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Stability of a liquid bridge under vibration

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We examine the stability of a vertical liquid bridge between two vertically vibrating, coaxial disks. Assuming that the vibration amplitude and period are much smaller than the mean distance between the disks and the global timescale, respectively, we employ the method of multiple scales to derive a set of asymptotic equations. The set is then used to examine the stability of a bridge of an almost cylindrical shape. It is shown that, if acting alone, gravity is a destabilizing influence, whereas vibration can weaken it or even eliminate altogether. Thus, counter-intuitively, vibration can stabilize an otherwise unstable capillary structure.

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I. INTRODUCTION

It is well known [1–25] that sufficiently slender liquid bridges between coaxial disks are unstable due to Plateau-Rayleigh instability. It has also been shown [2–4,7,10] that, for bridges with vertical axes, gravity strengthens the instability.

This paper examines vertical bridges between coaxial disks, experiencing high-frequency small-amplitude vertical vibration. Given the generally destabilizing nature of vibration, one would expect it to reduce the parameter range of stable bridges. It turns out, however, that this is not necessarily the case.

Mathematically, we shall demonstrate that the (fast) vibration and (slow) natural motions of the bridge can be separated asymptotically: the former is described by a mixed Dirichlet-Neumann problem for the Laplace equation in the domain occupied by the bridge, and the latter is governed by the Navier-Stokes equations with a vibration-induced force and pressure fields (the latter is applied at the free surface). It turns out that, if the vibration amplitude of the upper disk is greater than that of the lower disk, the vibration-induced pressure gradient is opposite to the hydrostatic one. As a result, the destabilizing effect of gravity can be weakened or even canceled altogether. A similar mechanism of stabilization has been observed experimentally using the radiation pressure of acoustic waves [20] and a surrounding flow of a different fluid [21,25].

This paper has the following structure. In Sec. II, we formulate the problem mathematically, and in Sec. III, derive asymptotic equations for a vertical liquid bridge affected by high-frequency, small-amplitude vibration. In Sec. IV, these equations are used to examine almost cylindrical bridges (which implies that surface tension is stronger than gravity and vibration, and the disks' radii do not differ much from the bridge's mean radius). This simple particular case has been extensively studied for static bridges (e.g., Refs. [2–4,11,12,24]), so it should be a reasonable departure point when studying vibrating ones.

In fact, it turns out to be sufficient for answering the most interesting question: Can vibration stabilize an otherwise unstable liquid bridge?

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II. FORMULATION

Consider a flow with velocity $\mathbf{u}$ and pressure p, in an incompressible liquid of density ρ, kinematic viscosity ν, and surface tension σ. Let the z axis of the cylindrical coordinate system (r,θ,z) be directed upward, and $\mathbf{u} = (u,v,w)$, where u, v, and w are the radial, azimuthal, and axial components, respectively.

The Navier-Stokes equations can be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\rho}\nabla p = -g\mathbf{e}_z + \nu\nabla\cdot\mathbf{S}, \quad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where t is the time, g is the acceleration due to gravity, $\mathbf{e}_z$ is the upward unit vector, and

$$\mathbf{S} = \begin{bmatrix} 2\frac{\partial u}{\partial r} & \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \frac{\partial v}{\partial r} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \frac{\partial v}{\partial r} & \frac{2}{r}\left(\frac{\partial v}{\partial \theta} + u\right) & \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} & \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} & 2\frac{\partial w}{\partial z} \end{bmatrix} \tag{2}$$

is the viscous stress tensor in cylindrical coordinates.

Let the liquid be bounded above and below by two horizontal coaxial disks and laterally, by a free surface (see Fig. 1). The disks are assumed to vibrate vertically with the same frequency ω, but different amplitudes $D_\pm$ and phases $\alpha_\pm$ (where the "+" corresponds to the upper disk). This implies that the disks are located at $z = \pm H + D_\pm \sin(\omega t + \alpha_\pm)$, where $2H$ is the mean distance between them, and the no-slip, no-through-flow condition takes the form

$$\mathbf{u} = \mathbf{e}_z D_\pm \omega \cos(\omega t + \alpha_\pm) \quad \text{at} \quad z = \pm H + D_\pm \sin(\omega t + \alpha_\pm). \tag{3}$$

Let the free surface be described by the equation $r = R(\theta,z,t)$ and assume that the contact lines are pinned to the disks' edges, i.e.,

$$R = R_\pm \quad \text{at} \quad z = \pm H + D_\pm \sin(\omega t + \alpha_\pm), \tag{4}$$

where $R_\pm$ are the disks' radii. Condition (4) is usually justified by the fact that the rapid change of the tangent at the disks' edges prevents the contact line from moving, which has also been ascertained experimentally [22]. Furthermore, a contact line can be pinned even to a flat part of the disk, provided the flow is week enough to not force the contact angle outside the hysteresis interval—which can indeed be assumed in a stability study, where linear perturbations imply *infinitesimal* change of the contact angle [26,27].

The kinematic condition at the free surface is

$$\frac{\partial R}{\partial t} + \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{at} \quad r = R, \tag{5}$$

whereas the dynamic condition can be written in the form

$$[(\sigma C - p)\mathbf{E} + \rho\nu\mathbf{S}]\cdot\mathbf{n} = \mathbf{0} \quad \text{at} \quad r = R, \tag{6}$$

where $\mathbf{E}$ is the unit matrix,

$$C = \frac{R^2 + 2\left(\frac{\partial R}{\partial\theta}\right)^2 - R\frac{\partial^2 R}{\partial\theta^2} + R^2\left(\frac{\partial R}{\partial z}\right)^2 - R^3\frac{\partial^2 R}{\partial z^2} - R\left[\left(\frac{\partial R}{\partial z}\right)^2\frac{\partial^2 R}{\partial\theta^2} - 2\frac{\partial R}{\partial\theta}\frac{\partial R}{\partial z}\frac{\partial^2 R}{\partial z\,\partial\theta} + \left(\frac{\partial R}{\partial\theta}\right)^2\frac{\partial^2 R}{\partial z^2}\right]}{\left[R^2 + \left(\frac{\partial R}{\partial\theta}\right)^2 + R^2\left(\frac{\partial R}{\partial z}\right)^2\right]^{3/2}}, \tag{7}$$

is the curvature of the free surface in terms of the cylindrical coordinates [28], and

$$\mathbf{n} = \begin{bmatrix} -1 \\ \frac{1}{R}\frac{\partial R}{\partial\theta} \\ \frac{\partial R}{\partial z} \end{bmatrix}. \tag{8}$$

The problem will be non-dimensionalized as follows:

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{H'}, \quad \tilde{t} = \omega t, \quad \tilde{\mathbf{u}} = \sqrt{\frac{\rho H'}{\sigma}}\mathbf{u}, \quad \tilde{p} = \frac{D}{\sigma}p,$$
$$\tilde{R} = \frac{R}{H'}, \quad \tilde{R}_\pm = \frac{R_\pm}{H'}, \quad \tilde{C} = CH',$$

where

$$H' = \pi^{-1}H, \quad D = \sqrt{D_-^2 + D_+^2}. \tag{9}$$

When rewritten in terms of the non-dimensional variables with the tildes omitted, expressions (2) and (7)–(8) do not change, whereas Eqs. (1) and (3)–(6) become

$$W\frac{\partial\mathbf{u}}{\partial t} + \varepsilon\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p = -\varepsilon\gamma\mathbf{e}_z + \varepsilon\mu\nabla\cdot\mathbf{S}, \tag{10}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{11}$$

$$\mathbf{u} = \mathbf{e}_z W_\pm\cos(t + \alpha_\pm)$$
$$\text{at} \quad z = \pm\pi + \varepsilon W_\pm\sin(t + \alpha_\pm), \tag{12}$$

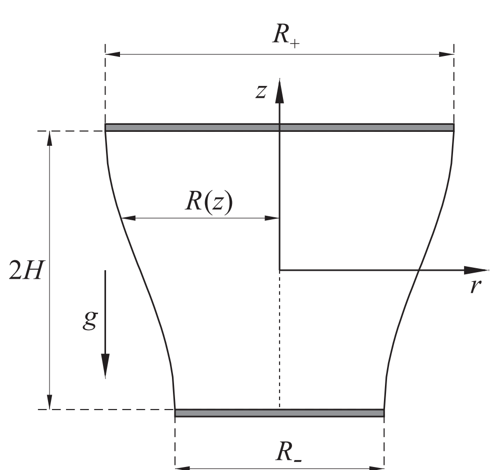

FIG. 1. The setting: a cross section of a liquid bridge between two coaxial disks. The disks experience high-frequency, small-amplitude vertical vibration.

$$R = R_\pm \quad \text{at} \quad z = \pm\pi + \varepsilon W_\pm\sin(t + \alpha_\pm), \tag{13}$$

$$[(\varepsilon C - p)\mathbf{E} + \varepsilon\mu\mathbf{S}]\cdot\mathbf{n} = \mathbf{0} \quad \text{at} \quad r = R, \tag{14}$$

$$W\frac{\partial R}{\partial t} + \varepsilon\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{at} \quad r = R, \tag{15}$$

where

$$\varepsilon = \frac{D}{H'}, \tag{16}$$

$$\gamma = \frac{\rho g H'^2}{\sigma}, \quad \mu = \nu\sqrt{\frac{\rho}{\sigma H'}},$$
$$W = \omega D\sqrt{\frac{\rho H'}{\sigma}}, \quad W_\pm = \omega D_\pm\sqrt{\frac{\rho H'}{\sigma}}. \tag{17}$$

The Bond number γ and the parameter μ (sometimes called the Ohnesorge number) characterize the effects of gravity and viscosity relative to surface tension. W and $W_\pm$, in turn, characterize the effect of vibration–again, relative to surface tension. Note also that Eq. (9) implies that $W = \sqrt{W_-^2 + W_+^2}$.

Equations (10)–(15), (2), and (7),(8) form a governing set for the unknowns $\mathbf{u}$, p, and R.

III. ASYMPTOTIC ANALYSIS OF THE GOVERNING EQUATIONS

Let the displacement of the disks due to vibration be much smaller than the average distance between them, in which case Eq. (16) implies

$$\varepsilon \ll 1.$$

No assumptions are made about γ, μ, W, and $W_\pm$ [defined by Eq. (17)]: they can be either small, or order-one, or large (but not as large as ε^{-1}).

The smallness of ε will be exploited using the method of multiple scales. In addition to the (fast) time t, a slow time is introduced, $T = W^{-1}\varepsilon t$ (where the order-one factor of W^{-1} is included for future convenience). Equations (10) and (15) should then be rewritten in the form

$$W\frac{\partial\mathbf{u}}{\partial t} + \varepsilon\left[\frac{\partial\mathbf{u}}{\partial T} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u})\right] + \nabla p$$
$$= -\varepsilon\gamma\mathbf{e}_z + \varepsilon\mu\nabla\cdot\mathbf{S}, \tag{18}$$

$$W\frac{\partial R}{\partial t} + \varepsilon\left(\frac{\partial R}{\partial T} + \mathbf{u}\cdot\mathbf{n}\right) = 0 \quad \text{at} \quad r = R. \tag{19}$$

Seek a solution as a series in ε,

$$(\mathbf{u},p,R) = (\mathbf{u},p,R)^{(0)} + \varepsilon(\mathbf{u},p,R)^{(1)} + \cdots .$$

The zeroth and first orders of the expansion will be presented in Secs. III A and III B, respectively, then Sec. III C will summarize the asymptotic equations derived.

A. The zeroth order

Equation (19) shows that $R^{(0)}$ is slow, i.e., independent of t,

$$R^{(0)} = \bar{R}(r,\theta,z,T)$$

with $\bar{R}$ satisfying the zeroth-order version of Eq. (13),

$$\bar{R} = R_\pm \quad \text{at} \quad z = \pm\pi. \tag{20}$$

In turn, the zeroth-order version of Eq. (18),

$$W\frac{\partial \mathbf{u}^{(0)}}{\partial t} + \nabla p^{(0)} = \mathbf{0}, \tag{21}$$

implies that, even if $\mathbf{u}^{(0)}$ involves both slow and fast components, $p^{(0)}$ can only be fast. Furthermore, since the fast flow is forced by the disks' motion, it inherits from it the sinusoidal dependence on t—hence,

$$\mathbf{u}^{(0)} = \bar{\mathbf{u}}(r,\theta,z,T) + \mathbf{u}_s^{(0)}(r,\theta,z,T)\,\sin t + \mathbf{u}_c^{(0)}(r,\theta,z,T)\,\cos t, \tag{22}$$

$$p^{(0)} = p_s(r,\theta,z,T)\,\sin t + p_c(r,\theta,z,T)\,\cos t \tag{23}$$

with Eq. (21) yielding

$$\mathbf{u}_s^{(0)} = -W^{-1}\nabla p_c, \quad \mathbf{u}_c^{(0)} = W^{-1}\nabla p_s. \tag{24}$$

Substitution of Eqs. (22)–(24) into Eqs. (11), (12), and (14) shows that the slow flow satisfies

$$\nabla\cdot\bar{\mathbf{u}} = 0, \tag{25}$$

$$\bar{\mathbf{u}} = 0 \quad \text{at} \quad z = \pm\pi, \tag{26}$$

whereas $p_{s,c}$ are fully determined by

$$\nabla^2 p_s = 0, \quad \nabla^2 p_c = 0, \tag{27}$$

$$p_s = p_c = 0, \quad \text{at} \quad r = \bar{R}, \tag{28}$$

$$\left.\begin{aligned} W^{-1}\frac{\partial p_c}{\partial z} &= -W_\pm \sin\alpha_\pm, \\ W^{-1}\frac{\partial p_s}{\partial z} &= W_\pm \cos\alpha_\pm \end{aligned}\right\} \quad \text{at} \quad z = \pm\pi. \tag{29}$$

Observe that, the no-slip condition for the fast flow was omitted [if it were not, the boundary-value problems for the Laplace equations (27) would be over-determined]. The omission is due to the fact that viscosity does not appear in the zeroth-order equation (21) and, thus, does not affect the fast flow.[1] The slow component, however, is governed by the next-order equations, where viscosity *will* be taken into account—which is why the no-slip condition has been retained for $\bar{\mathbf{u}}$ [it is a part of Eq. (26)].

B. The first order

In the first order, Eqs. (18), (19), and (14) yield

$$W\frac{\partial \mathbf{u}^{(1)}}{\partial t} + \frac{\partial \mathbf{u}^{(0)}}{\partial T} + \nabla\cdot(\mathbf{u}^{(0)}\otimes\mathbf{u}^{(0)}) + \nabla p^{(1)} = -\gamma\mathbf{e}_z + \mu\nabla\cdot\mathbf{S}^{(0)}, \tag{30}$$

$$W\frac{\partial R^{(1)}}{\partial t} + \frac{\partial \bar{R}}{\partial T} + \mathbf{u}^{(0)}\cdot\mathbf{n}^{(0)} = 0 \quad \text{at} \quad r = R^{(0)}, \tag{31}$$

$$\left[\left(sC^{(0)} - p^{(1)} - \frac{\partial p^{(0)}}{\partial r}R^{(1)}\right)\mathbf{E} + \mu\mathbf{S}^{(0)}\right]\cdot\mathbf{n}^{(0)} = \mathbf{0} \quad \text{at} \quad r = R^{(0)}, \tag{32}$$

where the expressions for $\mathbf{S}^{(0)}$, $C^{(0)}$, and $\mathbf{n}^{(0)}$ can be obtained from Eqs. (2), (7), and (8) by changing $(\mathbf{u},R)\to(\mathbf{u}^{(0)},R^{(0)})$. Note that the term $p^{(0)}\mathbf{n}^{(1)}$ has been omitted from Eq. (32), because $p^{(0)} = 0$ at $r = R^{(0)}$ [as follows from Eqs. (23) and (28)].

Equation (30) implies that the secular growth of $\mathbf{u}^{(1)}$ can be avoided only if all non-oscillating terms in this equation cancel— hence, taking into account Eqs. (22) and (24), one obtains

$$\frac{\partial\bar{\mathbf{u}}}{\partial T} + \nabla\cdot\left[\bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \frac{1}{2}\left(\nabla p_c\otimes\nabla p_c + \nabla p_s\otimes\nabla p_s\right)\right] + \nabla\bar{p} = -\gamma\mathbf{e}_z + \mu\nabla\cdot\bar{\mathbf{S}}, \tag{33}$$

where $\bar{p}$ is the slow component of $p^{(1)}$ and $\bar{\mathbf{S}}$ can be obtained from Eq. (2) by changing $\mathbf{u}\to\bar{\mathbf{u}}$. Equation (33) can be simplified using the incompressibility conditions (25) and (27), and thus becomes

$$\frac{\partial\bar{\mathbf{u}}}{\partial T} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} + \nabla\left[\frac{1}{4W^2}(|\nabla p_c|^2 + |\nabla p_s|^2) + \bar{p}\right] = -\gamma\mathbf{e}_z + \mu\nabla\cdot\bar{\mathbf{S}}. \tag{34}$$

Next, Eqs. (31) and (22) imply that the secular growth of $R^{(1)}$ can be avoided only if

$$\frac{\partial\bar{R}}{\partial T} + \bar{\mathbf{u}}\cdot\bar{\mathbf{n}} = 0 \quad \text{at} \quad r = \bar{R}, \tag{35}$$

where $\bar{\mathbf{n}}$ can be obtained from Eq. (8) by changing $R\to\bar{R}$. Then Eqs. (31), (22), and (24) yield

$$R^{(1)} = -W^{-1}(\nabla p_c\cdot\bar{\mathbf{n}}\cos t + \nabla p_s\cdot\bar{\mathbf{n}}\sin t) \quad \text{at} \quad r = \bar{R}.$$

Substituting this expression into Eq. (32), taking into account Eq. (23), and separating the fast and slow components, one obtains for the latter

$$\left\{\left[s\bar{C} - \bar{p} + \frac{1}{2W^2}\left(\frac{\partial p_c}{\partial r}\nabla p_c\cdot\bar{\mathbf{n}} + \frac{\partial p_s}{\partial r}\nabla p_s\cdot\bar{\mathbf{n}}\right)\right]\mathbf{E} + \mu\bar{\mathbf{S}}\right\}\cdot\bar{\mathbf{n}} = \mathbf{0} \quad \text{at} \quad r = \bar{R}, \tag{36}$$

where $\bar{C}$ can be obtained from Eq. (7) by changing $R\to\bar{R}$.

Equation (36) can be simplified: differentiating conditions (28) with respect to θ or z, and using the resulting

[1]To enforce the no-slip boundary condition for the fast flow, one needs to examine the near-wall boundary layers. The effect of these layers on the global flow, however, is weak [15].

identities as well as Eq. (33), one can show that

$$\frac{\partial p_c}{\partial r}\nabla p_c\cdot\bar{\mathbf{n}}+\frac{\partial p_s}{\partial r}\nabla p_s\cdot\bar{\mathbf{n}}=-|\nabla p_c|^2-|\nabla p_s|^2 \quad \text{at} \quad r=\bar{R},$$

hence, Eq. (36) becomes

$$\left\{\left[s\bar{C}-\bar{p}-\frac{1}{2W^2}(|\nabla p_c|^2+|\nabla p_s|^2)\right]\mathbf{E} +\mu\mathbf{S}^{(0)}\right\}\cdot\bar{\mathbf{n}}=\mathbf{0} \quad \text{at} \quad r=\bar{R}. \tag{37}$$

C. The full asymptotic set (summary)

It is convenient to introduce a complex variable characterizing the vibration-induced pressure,

$$q=W^{-1}(-p_s+ip_c).$$

Then, the asymptotic equations (20), (25)–(29), (34), (35), and (37) become (overbars omitted)

$$\nabla^2 q=0, \tag{38}$$

$$q=0 \quad \text{at} \quad r=R, \tag{39}$$

$$\frac{\partial q}{\partial z}=-W_\pm e^{i\alpha_\pm} \quad \text{at} \quad z=\pm\pi, \tag{40}$$

$$\frac{\partial \mathbf{u}}{\partial T}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\left(\frac{1}{4}|\nabla q|^2+p\right)=-\gamma\mathbf{e}_z+\mu\nabla\cdot\mathbf{S}, \tag{41}$$

$$\nabla\cdot\mathbf{u}=0, \tag{42}$$

$$\mathbf{u}=0 \quad \text{at} \quad z=\pm\pi, \tag{43}$$

$$\left[\left(C-p-\frac{1}{2}|\nabla q|^2\right)\mathbf{E}+\mu\mathbf{S}\right]\cdot\mathbf{n}=\mathbf{0} \quad \text{at} \quad r=R, \tag{44}$$

$$\frac{\partial R}{\partial T}+\mathbf{u}\cdot\mathbf{n}=0 \quad \text{at} \quad r=R, \tag{45}$$

$$R=R_\pm \quad \text{at} \quad z=\pm\pi, \tag{46}$$

where $\mathbf{S}$, C, and $\mathbf{n}$ are given by their original expressions (2), (7), and (8).

The equations derived can be subdivided into two groups: the Dirichlet-Neumann problem (38)–(40) describes the spatial distribution of the vibration amplitude q, whereas the modified Navier-Stokes set (41)–(46) governs the slow flow characterized by $\mathbf{u}$, p, and R. Observe that vibration affects the slow motion in two ways: firstly, it gives rise to a force with the potential $\frac{1}{4}|\nabla q|^2$ in Eq. (41) and, secondly, to a free-surface-applied pressure $-\frac{1}{2}|\nabla q|^2$ in condition (44). In principle, one of these terms (but not both) can be eliminated by an appropriate change of the pressure variable p.

Note that the inviscid equivalent of Eqs. (38)–(46) was derived in Ref. [29], and several asymptotic models for liquid films (but not bridges) under high-frequency vibration were examined in Refs. [30–37] . Vibrating bridges, in turn, were examined in Ref. [17], but the vibration frequency in this study was comparable to the frequency of the bridge's natural oscillations.

IV. ALMOST CYLINDRICAL BRIDGES UNDER WEAK GRAVITY AND VIBRATION

This section examines time-independent solutions of the asymptotic equations derived. It turns out that the solution bifurcates at a certain point of the parameter space, with the bifurcation signaling a change in stability. This way, a conclusion can be drawn about the stability of liquid bridges without solving the linearized problem for small perturbations (as has indeed been done many times for non-vibrating bridges—see Refs. [19,24], and references therein).

Due to viscosity, the slow flow in a steady liquid bridge dies out, whereas the oscillatory flow (forced by the vibrating disks) does not. Thus, substituting $\mathbf{u}=\mathbf{0}$ into Eq. (41), but keeping $q\neq 0$, one obtains

$$p=P-\gamma z-\tfrac{1}{4}|\nabla q|^2, \tag{47}$$

where P is a constant. Next, substitution of Eq. (47) and $\mathbf{u}=\mathbf{0}$ into Eq. (44) yields

$$C-P+\gamma z-\tfrac{1}{4}|\nabla q|^2=0 \quad \text{at} \quad r=R. \tag{48}$$

Since the bridge is axisymmetric, expression (7) for the curvature C can be simplified, reducing Eq. (48) to

$$\frac{1+\left(\frac{\partial R}{\partial z}\right)^2-R\frac{\partial^2 R}{\partial z^2}}{R\left[1+\left(\frac{\partial R}{\partial z}\right)^2\right]^{3/2}}-P+\gamma z -\frac{1}{4}\left[\left(\frac{\partial q}{\partial r}\right)^2+\left(\frac{\partial q}{\partial z}\right)^2\right]=0 \quad \text{at} \quad r=R. \tag{49}$$

One should keep in mind that R should also satisfy the boundary condition (46).

Next, the axisymmetric version of Eq. (38) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q}{\partial r}\right)+\frac{\partial^2 q}{\partial z^2}=0, \tag{50}$$

where q should satisfy the boundary conditions (39), (40). Finally, it is convenient to distinguish bridges by their net volumes—or, equivalently, by their mean-square radius R_*—i.e., requiring that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}R^2dz=R_*^2. \tag{51}$$

Thus, the set of the problem's control parameters comprises γ, R_*, $R_\pm$, $W_\pm$, and $\alpha_\pm$.

The main difficulty associated with the boundary-value problem (49)–(51), (46), (39), (40) stems from the fact that the Dirichlet–Neumann problem (50), (39), (40) admits a reasonably simple analytical solution only in a cylindrical domain. Thus, the simplest way to proceed consists in solving the problem asymptotically, under the assumption that the bridge is almost cylindrical—which implies the following restrictions:

$$\gamma\ll 1, \quad W_\pm\ll 1, \quad R_+\approx R_-\approx R_*. \tag{52}$$

Note that a *perfectly* cylindrical bridge is stable if and only if $R_*\geqslant 1$ (see Ref. [19], and references therein)—which implies that the stability of *almost* cylindrical bridges should be examined for

$$R_*\approx 1. \tag{53}$$

Indeed, if $|R_* - 1| = \mathcal{O}(1)$, the bridge is either clearly unstable or clearly stable, depending on whether $R_* < 1$ or $R_* > 1$, respectively.

Summarizing assumptions (52), (53), we assume

$$\begin{gathered} \gamma = \delta^3\hat{\gamma}, \quad R_* = 1 + \delta^2\hat{R}_*, \\ R_\pm = 1 + \delta\hat{R}_s \pm \delta^3\hat{R}_a, \quad W_\pm = \delta^{3/2}\hat{W}_\pm, \end{gathered} \tag{54}$$

where $\delta \ll 1$ whereas the parameters with hats are order-one. Observe the different scalings of different parameters (including those of the symmetric and antisymmetric parts of $R_\pm$): they were chosen by trial and error to ensure that they 'affect' the solution at the same order (as can be seen below).

Let

$$\begin{aligned} R &= 1 + \delta R_1 + \delta^2 R_2 + \delta^3 R_3 + \mathcal{O}(\delta^4), \\ P &= 1 + \delta P_1 + \delta^2 P_2 + \delta^3 P_3 + \mathcal{O}(\delta^4), \end{aligned} \tag{55}$$

$$q = \delta^{3/2}[q_0 + \mathcal{O}(\delta)]. \tag{56}$$

In what follows, static bridges, i.e., those without vibration, will be examined in Sec. IV A. This will enable us to assess the accuracy of our approach by comparing the asymptotic results to those obtained numerically from the exact equations (which are relatively simple to solve for static bridges). Vibrating bridges will be examined in Sec. IV B.

A. Static bridges

If $W_\pm = q = 0$ (no vibration), Eqs. (49)–(51), (46), and (54) reduce to

$$\frac{1 + \left(\frac{\partial R}{\partial z}\right)^2 - R\frac{\partial^2 R}{\partial z^2}}{R\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{3/2}} - P + \delta^3\hat{\gamma} z = 0, \tag{57}$$

$$R = 1 + \delta\hat{R}_s \pm \delta^3\hat{R}_a \quad \text{at} \quad z = \pm\pi, \tag{58}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R^2 dz = 1 + \delta^2\hat{R}_*. \tag{59}$$

Substituting expansions (55) into Eqs. (57)–(59) and resolving the first and second orders in δ, one obtains

$$R_1 = A\sin z - \hat{R}_s\cos z, \tag{60}$$

$$\begin{aligned} R_2 = {} & \frac{A^2 - \hat{R}_s^2}{4}(-1 + \cos 2z) + \frac{A\hat{R}_s}{2}\sin 2z \\ & + B\sin z + \left(\hat{R}_* - \frac{\hat{R}_s^2}{2}\right)(1 + \cos z), \end{aligned} \tag{61}$$

where the undetermined constants A and B are to be fixed in higher orders.

Next, consider the third-order approximation of Eqs. (57) and (58),

$$\begin{aligned} \frac{d^2R_3}{dz^2} + R_3 = {} & -R_1^3 + \frac{1}{2}\left(R_1 + 3\frac{d^2R_1}{dz^2}\right)\left(\frac{dR_1}{dz}\right)^2 \\ & + 2R_1R_2 - \frac{dR_1}{dz}\frac{dR_2}{dz} + \hat{\gamma}z - P_3, \end{aligned} \tag{62}$$

$$R_3(\pm\pi) = \pm R_a. \tag{63}$$

To determine A, multiply Eq. (62) by $\sin z$ and integrate from $z = -\pi$ to $z = \pi$. Integrating the term involving d^2R_3/dz^2 by parts twice and taking into account Eqs. (63) and (60),(61), one obtains, after some straightforward algebra,

$$\frac{3}{4}A^3 - \left(\hat{R}_* - \frac{3}{4}\hat{R}_s^2\right)A - \hat{\gamma} + \frac{\hat{R}_a}{\pi} = 0. \tag{64}$$

This equation determines the parameter A and, thus, completes the first-order solution (60).

It is interesting to examine the dependence of A on $\hat{R}_*$, as the latter characterizes the slenderness of the bridge (a decrease in $\hat{R}_*$ corresponds to the disks being slowly pulled apart). One can readily show that Eq. (64) has either one or two, or three real roots (see an example in Fig. 2), with more than one existing if and only if

$$\hat{R}_* \geqslant \left[\frac{9}{4}\left(\hat{\gamma} - \frac{\hat{R}_a}{\pi}\right)\right]^{2/3} + \frac{3\hat{R}_s^2}{4}. \tag{65}$$

It is well known (see Ref. [19] and references therein) that only the middle root is stable (if it exists)—hence, Eq. (65) is effectively the condition of stability. Note also that, for $\hat{R}_s = 0$, Eq. (65) is equivalent to the corresponding reduction (no lateral gravity, no eccentricity) of condition (7) of Ref. [19].

Equation (65) shows that $\hat{R}_s$ (the symmetric part of the deviations of the disks' radii from unity) is a destabilizing influence, as it reduces the range of stable $\hat{R}_*$. The antisymmetric part $\hat{R}_a$, in turn, can be either destabilizing or stabilizing—depending on whether $|\hat{\gamma} - \hat{R}_a/\pi|$ is greater or smaller than $\hat{\gamma}$—which coincides with conclusions of Refs. [6,14].

The above asymptotic results have been compared, for a wide range of parameters, with numerical solutions of the exact problem (57)–(59). It appears that, if $\delta \lesssim 0.03$, the first-order solution provides a good approximation for the profile of the bridge (see Fig. 3).

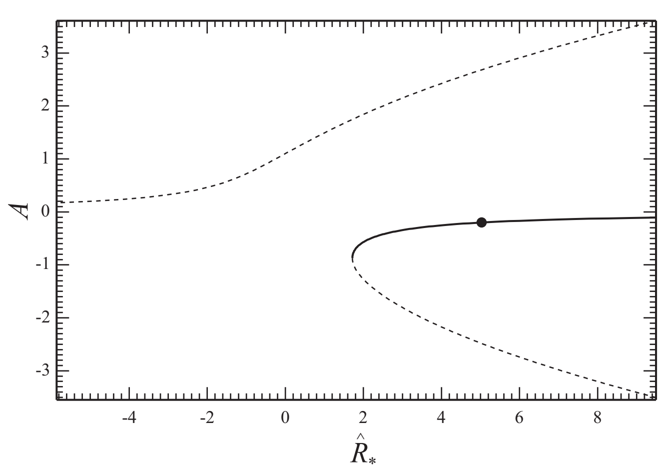

FIG. 2. The roots of Eq. (64) vs. $\hat{R}_*$, for $\hat{\gamma} = 1$ and $\hat{R}_s = \hat{R}_a = 0$. The stable and unstable roots are shown by the solid and dotted lines, respectively. The black dot corresponds to the liquid-bridge solution shown in Fig. 3.

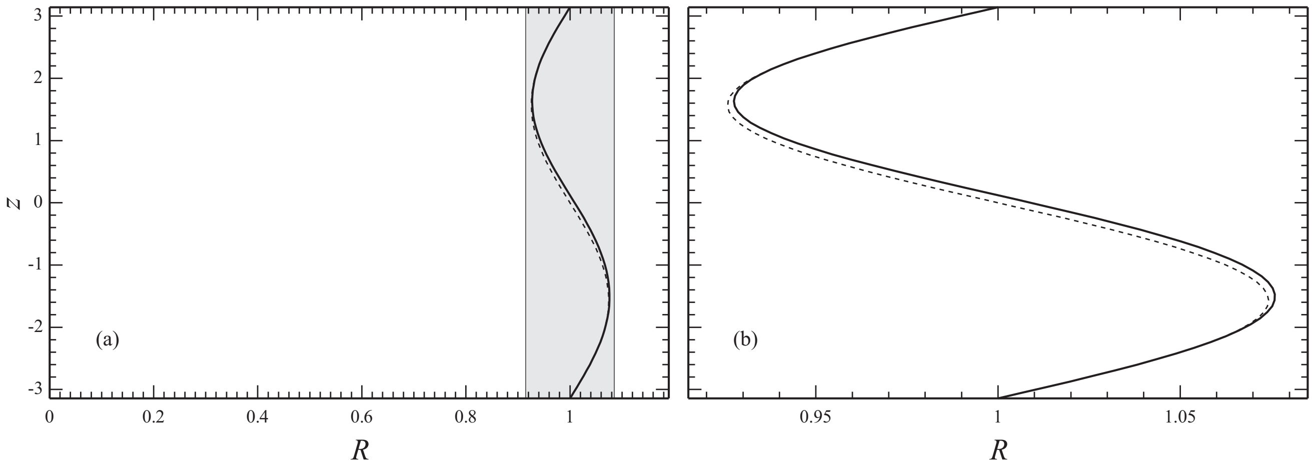

FIG. 3. The profile of a static (non-vibrating) liquid bridge with $\epsilon = 0.03$, $\hat{\gamma} = 1$, $\hat{R}_* = 5$, and $\hat{R}_s = \hat{R}_a = 0$. The numerical solution of the exact problem (57)–(59) and the first-order asymptotic solution (55), (60), (64), are shown in solid and dotted lines, respectively. The solutions presented correspond to the middle (stable) root of Eq. (64), with the parameter values marked in Fig. 2 by a black dot. (b) is a blow-up of the shaded region of (a).

B. Vibrating bridges

An asymptotic expansion similar to the one obtained previously yields the following extension of the static stability criterion:

$$\hat{R}_* \geqslant \left[\frac{9}{4}\left(\hat{\gamma} - \frac{\hat{R}_a}{\pi} - \frac{I}{8\pi}\right)\right]^{2/3} + \frac{3\hat{R}_s^2}{4}, \tag{66}$$

where

$$I = \int_{-\pi}^{\pi}\left(\left|\frac{\partial q_0}{\partial r}\right|^2 + \left|\frac{\partial q_0}{\partial z}\right|^2\right)_{r=1} \sin z\, dz, \tag{67}$$

and q_0 is determined by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_0}{\partial r}\right) + \frac{\partial^2 q_0}{\partial z^2} = 0, \tag{68}$$

$$q_0 = 0 \quad \text{at} \quad r = 1, \tag{69}$$

$$\frac{\partial q_0}{\partial z} = -\hat{W}_\pm e^{i\alpha_\pm} \quad \text{at} \quad z = \pm\pi. \tag{70}$$

The solution of problem (68)–(70) is presented in Appendix A. Substituting Eqs. (A1), (A5) into Eq. (67) and carrying out the integration with respect to z, one obtains

$$I = k(\hat{W}_+^2 - \hat{W}_-^2), \tag{71}$$

where

$$k = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{a_n a_m}\left[\frac{\tanh a_n\pi\ \coth a_m\pi + 1}{1 + (a_m + a_n)^2} + \frac{\tanh a_n\pi\ \coth a_m\pi - 1}{1 + (a_m - a_n)^2}\right]$$

and a_n are the zeros of the Bessel function of the first kind. k can be readily computed numerically,

$$k \approx 0.024794. \tag{72}$$

Criterion (66), (71)–(72) shows that, for any γ and $\hat{R}_a$, the effects of gravity and the disks' different radii on the bridge's stability can be canceled by choosing appropriate values of the vibration amplitudes $W_\pm$. In the case of disks of equal radii, $\hat{R}_a = 0$, it follows from Eq. (66), (71) that the effect of gravity is compensated if

$$k(\hat{W}_+^2 - \hat{W}_-^2) = 8\pi\hat{\gamma}. \tag{73}$$

This result can be interpreted as follows: if the vibration amplitude of the upper disk is greater than that of the lower one, the vibration-induced and hydrostatic pressure gradients are of opposite signs and, thus, the destabilizing influence of the latter can be canceled by the former.

V. SUMMARY AND CONCLUDING REMARKS

The main result of this paper is the stability criterion (66) for an almost cylindrical liquid bridge. It will now be formulated in terms of physical (dimensional) variables, and further prospects of the results obtained will be outlined.

Since the parameter δ has already played its role of an 'indicator' of small terms, one can now set

$$\delta = 1. \tag{74}$$

Recalling that the disks' radii $R_\pm$ were non-dimensionalized by H' [defined by Eq. (9)] and taking into account Eqs. (17), (54), and (74), one can rewrite criterion (66) in the form

$$\frac{\pi V}{2H^3} - 1 \geqslant \left[\frac{9}{4}\left(\frac{\rho g H^2}{\pi^2\sigma} - \frac{R_+ - R_-}{2H} - \frac{k\omega^2\left(D_+^2 - D_-^2\right)\rho H}{8\pi^2\sigma}\right)\right]^{2/3} + \frac{3}{4}\left[\frac{\pi(R_+ + R_-)}{2H} - 1\right]^2, \tag{75}$$

where V is the bridge's volume, H is the mean half-distance between the disks, $D_\pm$ are the amplitudes of the disks' vibrations, k is given by Eq. (72), and ρ and σ are the liquid's density and surface tension. In the case of disks of equal radii,

$R_+ = R_-$, Eq. (75) shows that vibration compensates gravity if

$$k\omega^2(D_+^2 - D_-^2) = 8gH.$$

This equality is the dimensional analog of Eq. (73).

Note that, historically, this work was prompted by experiments carried out in the Aerosol Research Laboratory (ARL) of the Technion Institute of Technology, Israel. A rod was dipped in a large vessel with liquid and then slowly lifted up, so that a liquid bridge would rise attached to the rod. Two sets of experiments were carried out, with vibrating and non-vibrating rods, and it turned out that the former would reach a noticeably higher height before the bridge would rapture [38]. Note also that the parameters of the ARL experiments agree with the assumptions used in the derivation of the asymptotic equations (38)–(46).

Even though bridges originating from an unbounded vessel are not cylindrical, one can conjecture that the mechanism of this effect is the same as that examined in this work. To verify the conjecture, one needs to develop an effective numerical tool for the asymptotic equations derived in Sec. III, as they are unlikely to admit an analytical solution for non-cylindrical geometry. In fact, even the non-vibrating analog of this problem [27,39,40] turned out to be less than straightforward.

Alternatively, an experiment could be carried out within the standard liquid-bridge formulation, i.e., exactly as considered in the present work. In fact, such experiments have already been conducted in Refs. [18,23]—but, unfortunately, its authors were interested in issues other than stability.

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APPENDIX: THE SOLUTION OF PROBLEM (68)–(70)

It can be readily shown that Eq. (68) and the boundary condition (69) are satisfied by

$$q_0 = \sum_{n=1}^{\infty} \mathrm{J}_0(a_n r)(B_n \cosh a_n z + C_n \sinh a_n z), \qquad \text{(A1)}$$

where $\mathrm{J}_0(a)$ is the Bessel function of the first kind and a_n are its zeros. Substituting Eq. (A1) into the boundary condition (70), one obtains

$$-\sum_{n=1}^{\infty} \mathrm{J}_0(a_n r)(\pm B_n \sinh a_n\pi + C_n \cosh a_n\pi)a_n = W_\pm e^{i\alpha_\pm}. \qquad \text{(A2)}$$

The following orthogonality conditions can be readily derived from the Bessel equation:

$$\int_0^1 r\,\mathrm{J}_0(a_m r)\,dr = -\frac{1}{a_m}\,\mathrm{J}_0'(a_m), \qquad \text{(A3)}$$

$$\int_0^1 r\,\mathrm{J}_0(a_n r)\,\mathrm{J}_0(a_m r)\,dr = \frac{1}{2}[\mathrm{J}_0'(a_m)]^2\delta_{nm}, \qquad \text{(A4)}$$

where δ_{nm} is the Kronecker delta and $\mathrm{J}_0'(a)$ is the derivative of $\mathrm{J}_0(a)$.

Now, multiplying Eq. (A1) by $r\,\mathrm{J}_0(a_m r)$, integrating from $r=0$ to $r=1$, taking into account Eqs. (A3)–(A4), and solving the resulting equations for B_m and C_m, one obtains

$$B_m = \frac{W_+e^{i\alpha_+} - W_-e^{i\alpha_-}}{a_m^2\,\mathrm{J}_0'(a_m)\sinh a_m\pi}, \quad C_m = \frac{W_+e^{i\alpha_+} + W_-e^{i\alpha_-}}{a_m^2\,\mathrm{J}_0'(a_m)\cosh a_m\pi}. \qquad \text{(A5)}$$

Equations (A1) and (A5) deliver the desired solution of problem (68)–(70).

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