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Depth-averaged model for hydraulic jumps on an inclined plate

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We examine the dynamics of a layer of viscous liquid on an inclined plate. If the layer's upstream depth h_- exceeds the downstream depth h_+, a smooth hydraulic jump (bore) forms and starts propagating down the slope. If the ratio $\eta = h_+/h_-$ is sufficiently small and/or the plate's inclination angle is sufficiently large, the bore overturns and no smooth steadily propagating solution exists in this case. In this work, the dynamics of bores is examined using a heuristic depth-averaged model where the vertical structure of the flow is approximated by a polynomial. It turns out that even the simplest version of the model (based on the parabolic approximation) is remarkably accurate, producing results which agree, both qualitatively and quantitatively, with those obtained through the Stokes equations. Furthermore, the depth-averaged model allows one to derive a sufficient criterion of bore overturning, which happens to be valid for the exact model as well. Physically, this criterion reflects the fact that, for small η, a stagnation point appears in the flow, causing wave overturning.

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I. INTRODUCTION

The main difficulty when studying hydraulic jumps (bores) in a liquid with a free surface stems from the fact that they are described by a boundary-value problem in a region of evolving shape. This difficulty is usually dealt with by assuming that the slope of the free surface, and sometimes its displacement, are small—in which case the exact boundary-value problem can be reduced to a single asymptotic equation for the liquid's depth [1–14].

Bores with order-one slope and displacement were examined in Ref. [15] for the case of a viscous liquid flowing down a plate inclined at an angle α, with the upstream depth h_- exceeding the downstream depth h_+ (see Fig. 1). Since this problem does not involve small parameters, it was examined numerically for a wide range of α and $\eta = h_+/h_-$. For the limit of zero Reynolds number, a parameter region was identified in the (η,α) plane where smooth steadily translating bores exist and outside of which all bores overturn. A similar result has been obtained for a flow inside a rotating horizontal cylinder [16].

In the present paper, bores are examined using a heuristic depth-averaged (DA) model obtained from the Stokes equations in the spirit of the classical Saint-Venant model [17]. Such models are viewed as relatively simple tools for clarifying the flow's qualitative features.

The DA model proposed in this work goes considerably further. For $\alpha \gtrsim 30^\circ$, it agrees qualitatively and *quantitatively* with the numerical solution of the exact equations obtained in Ref. [15]. For $\alpha < 30^\circ$, however, the two approaches disagree, but it turns out that the DA results are actually correct, whereas the simulations of Ref. [15] lack accuracy due to insufficient resolution (for more details, see Ref. [18]).

This paper has the following structure. In Sec. II we formulate the equations governing a viscous flow on an inclined plate and, in Sec. III derive the DA model for the same. In Sec. IV the DA model is solved numerically, and the results are compared to those obtained through the exact equations. In Sec. V the DA model is used to derive a necessary condition of existence of steady bores (or, equivalently, a sufficient condition for their overturning) and clarify the physical effect causing the overturning.

II. FORMULATION

Consider a two-dimensional layer of liquid (of density ρ and dynamic viscosity μ) on a plate inclined at an angle α. Let the x and z axes be directed along, and perpendicular to, the plate (see Fig. 1), so that the free surface is described by the equation $z = h(x)$ where h is the thickness of the liquid. The spatial coordinates are assumed to be nondimensionalized by the liquid's characteristic depth H. The x and z components of the velocity, u and w, are in turn nondimensionalized by $H^2\rho g\cos\alpha/\mu$, and the pressure p by $\mu/H\rho g\cos\alpha$ (g is the acceleration due to gravity). It is also convenient to introduce the plate's slope s and the downstream-to-upstream depth ratio η:

$$s = \tan\alpha, \quad \eta = h_+/h_-.$$

In terms of the nondimensional variables introduced, the Stokes equations are

$$\frac{\partial p}{\partial x} = s + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$\frac{\partial p}{\partial z} = -1 + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{3}$$

Let U be the bore's nondimensional speed relative to the plate. Then it is convenient to have the plate moving in the opposite direction with the matching speed, which makes the bore stationary and implies the following boundary conditions:

$$u = -U \quad \text{at} \quad z = 0, \tag{4}$$

$$w = 0 \quad \text{at} \quad z = 0. \tag{5}$$

At the free surface we apply the standard conditions for the stress tensor, one which is rewritten as an expression for the

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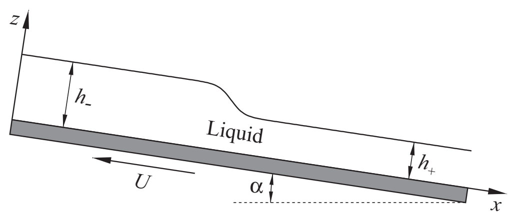

FIG. 1. The setting: a hydraulic jump (bore) propagating down an inclined plate which moves in the opposite direction.

pressure

$$p = 2\frac{\partial w}{\partial z} - \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{dh}{dx} \quad \text{at} \quad z = h, \tag{6}$$

and the other, as an equation *not* including p,

$$2\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\frac{dh}{dx} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left[1 - \left(\frac{dh}{dx}\right)^2\right] \quad \text{at } z = h. \tag{7}$$

In this work we are concerned with steady solutions, for which the x component of the flux is conserved, i.e.,

$$\int_0^h u\, dz = q, \tag{8}$$

where q is a constant. Since we are modeling bores, we assume

$$h \to h_\pm \quad \text{as} \quad x \to \pm\infty, \tag{9}$$

where h_- (h_+) is the liquid's upstream (downstream) depth; see Fig. 1. Note that, without loss of generality, one can set $h_- = 1$, which implies that the scale H used in the nondimensionalization is the upstream depth.

Equations (1)–(7) and the boundary conditions (9) imply

$$u \to s\left(\tfrac{1}{2}z^2 - zh_\pm - U\right) \quad \text{as} \quad x \to \pm\infty. \tag{10}$$

Considering the limits $x \to \pm\infty$ in Eq. (8) and taking into account (10), one can relate the bore's velocity and flux to $h_\pm$:

$$U = \tfrac{1}{3}s(h_-^2 + h_-h_+ + h_+^2), \tag{11}$$

$$q = -\tfrac{1}{3}sh_-h_+(h_- + h_+). \tag{12}$$

III. THE DEPTH-AVERAGED MODEL

A. The general approach

The model proposed in this work is based on approximating the vertical structure of the flow by a polynomial in z, with coefficients depending on x and t,

$$u = -U + \sum_{n=1}^{N} z^n u_n(x,t), \tag{13}$$

where the first term is fixed to satisfy the boundary condition (4). The assumed dependence of u on z makes Eq. (1) (which governs u) overdetermined, so, to avoid this, (1) is replaced with $N-1$ depth-averaged equations,

$$\int_0^h z^n \frac{\partial p}{\partial x} dx = \int_0^h z^n \left(s + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) dx, \tag{14}$$

where $n = 0,1,\ldots,N-2$. Equations (2)–(3), in turn, will be used in their exact form, together with the boundary conditions (6) and (5), to express p and w in terms of u_n. Finally, the expressions for u, w, and p should be substituted into conditions (7)–(8) and (14), which form a closed set for $u_n(x)$ and $h(x)$.

B. The difference between the present and existing DA models

Generally, all depth-averaged models are based on approximating the velocity profile by a polynomial and integrating various combinations of the governing equations with respect to the vertical variable. This approach has been first suggested for free-surface flows by Saint-Venant [17], with Ref. [19] putting it on a more rigorous footing. A DA model based on replacing one of the governing equations by the momentum-balance equation has been suggested in Ref. [20], whereas Ref. [21] argued that a modification based on the kinetic-energy balance yields better results. Various approaches based on a more refined approximation of the vertical structure of the flow have been developed in Refs. [22–25], and similar models have been applied to pipe flows (e.g., Ref. [26]) and general boundary-layer equations (the latter application is known as the Kármán-Pohlhausen method [27]).

Unlike the existing DA models (where all governing equations are depth-averaged), our approach implies averaging of only one equation, whereas the other two are used in their exact form. The exact interconnections between the pressure and velocity components should help to capture the correct dynamics, especially, if the flow has an oscillatory structure (which will be shown to be indeed the case for the problem at hand).

C. The case $N = 2$

In this work, the simplest version of the DA model is examined, where (13) is truncated at $N = 2$, i.e.,

$$u = -U + zu_1 + z^2u_2. \tag{15}$$

Then (2), (6) and (3), (5) yield the pressure and vertical velocity:

$$p = h - h\frac{du_1}{dx} + \frac{h^3}{6}\frac{d^3u_1}{dx^3} - h^2\frac{du_2}{dx} + \frac{h^4}{12}\frac{d^3u_2}{dx^3} - \left(u_1 - \frac{h^2}{2}\frac{d^2u_1}{dx^2} + 2hu_2 - \frac{h^3}{3}\frac{d^2u_2}{dx^2}\right)\frac{dh}{dx} - z\left(1 + \frac{du_1}{dx}\right) - z^2\frac{du_2}{dx} - \frac{z^3}{6}\frac{d^3u_1}{dx^3} - \frac{z^4}{12}\frac{d^3u_2}{dx^3}, \tag{16}$$

$$w = -\frac{z^2}{2}\frac{du_1}{dx} - \frac{z^3}{3}\frac{du_2}{dx}. \tag{17}$$

Substituting (15) and (17) into Eqs. (7)–(8), we obtain

$$\left(u_1 - \frac{h^2}{2}\frac{d^2u_1}{dx^2} + 2hu_2 - \frac{h^3}{3}\frac{d^2u_2}{dx^2}\right)\left[1 - \left(\frac{dh}{dx}\right)^2\right] - 4h\left(\frac{du_1}{dx} + h\frac{du_2}{dx}\right)\frac{dh}{dx} = 0, \tag{18}$$

$$-Uh + \frac{h^2}{2}u_1 + \frac{h^3}{3}u_2 = q. \tag{19}$$

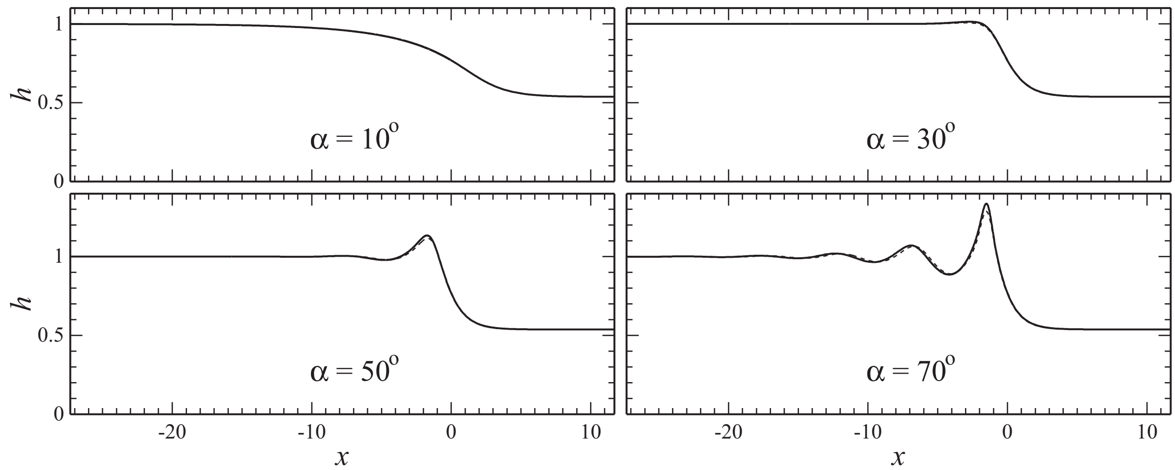

FIG. 2. Examples of steady bores for $\eta = 0.5377$ and various values of the inclination angle α. The solutions of the exact and depth-averaged equations are shown in solid and dotted lines, respectively (the former have been computed in Ref. [15]).

Finally, substitution of (15)–(16) into Eq. (14) with $n = 0$ yields, after straightforward algebra,

$$-2h\frac{d^2u_1}{dx^2} + \frac{h^3}{8}\frac{d^4u_1}{dx^4} - 2u_2 - \frac{5h^2}{3}\frac{d^2u_2}{dx^2} + \frac{h^4}{15}\frac{d^4u_2}{dx^4} + \left(1 - 2\frac{du_1}{dx} + h^2\frac{d^3u_1}{dx^3} - 4h\frac{du_2}{dx} + \frac{2h^3}{3}\frac{d^3u_2}{dx^3}\right)\frac{dh}{dx} + \left(h\frac{d^2u_1}{dx^2} - 2u_2 + h^2\frac{d^2u_2}{dx^2}\right)\left(\frac{dh}{dx}\right)^2 - \left(u_1 - \frac{h^2}{2}\frac{d^2u_1}{dx^2} + 2hu_2 - \frac{h^3}{3}\frac{d^2u_2}{dx^2}\right)\frac{d^2h}{dx^2} = s. \quad (20)$$

Equations (18)–(20) form a closed set for $h(x)$, and $u_{1,2}(x)$. Even though the DA model is highly nonlinear and does not admit obvious analytical solutions, it is still incomparably simpler than the original boundary-value problem for both numerical integration and qualitative analysis.

Note that, if the plate's slope is small, $s \ll 1$, Eqs. (18)–(20) can be simplified using the lubrication approximation, and the resulting asymptotic equation happens to precisely coincide with its counterpart derived from the exact set (1)–(8).

IV. NUMERICAL RESULTS

Let the boundary condition for h be as in (9). The corresponding conditions for $u_{1,2}$ [the DA equivalent of (10)] can be obtained by substituting (9) into Eqs. (18)–(20), which yields

$$u_1 \to sh_\pm, \quad u_2 \to -\tfrac{1}{2}s \quad \text{as} \quad x \to \pm\infty. \quad (21)$$

It can also be shown that, subject to the boundary conditions (9), U and q remain the same as those in the exact case, i.e., (11), (12).

Equations (18)–(20) and the boundary conditions (9), (21) were solved numerically by discretizing them on a suitable mesh in a large but finite domain, and solving the resulting set of algebraic equations using the MATLAB function FSOLVE. Examples of the solutions found are shown in Fig. 2, together with the corresponding numerical solutions of the exact set (1)–(9) (computed in Ref. [15]). One can see that the DA and exact results agree well even for large α (for small α, the agreement is expected as, in this limit, both models reduce to the same lubrication equation).

We have also computed the region of existence of steady bores described by the DA model on the (η,α) plane. This has been done by fixing α and computing the solution for a sequence of decreasing values of η until the iterative process in FSOLVE would cease to converge. Then the smallest value of η for which it does converge was taken to represent the boundary of the existence region.

The results obtained through this approach were compared to those obtained through the exact equations (again, computed in Ref. [15]). As Fig. 3 shows, the two models agree well for $\alpha \gtrsim 30°$, but they *dis*agree for $\alpha < 30°$.

In what follows, we shall derive a necessary condition of existence of bores described by the DA model and clarify which physical effect eliminates them. This will help us to resolve the discrepancy between the DA and exact results.

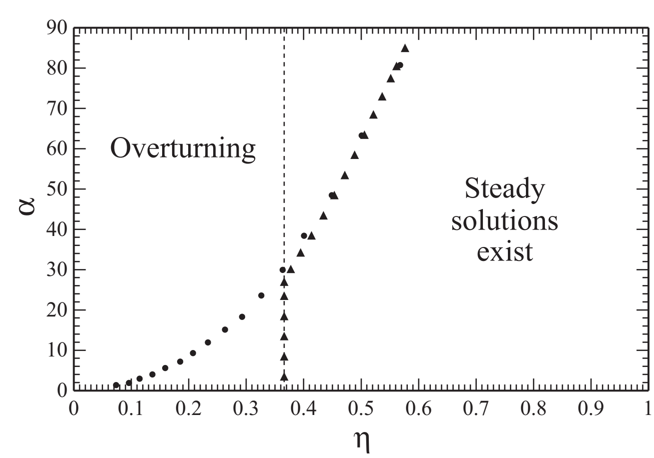

FIG. 3. Existence of steady bores on the (η,α) plane. Circles and triangles correspond to the exact and the depth-averaged equations, respectively (the former results have been computed in Ref. [15]). The dotted line corresponds to condition (32).

V. PROPERTIES OF THE DA MODEL

A. A necessary condition of existence of steady bores

It turns out that a condition of existence of steady smooth bores can be deduced from the long-distance asymptotics of Eqs. (18)–(20).

As $x \to \pm\infty$, the unknowns become close to the values determined by the boundary conditions; i.e., one can assume

$$h = h_\pm + \tilde{h}(x), \quad u_1 = sh_\pm + \tilde{u}_1(x), \quad u_2 = -\tfrac{1}{2}s + \tilde{u}_2(x),$$

where the tilded variables describe a small perturbation. Linearizing (18)–(20),

$$\tilde{u}_1 - \frac{h_\pm^2}{2}\frac{d^2\tilde{u}_1}{dx^2} + 2\left(h_\pm\tilde{u}_2 - \frac{s\tilde{h}}{2}\right) - \frac{h_\pm^3}{3}\frac{d^2\tilde{u}_2}{dx^2} = 0, \tag{22}$$

$$-2h_\pm\frac{d^2\tilde{u}_1}{dx^2} + \frac{h_\pm^3}{8}\frac{d^4\tilde{u}_1}{dx^4} - 2\tilde{u}_2 - \frac{5h_\pm^2}{3}\frac{d^2\tilde{u}_2}{dx^2} + \frac{h_\pm^4}{15}\frac{d^4\tilde{u}_2}{dx^4} + \frac{d\tilde{h}}{dx} = 0, \tag{23}$$

$$\frac{h_\pm^2}{2}\tilde{u}_1 + \frac{h_\pm^3}{3}\tilde{u}_2 + \left(\frac{sh_\pm^2}{2} - U\right)\tilde{h} = 0, \tag{24}$$

we seek solutions of the form

$$[\tilde{h},\tilde{u}_1,\tilde{u}_2] = [A_\pm,B_{1\pm},B_{2\pm}]\exp a_\pm x, \tag{25}$$

where $A_\pm$, $B_{1\pm}$, and $B_{2\pm}$ are undetermined constants. These solutions represent the long-distance asymptotics of a bore only if they decay at the corresponding infinity, i.e.,

$$\operatorname{Re} a_- > 0, \quad \operatorname{Re} a_+ < 0.$$

In what follows, such solutions will be referred to as *meaningful* (as in "physically meaningful"). It can be shown that set (22)–(24) is of the sixth order, hence, the numbers of meaningful solutions for plus and minus infinities, n_- and n_+, cannot exceed 6, i.e., $n_- + n_+ \leqslant 12$.

Most importantly, the full set (18)–(20) may have a solution only if the linearized set (22)–(24) has sufficiently many meaningful solutions, namely, $n_- + n_+ \geqslant 7$.

To understand why, define two particular solutions, $\hat{h}_-$ and $\hat{h}_+$, of the full set by fixing their long-distant (linearized) asymptotics,

$$\hat{h}_\pm \sim h_\pm + \sum_{n=1}^{n_\pm} A_\pm^{(n)} \exp(a_\pm^{(n)}x) \quad \text{as} \quad x \to \mp\infty, \tag{26}$$

where $A_\pm^{(n)}$ are coefficients. Since the problem at hand is translationally invariant, one can change x to $x + \text{const}$ and chose the const that makes one of the coefficients in (26) equal to unity, say,

$$A_-^{(1)} = 1.$$

Since solution $\hat{h}_-$ satisfies the boundary conditions as $x \to -\infty$ and $\hat{h}_+$ satisfies these as $x \to +\infty$, one can construct a solution satisfying both boundary conditions by choosing the remaining coefficients in (26) such that $\hat{h}_-$ and $\hat{h}_+$ match at, say, $x = x_0$:

$$\hat{h}_- = \hat{h}_+, \quad \frac{d\hat{h}_-}{dx} = \frac{d\hat{h}_+}{dx}, \quad \dots \quad \frac{d^5\hat{h}_-}{dx^5} = \frac{d^5\hat{h}_+}{dx^5} \quad \text{at } x = x_0. \tag{27}$$

These equalities should be viewed as equations for the coefficients $A_-^{(2)}$, $A_-^{(3)}$, ... Given that there are $n_- + n_+ - 1$ of these, the *six* equations (27) can be satisfied only if there exist at least *seven* meaningful solutions.

To determine the number of meaningful solutions in the problem at hand, substitute (25) into (22)–(24) and, thus, obtain a set of linear algebraic equations for $A_\pm$, $B_{1\pm}$, and $B_{2\pm}$. This set has a solution only if the matrix of its coefficients has zero determinant, which yields two equations for a_- and a_+. Introducing $b_\pm = h_\pm a_\pm$, one can write these equations in the form

$$(s - 2V_-)b_-^6 - 4(s - V_-)b_-^4 + 40(9s - 20V_-)b_-^2 + 160b_- - 480(s - V_-) = 0, \tag{28}$$

$$(s - 2V_+)b_+^6 - 4(s - V_+)b_+^4 + 40(9s - 20V_+)b_+^2 + 160b_+ - 480(s - V_+) = 0, \tag{29}$$

where

$$V_\pm = \frac{U}{h_\pm^2}. \tag{30}$$

It can be demonstrated that, as long as $h_- > h_+ > 0$ and U is given by (11), Eq. (29) has three roots with negative real parts, i.e., $n_+ = 3$. Equation (28), in turn, has three roots with positive real parts if

$$s - 2V_- \leqslant 0, \tag{31}$$

and four such roots otherwise (it can be shown that the imaginary parts of two meaningful roots tend to infinity as $V_- \to s/2 - 0$, with only one root "coming back" for $V_- > s/2$). Using (30) and (11) to relate V_- to $h_\pm$, one can reduce (31) to

$$\eta \leqslant \tfrac{1}{2}(\sqrt{3} - 1). \tag{32}$$

If this condition holds (and, thus, $n_+ + n_- = 6$), no steady-bore solutions exist in the problem. Hence, (32) can be viewed as a sufficient condition of nonexistence of solutions of the boundary-value problem (18)–(20), (9), (21).

Comparison of (32) with the "de facto" nonexistence region on the (η,α) plane (see Fig. 3) brings up the following issues:

(1) Observe that the boundary of the de facto existence region deviates from condition (32) for $\alpha \gtrsim 30°$. Mathematically, this poses no contradiction [as (32) is not a *necessary* condition], but it would be interesting to clarify the physical mechanism eliminating bores in a region where they could potentially exist.

(2) For $\alpha < 30°$, condition (32) is evidently effectively necessary as well as sufficient. Is there a physical reason for this?

(3) Condition (32) confirms the numerical results presented in the previous section, and, hence, confirms the discrepancy between the exact and DA models for $\alpha < 30°$ (see Fig. 3). This discrepancy has yet to be explained.

These issues will be resolved in the next subsections.

B. Stagnation points

It is well known [28] that a steady flow with a stagnation point (SP) is unstable. In the problem at hand, however, SPs seem to eliminate steady solutions altogether. Firstly, this is suggested by the fact that none of the existing steady bores contains an SP (as checked numerically). Secondly, if one considers a sequence of solutions with parameters approaching the boundary of the existence region, one observes a rapid decrease of the velocity in a certain part of the flow, near the free surface.

The latter feature is best illustrated using the surface velocity,

$$u_s = \sqrt{(-U + hu_1 + h^2 u_2)^2 + \left(\frac{h^2}{2}\frac{du_1}{dx} + \frac{h^3}{3}\frac{du_2}{dx}\right)^2}, \quad (33)$$

derived from expressions (15)–(16) with $z = h$.

It turns out that, for $\alpha < 30^\circ$ and $\alpha \gtrsim 30^\circ$, SPs emerge differently. In the former case, the SP emerges at minus-infinity [see Fig. 4(b)], whereas, in the latter case, it emerges in the bore's core [see Fig. 4(a)], near its tallest peak.

Note that the former scenario corresponds to the precise equality in condition (32). Indeed, taking the limit $x \to -\infty$ in (33), substituting the limiting values of $u_{1,2}$ from the boundary condition (21), and equating the resulting value of $(u_s)_{x\to-\infty}$ to zero, we obtain

$$-U + \tfrac{1}{2} s h_-^2 = 0.$$

Then, recalling that U is given by (11), one can reduce the above equality to criterion (32) with "=" instead of "⩽."

We can now clarify the three issues raised in the end of the previous section:

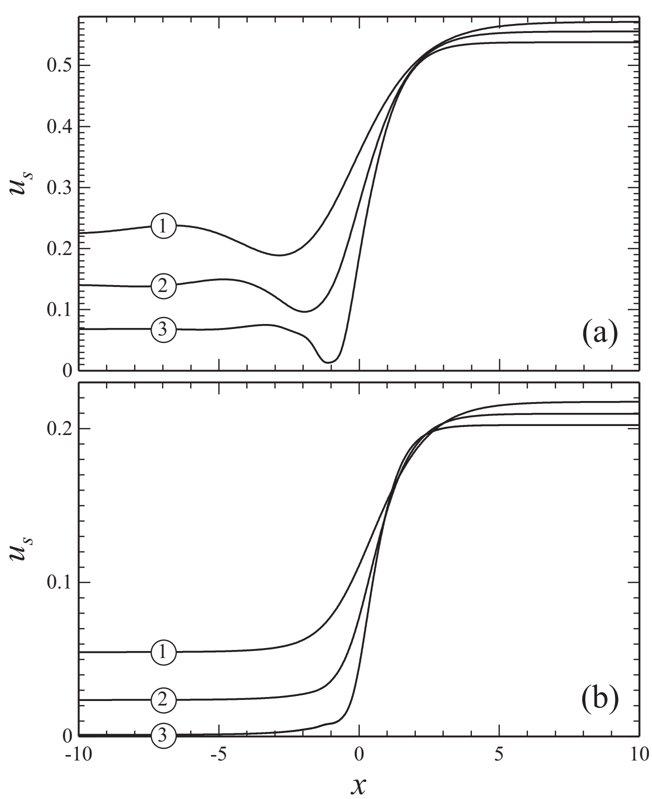

FIG. 4. The velocity u_s at the free surface [given by (33)] vs x. (a) $\alpha = 50^\circ$, curves 1, 2, 3 correspond to $\eta = 0.65, 0.55, 0.46$; (b) $\alpha = 25^\circ$, curves 1, 2, 3 correspond to $\eta = 0.55, 0.45, 0.38$.

(1) For $\alpha \gtrsim 30^\circ$, the SP emerges near the tallest peak of the bore *before* the minus-infinity SP has a chance to emerge, hence, steady solutions disappear *before* the sufficient criterion (32) begins to hold.

(2) For $\alpha < 30^\circ$, the SP emerges at minus-infinity [as in Fig. 4(b)]. Accordingly, criterion (32), which deals with precisely this kind of SPs, becomes *necessary* and sufficient (and makes the correspond part of the boundary of the existence region strictly vertical).

(3) Note that solutions of the *exact* equations (1)–(9) have the same long-distance asymptotics as those of the DA model. As a result, if an exact solution satisfies condition (32), it has an SP [as before, the precise equality in (32) corresponds to an SP at minus-infinity]. This suggest that, contrary to the results of Ref. [15], the exact equations may *not* admit a steady solution when (32) holds.

Numerical solutions of the exact equations (1)–(9) were reexamined in Ref. [18], which concluded that the steady bores with SPs found in Ref. [15] are artifacts resulting from insufficient resolution near the SPs.

VI. CONCLUDING REMARKS

We have examined a depth-averaged (DA) model for bores in a liquid on an inclined plate. In some cases, the DA results turned out to agree, both quantitatively and qualitatively, with those obtained through numerical simulations of the exact equations. In other cases, where the two approaches *dis*agree, the former has helped to detect a problem (insufficient resolution) in the latter. The DA model has also helped to clarify the physical mechanism causing overturning of bores, namely, the emergence of a stagnation point.

Finally, we briefly discuss possible extensions of the DA model.

Firstly, it can be readily modified to account for surface tension [which only requires the inclusion of the capillary component of the pressure in Eq. (20)].

Secondly, the DA model can be extended from steady solutions to evolving ones. To do so, one needs to replace (19) with

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(-Uh + \frac{h^2}{2}u_1 + \frac{h^3}{3}u_2\right) = 0,$$

where the time t is nondimensionalized by $H\rho g \cos\alpha$. Equations (18) and (20) remain the same as before.

Thirdly, the DA model can be modified to include inertial effects. To do so, one should replace the Stokes equations (1)–(2) with the Navier-Stokes equations, but the derivation of the DA model remains more or less the same. This derivation has actually been carried out, and it has turned out that the resulting equations describe the inertial instability of flows on inclined plates with a good accuracy.

Generally, the "averaged" approach can be helpful in all physical settings where a wave or a flow has a nontrivial transverse structure, such as avalanches and landslides, flows and shock waves in pipes, or rimming flows (in the last case, a classical version of the DA approach has already been used in Refs. [29,30]).

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