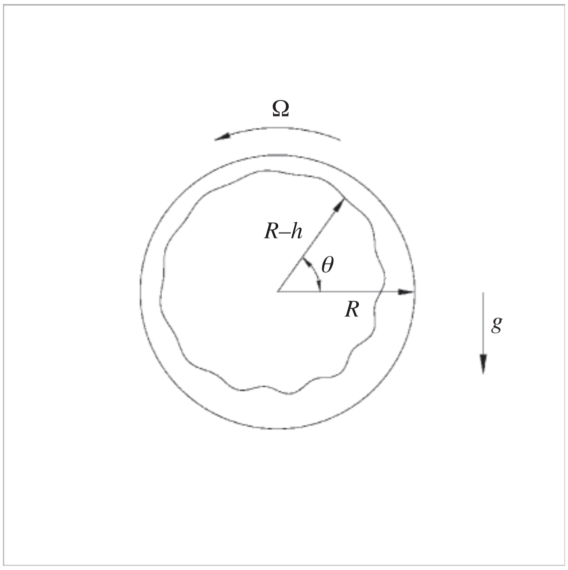

Fig. 1 Formulation of the problem

2. Formulation

2.1 *The governing equation*

Consider a thin film of liquid on the inside surface of a cylinder of radius R, with horizontal axis, rotating about this axis with constant angular velocity Ω (see Fig. 1). The film is characterized by its density ρ, kinematic viscosity ν, and surface tension γ. In order to examine the two-dimensional motion of the film, we shall use polar coordinates, so the film's thickness $\hat{h}$ depends on the polar angle $\hat{\theta}$ and time $\hat{t}$—or, non-dimensionally,

$$\theta = \hat{\theta}, \qquad t = \Omega \hat{t}, \qquad h = \frac{\hat{h}}{\varepsilon R} - \frac{\hat{h}^2}{\varepsilon R^2}, \tag{2.1}$$

where

$$\varepsilon = \left(\frac{\nu \Omega}{g R} \right)^{1/2} \tag{2.2}$$

and g is acceleration due to gravity.

The evolution of the film is governed by a large number of parameters, depending on which particular asymptotic equations are derived. We shall use the equation derived in (**8**) using so-called lubrication theory (LT). For two-dimensional flows, this equation can be written in the form (see (**9**))

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta} \left[h - \tfrac{1}{3} h^3 \cos\theta + \tfrac{1}{3} \varepsilon h^3 \frac{\partial h}{\partial \theta} \sin\theta + \tfrac{1}{3} \beta h^3 \left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3} \right) \right] = 0, \tag{2.3}$$

where

$$\beta = \frac{\gamma}{\rho g R^2}\left(\frac{\nu\Omega}{gR}\right)^{1/2}. \tag{2.4}$$

Note that, in the LT, the second term in the expression for h (see (2.1)) is much smaller than the first one—that is, h is, essentially, the non-dimensional thickness of the film.

In equation (2.3), the first term in square brackets describes viscous entrainment of the film by the cylinder's rotation, and the second term describes the effect of gravity (in the LT, both are leading-order effects). The third term describes the effect of hydrostatic pressure—in the LT, this is small. Finally, the fourth term describes surface tension effects—depending on the parameters involved, it may or may not be of leading order.

2.2 *Steady states and disturbances*

Let the solution of equation (2.3) be independent of time, $h(\theta, t) = \bar{h}(\theta)$, in which case (2.3) yields

$$\bar{h} - \tfrac{1}{3}\bar{h}^3\cos\theta + \tfrac{1}{3}\varepsilon\bar{h}^3\frac{\partial\bar{h}}{\partial\theta}\sin\theta + \tfrac{1}{3}\beta\bar{h}^3\left(\frac{\partial\bar{h}}{\partial\theta} + \frac{\partial^3\bar{h}}{\partial\theta^3}\right) = q, \tag{2.5}$$

where q is a constant of integration (physically, q is the non-dimensional mass flux). Various particular cases of equation (2.5) have been previously examined in (**10** to **14**).

In order to examine the steady state $\bar{h}$ for stability, assume that

$$h(\theta, t) = \bar{h}\,(\theta) + h'(\theta, t), \tag{2.6}$$

where h' represents the disturbance. Substituting (2.6) into (2.5) and omitting the nonlinear terms, we obtain (primes omitted)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial\theta}\left[C(\theta)\,h + D(\theta)\frac{\partial h}{\partial\theta} + B(\theta)\left(\frac{\partial h}{\partial\theta} + \frac{\partial^3 h}{\partial\theta^3}\right)\right] = 0, \tag{2.7}$$

where

$$C(\theta) = 1 - \bar{h}^2\cos\theta + \varepsilon\bar{h}^2\frac{d\bar{h}}{d\theta}\sin\theta + \beta\bar{h}^2\left(\frac{d\bar{h}}{d\theta} + \frac{d^3\bar{h}}{d\theta^3}\right), \tag{2.8}$$

$$D(\theta) = \tfrac{1}{3}\varepsilon\bar{h}^3\sin\theta, \qquad B(\theta) = \tfrac{1}{3}\beta\bar{h}^3. \tag{2.9}$$

Equations similar to (2.7) were previously examined in (**1**, **8**, **9**, **15**).

The main difficulty associated with equation (2.7) is that the explicit form of its coefficients is unknown (recall that $\bar{h}$ is determined by (2.5)). The only exception is the limit of small q, in which case (2.5) yields an explicit asymptotic solution

$$\bar{h} = q + \tfrac{1}{3}q^3\cos\theta + O(q^5) \qquad \text{as} \qquad q \to 0. \tag{2.10}$$

Physically, this limit corresponds to viscous forces dominating gravity—as a result, the asymmetry introduced by the latter is negligible, and the film's thickness is almost constant. Note also that formula (2.10) (together with definition (2.1) of h) relates q to dimensional quantities:

$$q \approx \frac{\hat{\hat{h}}}{\varepsilon R}, \tag{2.11}$$

where $\hat{\hat{h}}$ is the dimensional thickness of the film. The relation (2.11) shows that, if ε is small, the small-q assumption is stronger than the usual thin-film approximation.

Substituting (2.10) into (2.7) to (2.9), assuming

$$\varepsilon q^3, \beta q^3 \gg q^4,$$

and retaining terms up to $O(q^2, \varepsilon q^3, \beta q^3)$, we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left[\left(1 - q^2 \cos\theta\right) h + \tfrac{1}{3}\varepsilon q^3 \sin\theta \frac{\partial h}{\partial \theta} + \tfrac{1}{3}\beta q^3 \left(\frac{\partial h}{\partial \theta} + \frac{\partial^3 h}{\partial \theta^3}\right)\right] = 0. \tag{2.12}$$

In the remainder of this paper, we shall examine equation (2.12).

3. Eigenmodes

First, we shall examine the eigenmodes, that is, solutions with harmonic dependence on time,

$$h(\theta, t) = \phi(\theta)\, e^{-i\omega t}, \tag{3.1}$$

where ω is the (complex) frequency. Substitution of (3.1) into (2.12) yields

$$\frac{d}{d\theta}\left[\left(1 - q^2 \cos\theta\right) \phi + \epsilon \sin\theta \frac{d\phi}{d\theta} + b \left(\frac{d\phi}{d\theta} + \frac{d^3\phi}{d\theta^3}\right)\right] - i\omega\phi = 0, \tag{3.2}$$

where

$$\epsilon = \tfrac{1}{3}\varepsilon q^3, \qquad b = \tfrac{1}{3}\beta q^3. \tag{3.3}$$

These equations should be supplemented by the periodicity condition

$$\phi(\theta + 2\pi) = \phi(\theta). \tag{3.4}$$

Equations (3.2), (3.4) form an eigenvalue problem, where ω is the eigenvalue and ϕ is the eigenfunction. If Im $\omega > 0$, the disturbance is unstable.

3.1 *The case* $b = 0$ (*no surface tension*)

With an extra assumption $q^2 = 0$ (infinitesimally thin film), this particular case was examined in (**1**)—in the present paper we extend their results to $q^2 \neq 0$. The two cases turned out to be very similar, both methodologically and conclusions-wise—thus, there is no need to dwell on this case in detail.

If $b = 0$, the eigenvalue problem (3.2), (3.4) has infinitely many eigenmodes with neutrally stable eigenvalues ($\operatorname{Im}\omega = 0$) and symmetric eigenfunctions:

$$\phi(-\theta) = \phi^*(\theta),$$

where the asterisk denotes complex conjugate. To understand the physical meaning of this result, observe that the second term in square brackets in equation (3.2),

$$\frac{d}{d\theta}\left[\epsilon \sin\theta \frac{d\phi}{d\theta}\right],$$

involves a diffusion-type operator, with $-\epsilon \sin\theta$ being the diffusivity. Thus, in the lower half of the cylinder ($\pi < \theta < 2\pi$), the diffusivity is positive, and, in the upper half, negative (because of the 'reversed' gravity). Accordingly, in the latter, the disturbance grows, and in the former, decays. However, due to the symmetry of the eigenfunction, the growth and decay are in perfect balance, adding up to neutral stability.

To quantify this argument, multiply (3.2) by ϕ^*, integrate over $0 < \theta < 2\pi$, and take the real part. After straightforward algebra involving integration by parts, we obtain

$$(\operatorname{Im}\omega)\int_0^{2\pi} |\phi|^2\, d\theta = \int_0^{2\pi}\left[\sin\theta\left(\epsilon\left|\frac{d\phi}{d\theta}\right|^2 - \tfrac{1}{2}q^2\,|\phi|^2\right) + b\left(\left|\frac{d\phi}{d\theta}\right|^2 - \left|\frac{d^2\phi}{d\theta^2}\right|^2\right)\right] d\theta. \quad (3.5)$$

It then becomes clear that, for $b = 0$ and for even $|\phi(\theta)|$, the right-hand side of this expression is zero—hence, $\operatorname{Im}\,\omega = 0$ (neutral stability).

3.2 *The case $b \neq 0$: examples*

Given the generally stabilizing nature of surface tension, one would expect it to make the film *asymptotically* stable, that is, the eigenmodes would not just remain bounded as $t \to \infty$ (as they do in neutrally stable systems), but would actually decay. This prediction, however, is realized only if surface tension is sufficiently strong and q^2 is sufficiently large—otherwise, capillary effects *de*stabilize the film!

To illustrate this conclusion, we have computed the eigenmodes for several particular cases with the same values of q^2 and ϵ, but different b:

$$q^2 = 0, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0050, \quad (3.6)$$

$$q^2 = 0, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0010, \quad (3.7)$$

$$q^2 = 0, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0005, \quad (3.8)$$

$$q^2 = 0, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0001. \quad (3.9)$$

The results are shown in Fig. 2—one can see two distinct regions of instability:

- for smaller b, some of the high modes are unstable, and
- in all cases, the first mode is unstable.

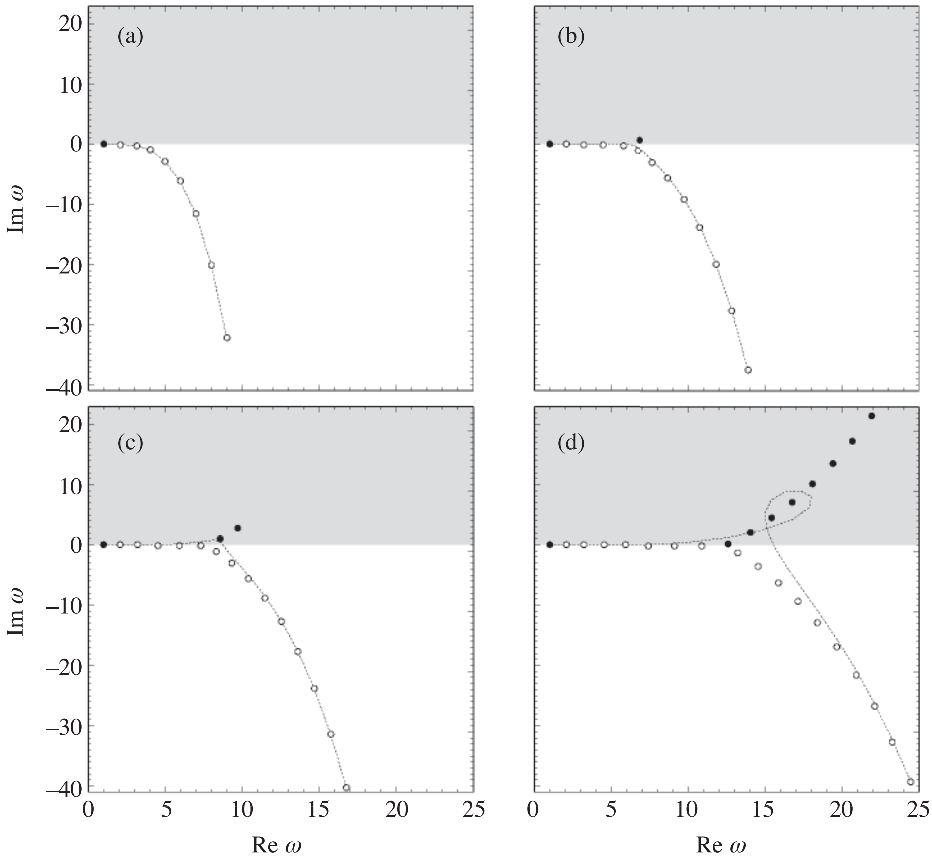

Fig. 2 Eigenvalues of problem (3.2), (3.4). Panels (a) to (d) correspond to particular cases (3.6) to (3.9). The region of instability is shaded, the dotted line corresponds to asymptotic formula (3.21). The unstable modes are shown by black circles

Observe that, with decreasing b, the number of unstable modes grows, their growth rates (Im ω) increase, while the eigenvalues move towards the high-frequency end of the spectrum (that is, their Re ω grow). In the limit $b \to 0$, the unstable region moves to infinity (Re $\omega \to \infty$) and disappears, while the (asymptotically) stable modes approach the real axis and become neutrally stable.

Before we proceed, note that the eigenmodes can be numbered using

$$n = \frac{1}{2\pi} \arg\left[\frac{\phi(2\pi)}{\phi(0)}\right],$$

which assigns a (usually unique) integer to each eigenfunction.[†] Then, in the cases considered above, the modes in Table 1 are unstable. Note that the first mode's growth rate is very small—in

[†] The fact that n is an integer follows from ϕ's periodicity.

Table 1 Unstable modes

Equation	Mode numbers
(3.6)	1
(3.7)	1, 6
(3.8)	1, 7, 8
(3.9)	1, 9 to 14

case (3.6), for example, it is

$$(\mathrm{Im}\ \omega_1)_{(3.6)} = 1{\cdot}01 \times 10^{-5}.$$

To illustrate how the instability depends on q^2, we compared (3.7) to other particular cases, with the same ϵ and b, but different q^2,

$$q^2 = 0{\cdot}05, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0010, \tag{3.10}$$

$$q^2 = 0{\cdot}1, \qquad \epsilon = 0{\cdot}1, \qquad b = 0{\cdot}0010. \tag{3.11}$$

It turned out that the sixth eigenmode, which was unstable for case (3.7), is still unstable, and with a similar growth rate:

$$(\mathrm{Im}\ \omega_6)_{(3.7)} \approx 0{\cdot}068, \qquad (\mathrm{Im}\ \omega_6)_{(3.10)} \approx 0{\cdot}065, \qquad (\mathrm{Im}\ \omega_6)_{(3.11)} = 0{\cdot}060.$$

The first mode, however, becomes more stable:

$$(\mathrm{Im}\ \omega_1)_{(3.7)} \approx 1{\cdot}01 \times 10^{-4},$$

$$(\mathrm{Im}\ \omega_1)_{(3.10)} \approx 0{\cdot}64 \times 10^{-4},$$

$$(\mathrm{Im}\ \omega_1)_{(3.11)} \approx 1{\cdot}55 \times 10^{-15}.$$

We conclude that increasing q^2 stabilizes the first mode, but its effect on higher modes is negligible.

To clarify the mechanism of instability, note that surface tension breaks the symmetry of equation (3.2), as the capillary term in (3.2) is invariant with respect to the change

$$\theta \to -\theta, \qquad \phi \to \phi^*,$$

while the other terms alternate their signs. As a result, $|\phi|$ is no longer symmetric; see Fig. 3, where three eigenfunctions (the unstable no. 6 and the two neighbouring ones) are graphed for case (3.11). The maxima of these and other modes are shown in Fig. 4—observe that the maximum of the unstable mode is located well inside the region of negative diffusivity ($0 < \theta < \pi$), whereas the maxima of the stable modes are located either near the boundary of the regions, or inside the positive-diffusivity region.

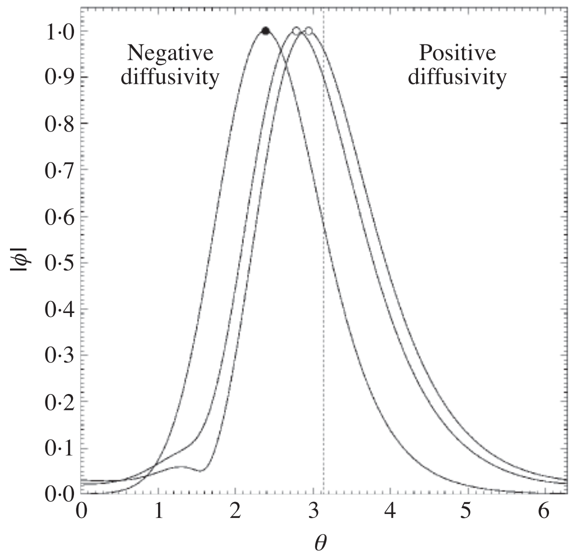

Fig. 3 The eigenfunctions of problem (3.2), (3.4) for the particular case (3.7). The unstable mode (no. 6) is marked with a black circle, the stable modes (5 and 7) are marked with 'empty' circles

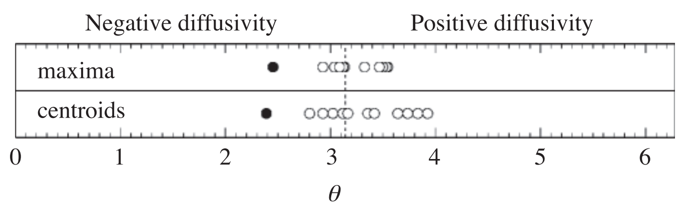

Fig. 4 The centroids (see formula (3.12)) and maxima of the eigenfunctions for case (3.11)

To reinforce this conclusion, we have also computed

$$\Theta = \int_0^{2\pi} \theta\, |\phi|^2\, d\theta \Bigg/ \int_0^{2\pi} |\phi|^2\, d\theta, \tag{3.12}$$

which can be interpreted as the 'centroid' of an eigenmode. Figure 4 shows that the centroids follow the same pattern as the maxima of the eigenfunctions.

Next, observe that the contribution of surface tension to the growth/decay rate (3.5) is always negative (see Appendix B), that is, stabilizing. The contribution of gravity, in turn, depends on the eigenfunction: for the unstable modes (which are 'shifted' into the region of negative diffusivity), the contribution of gravity is also negative; see the term involving q^2 in (3.5). On the other hand, the contribution of hydrostatic pressure (the term involving ϵ) for the unstable modes is positive. Hence,

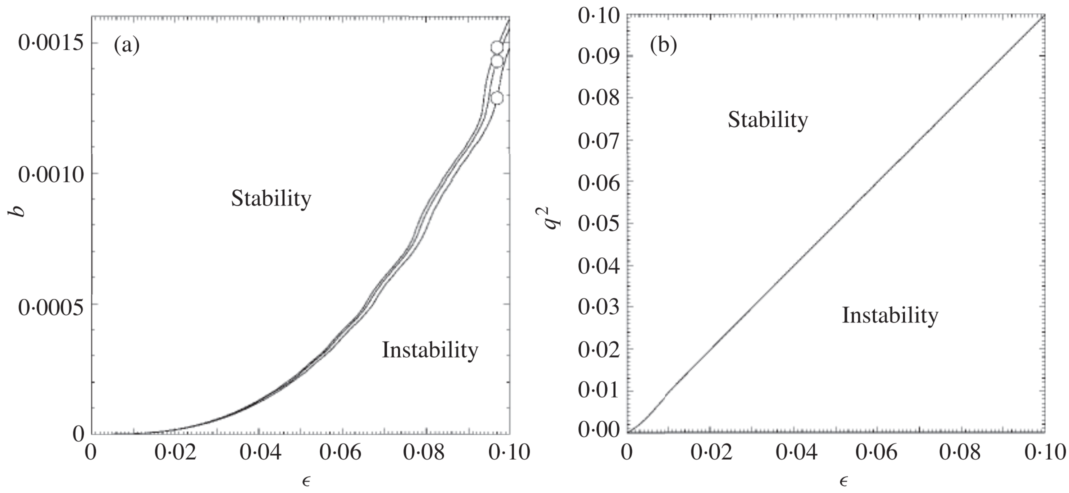

Fig. 5 The curves of marginal stability: (a) the high-n instability on the (ϵ, b)-plane, (b) the first-mode instability on the $\left(\epsilon, q^2\right)$-plane

the film's stability is determined by the competition between the destabilizing effect of hydrostatic pressure and stabilizing effect of surface tension/gravity.

We conclude that the influence of surface tension on the eigenmodes is two-fold.

1. Surface tension moves some of the eigenmodes into the upper half of the cylinder (region of negative diffusivity), where they are destabilized by hydrostatic pressure and stabilized by gravity.
2. Surface tension also stabilizes all eigenmodes 'by itself' (regardless of gravity).

We have also computed the marginal stability curve of the high-n instability, on the (ϵ, b)-plane for various values of q^2 (Fig. 5a), and that of the first-mode instability, on the (ϵ, q^2)-plane for various values of b (Fig. 5b). It turned out that the former is not very sensitive to q^2 (as long as that remains small), whereas the latter is almost independent of b and can, with a very high accuracy, be approximated by

$$q^2 \approx \epsilon$$

(the curve shown in Fig. 5b has been computed for $b = 0{\cdot}001$, but it is very much the same for any $b \lesssim 0{\cdot}5$).

Finally, note that the case of finite q has been previously considered in (**8**). Both stable and unstable eigenmodes were found—which is corroborated here. However, it was incorrectly conjectured in (**8**) that surface tension has only a stabilizing effect upon the eigenmodes, with the implication that the zero surface tension case, $b = 0$, is always unstable. In fact, as we have shown above, the case $b = 0$ is neutrally stable—but, nevertheless, adding surface tension can paradoxically give rise to unstable solutions, via the interaction between the surface tension and (anti)diffusion terms.

3.3 *The case* $b \neq 0$*: analytical results*

Problem (3.2), (3.4) can be solved asymptotically, by expanding its solution in powers of the small parameters q^2 and ϵ. For simplicity, we shall assume them to be of the same order, that is, put

$$q^2 = \epsilon Q,$$

where Q is a constant of order one. Then, seek a solution to (3.2), (3.4) in the form

$$\phi = \phi^{(0)} + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots, \qquad \omega = \omega^{(0)} + \epsilon\omega^{(1)} + \epsilon^2\omega^{(2)} + \cdots.$$

At the zeroth order, we have

$$\frac{d\phi^{(0)}}{d\theta} + b\left(\frac{d^2\phi^{(0)}}{d\theta^2} + \frac{d^4\phi^{(0)}}{d\theta^4}\right) - i\omega^{(0)}\phi^{(0)} = 0, \qquad \phi^{(0)}(\theta + 2\pi) = \phi^{(0)}(\theta),$$

which yields

$$\phi^{(0)} = e^{in\theta}, \tag{3.13}$$

$$\omega^{(0)} = n - ib\left(n^4 - n^2\right), \tag{3.14}$$

where n is an integer (the mode number). In the first order, we have

$$\frac{d\phi^{(1)}}{d\theta} + b\left(\frac{d^2\phi^{(1)}}{d\theta^2} + \frac{d^4\phi^{(1)}}{d\theta^4}\right) - i\omega^{(0)}\phi^{(1)} = i\omega^{(1)}\phi^{(0)} + \frac{d}{d\theta}\left(Q\frac{e^{i\theta} + e^{-i\theta}}{2}\phi^{(0)} - \frac{e^{i\theta} - e^{-i\theta}}{2i}\frac{d\phi^{(0)}}{d\theta}\right). \tag{3.15}$$

Substituting (3.13) and (3.14) for $\phi^{(0)}$ and $\omega^{(0)}$, one can verify that (3.15) only has a periodic solution if no terms involving $e^{in\theta}$ appear on the right-hand side. This yields

$$\omega^{(1)} = 0. \tag{3.16}$$

Then the first-order eigenfunction is

$$\phi^{(1)} = \frac{1}{2}\left[\frac{(n - Q)(n + 1)e^{i(n+1)\theta}}{2i\beta n(n + 1)(2n + 1) - 1} + \frac{(n + Q)(n - 1)e^{i(n-1)\theta}}{2i\beta n(n - 1)(2n - 1) - 1}\right]. \tag{3.17}$$

At the next order, we have

$$\frac{d\phi^{(2)}}{d\theta} + b\left(\frac{d^2\phi^{(2)}}{d\theta^2} + \frac{d^4\phi^{(2)}}{d\theta^4}\right) - i\omega^{(0)}\phi^{(2)} = i\left(\omega^{(1)}\phi^{(1)} + \omega^{(2)}\phi^{(0)}\right) + \frac{d}{d\theta}\left(Q\frac{e^{i\theta} + e^{-i\theta}}{2}\phi^{(1)} - \frac{e^{i\theta} - e^{-i\theta}}{2i}\frac{d\phi^{(1)}}{d\theta}\right). \tag{3.18}$$

Now, substitute (3.13), (3.14), (3.16), (3.17) into the right-hand side of (3.18) and require that $\phi^{(2)}$ be periodic (which essentially implies that all terms involving $e^{in\theta}$ on the right-hand side cancel out). As a result, we obtain the following expression for $\omega^{(2)}$:

$$\omega^{(2)} = \frac{(n-1-Q)\,(n+Q)\,(n-1)}{8i\beta\,(n-1)\,(2n-1)-1} - \frac{(n+1+Q)\,(n-Q)\,(n+1)}{8i\beta\,(n+1)\,(2n+1)-1}. \tag{3.19}$$

The next-order frequency turns out to be zero:

$$\omega^{(3)} = 0, \tag{3.20}$$

as well as all further odd-numbered corrections to ω. Then, collecting (3.14), (3.16) and (3.19), (3.20), we obtain

$$\begin{aligned}\omega = {} & n + ib\left(n^2 - n^4\right) \\ & + \frac{\left[\epsilon\,(n-1) - q^2\right]\left(\epsilon n + q^2\right)(n-1)}{8i\beta\,(n-1)\,(2n-1)-1} - \frac{\left[\epsilon\,(n+1) + q^2\right]\left(\epsilon n - q^2\right)(n+1)}{8i\beta\,(n+1)\,(2n+1)-1} \\ & + O\left(\epsilon^4, \epsilon^2 q^4, q^8\right).\end{aligned} \tag{3.21}$$

3.4 *Discussion*

1. Note that (3.21) has been formally derived for $b = O(1)$, in which case it is valid for all n. If, however, b is small, then, for some n, the leading-order terms in equation (3.2) may become comparable to the perturbation terms. Estimating the former

$$\phi = O(1), \qquad b\left(\frac{d\phi}{d\theta} + \frac{d^3\phi}{d\theta^3}\right) = O(bn^3)$$

and the latter

$$\left(q^2\cos\theta\right)\phi = O(q^2), \qquad \epsilon\sin\theta\,\frac{d\phi}{d\theta} = O(\epsilon n),$$

one can show that

- if $b \gg \epsilon^3$, (3.21) is valid for all n,
- if, however,

$$b \lesssim \epsilon^3, \tag{3.22}$$

then (3.21) does not work for n such that

$$\frac{1}{\epsilon} \lesssim n \lesssim \left(\frac{\epsilon}{b}\right)^{1/2}. \tag{3.23}$$

Unfortunately, the above restriction makes the asymptotic formula (3.21) inapplicable to the high-n instability, as all unstable modes resulting from (3.21) satisfy conditions (3.22), (3.23). Accordingly, the numerical and asymptotic results in Fig. 2 disagree for all unstable modes but the first one.

2. Substituting $n = 1$ into (3.21), we obtain

$$\operatorname{Im}\omega_1 \approx \frac{10\beta\left(2\epsilon + q^2\right)\left(\epsilon - q^2\right)}{400\beta^2 + 1}. \tag{3.24}$$

   This result shows that the first mode is unstable if $\epsilon > q^2$—which agrees with Fig. 5b.
3. It can also be shown that, for the high-n modes, the hydrostatic-pressure term in equation (3.2) (the one involving ϵ) is comparable to the leading-order term (the first term in the square brackets)—as a result, lubrication theory, from which (3.2) was derived, becomes inapplicable. Thus, not only is our asymptotic solution not valid for equation (3.2), but the latter is not valid itself!

   Despite that, the fact that a dissipative term can cause instability is still of significant mathematical interest—even though the model where this occurs has no immediate physical applications.
4. Finally, note that the first-mode instability found here is different from a similar phenomenon examined in (**9**). The latter is caused by inertia, whereas the former is a result of the combined effect of hydrostatic pressure and surface tension.

4. The effect of surface tension on exploding solutions

In this section, we shall examine how surface tension affects the exploding solutions, that is, those solutions of equation (2.12) that would develop a singularity for $b = 0$. As shown in (**1**), these solutions result from an initial condition corresponding to a narrow Gaussian pulse—which, under the constraint of periodicity, can be modelled by

$$h(\theta, 0) = \exp\left(-\frac{1 - \cos\theta}{W_0^2}\right), \tag{4.1}$$

where W_0 is the width of the pulse. If $b = 0$ and $W_0 > 2\epsilon$, the solution of the initial-value problem (2.12), (4.1) is smooth and periodic in time, with a period of 2π; see (**1**). If, however, the initial pulse is sufficiently narrow ($W_0 \leqslant 2\epsilon$), it explodes in a finite time (its width collapses and its amplitude shoots to infinity).

In order to examine how such behaviour is modified by surface tension ($b \neq 0$), several particular cases have been computed numerically. The numerical method consisted in a fourth-order Runge–Kutta scheme for the time derivative and symmetric differences for the spatial derivatives.

We shall present the results for

$$W_0 = 0{\cdot}1, \qquad \epsilon = 0{\cdot}1, \qquad q^2 = 0{\cdot}1, \qquad b = 0{\cdot}0010. \tag{4.2}$$

The evolution of the initial condition (4.1) for parameters (4.2) is shown in Fig. 6. One can see that even a single rotation of the cylinder triggers off an unstable disturbance which exceeds the pulse's initial amplitude by an order of magnitude.

It is also instructive to introduce the amplitude of the solution,

$$h_{\max}(t) = \max_{0<\theta\leqslant 2\pi}\{h(\theta, t)\}. \tag{4.3}$$

Then, scaling the solution by $h_{\max}$, we can show that the disturbance has the shape of the most

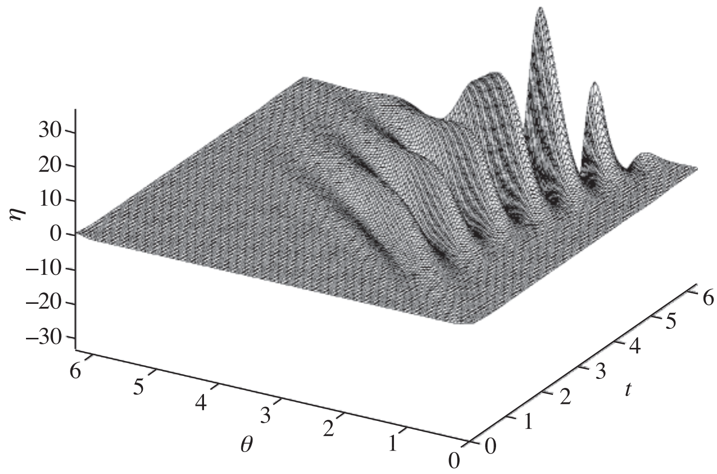

Fig. 6 The solution $h(\theta, t)$ of the initial-value problem (2.12), (4.1) for parameters (4.2) and $t \in [0, 2\pi]$

unstable eigenmode (see Fig. 7), while the growth of $h_{\max}$ is consistent with the eigenmode's growth rate (see Fig. 8). Note also that the pattern emerging in our simulations can be interpreted as a gravity-modified version of the 'multi-petal' structures observed in (**7**, Fig. 4). These structures, however, were observed for zero gravity, which was why the 'petals' there were of the same amplitude—whereas the extrema of the eigenfunction shown in Fig. 7 below are all different (depending on their locations within the cylinder).

We have also simulated initial condition (4.1) for larger q^2 and b, such that no unstable modes exist. In such cases, the initial disturbance grew initially (while travelling through the region of negative diffusivity), but eventually always decayed.

We conclude that surface tension transforms explosive disturbances into the usual harmonic disturbances (which grow/decay exponentially).

5. Summary and concluding remarks

Thus, we have examined the effect of surface tension on the instability of a viscous film inside a rotating cylinder. There are three governing parameters in this problem: the non-dimensional thickness q of the film, hydrostatic-pressure parameter ϵ, and non-dimensional capillary coefficient b. Summarizing (2.11), (3.3), (2.2) and (2.4), we obtain

$$q = \left(\frac{gR}{\nu\Omega}\right)^{1/2} \frac{\hat{\hat{h}}}{R}, \qquad \epsilon = \frac{q^3}{3}\left(\frac{\nu\Omega}{gR}\right)^{1/2}, \qquad b = \frac{\gamma q^3}{3\rho g R^2}\left(\frac{\nu\Omega}{gR}\right)^{1/2},$$

where $\hat{\hat{h}}$ is the dimensional thickness, R and Ω are the radius of the cylinder and the angular velocity of its rotation, ρ, γ, and ν are the density, surface tension, and kinematic viscosity, and g is acceleration due to gravity.

It has been shown that, if $b \neq 0$, capillary effects eliminate the exploding solutions (which would

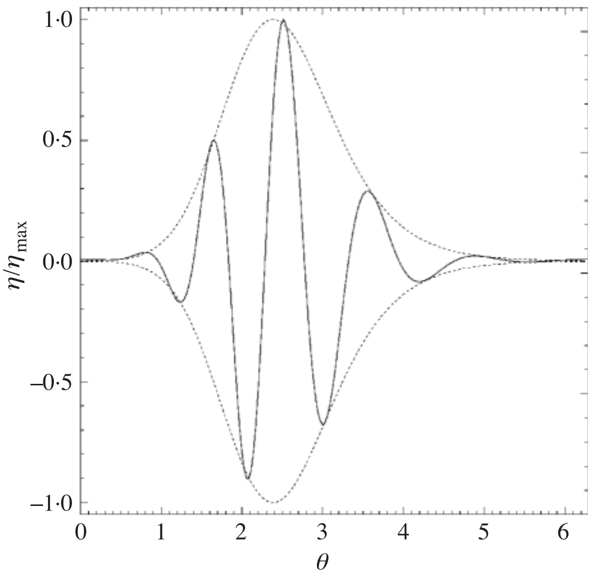

Fig. 7 A snapshot, for $t = 2\pi$, of the solution of the initial-value problem (2.12), (4.1) for parameters (4.2). The amplitude h_{max} of the solution is defined by (4.3). The dotted line shows $\pm|\phi(\theta)|$ for the most unstable eigenmode ($n = 6$)

exist otherwise) and stabilize most of the eigenmodes. For some eigenmodes, however, surface tension is a *de*stabilizing influence, with the instability observed in two parameter regions.

1. If $\epsilon \gtrsim q^2$, it has been shown, both analytically and numerically, that the first ($n = 1$) eigenmode is unstable (see Fig. 5b).
2. If b is smaller than a certain threshold value

$$b < b_{\text{threshold}} = O\left(\epsilon^3\right),$$

then it has been shown numerically that there are several unstable eigenmodes, such that

$$\frac{1}{\epsilon} \lesssim n \lesssim \left(\frac{\epsilon}{b}\right)^{1/2}$$

(see Figs 2 and 5a).

The counter-intuitive conclusion with regard to the destabilizing effect of surface tension can be interpreted on the basis of the asymmetry of the corresponding eigenfunctions; see section 3.2.

Finally, note that our main equation (2.12) is not valid for the high-n (unstable) modes, as those violate the lubrication theory (LT). Still, the fact that a dissipative term can cause instability is of a significant mathematical interest, despite the fact that the model where this occurs has no immediate physical applications. The physically motivated reader, however, should wait until the problem is re-examined for non-LT disturbances—which will be a much harder task, but not a hopeless one. After all, the film's steady state still remains within the limits of the LT—hence, the problem should be tractable by asymptotic means.

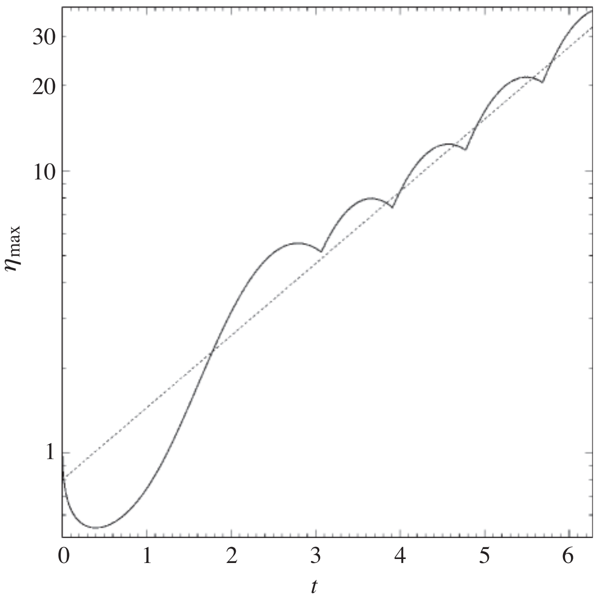

Fig. 8 The amplitude $h_{\max}$ (defined by (4.3)) of the solution of the initial-value problem (2.12), (4.1) versus t, for parameters (4.2). The dotted line shows the growth rate of the most unstable eigenmode ($n = 6$).

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APPENDIX A

Numerical method for eigenvalue problem (3.2)

The numerical method for solving (3.2), (3.4) is based on representing the solution by its complex Fourier series,

$$\phi = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}. \tag{A.1}$$

Substitution of (A.1) into (3.2) and routine algebra yield the following equation for the Fourier coefficients c_k:

$$\sum_{k=-\infty}^{\infty} A_{l,k} c_k = \omega c_l,$$

where

$$A_{l,k} = \left[k - ib\left(k^4 - k^2\right)\right]\delta_{l,k} + \frac{l\left[\epsilon\,(l-1) - q^2\right]}{2}\delta_{k,l-1} - \frac{l\left[\epsilon\,(l+1) + q^2\right]}{2}\delta_{k,l+1},$$

and $\delta_{l,k}$ is the Kronecker delta. Thus, the problem is reduced to finding the eigenvalues of an infinite tridiagonal matrix $A_{l,k}$. In practice, $A_{l,k}$ is truncated at a large but finite size, and its eigenvalues are computed using a suitable numerical algorithm (we used MATLAB's EIG function for sparse matrices).

APPENDIX B

Contribution of surface tension to growth/decay rate

To demonstrate that the term involving b in equation (3.5) is negative, we shall make use of the complex Fourier series (A.1). After straightforward algebra, we obtain

$$b\int_0^{2\pi}\left(\left|\frac{d\phi}{d\theta}\right|^2 - \left|\frac{d^2\phi}{d\theta^2}\right|^2\right)d\theta = 2\pi b\sum_{k=-\infty}^{\infty}\left(k^2 - k^4\right)|c_k|^2,$$

which is negative, as required.