Weakening of magnetohydrodynamic interchange instabilities by Alfvén waves

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Alfvén waves, made to propagate along an ambient magnetic field and polarized transverse to a gravitational field **g**, with wave amplitude stratified along **g**, are shown to reduce the growth rate of interchange instability by increasing the effective inertia by a factor of $1+(\bar{B}_y'/\bar{B}_z k_z)^2$, where $\bar{B}_z$ is the ambient magnetic field, k_z is the wavenumber, and $\bar{B}_y'$ is the wave amplitude shear. Appropriately placed Alfvén wave power could thus be used to enhance the stability of interchange and ballooning modes in tokamaks and other interchange-limited magnetically confined plasmas. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2837516]

It is well known that magnetic shear stabilizes (or, at least, weakens) interchange instability of magnetized plasmas.[1,2] The convection cell of the unstable mode tends to align itself in the direction of the dominant magnetic field and forms so-called "flutes," minimizing field line bending—but any transverse fields will inhibit the release of the gravitational potential energy. The resulting growth rate γ is, schematically, $\gamma^2 = g/L - k^2 v_A^2$, where g is the gravitational field, L is the scale of the density gradient, v_A is the Alfvén speed corresponding to the transverse field, and k is the transverse wavenumber. For tokamaks, the effective gravitational field is proportional to the plasma temperature, and the achievable stored thermal energy for fusion is limited by the strongest transverse field that can be designed into the configuration geometry. This establishes the so-called β limit for tokamaks,[3] and any enhancements in β are highly desirable.

Static magnetic fields, however, may not be the only means of optimizing the geometry of the field lines to contain plasmas. A fluctuating field—in particular, a sheared wave—could also be configured to provide an effective transverse field. A shear Alfvén wave, for example, polarized in the transverse direction, can be made to propagate along the dominant direction of the magnetic field. The fluctuating field could then inhibit interchanges, subject presumably to $k\tilde{v}_A$ exceeding the growth rate $(g/L)^{1/2}$ (where $\tilde{v}_A$ is the Alfvén speed corresponding to a time-averaged wave amplitude).

The above possibility is examined in the present brief communication. We consider the simplest case, that of a layer of Alfvén waves of amplitude $\tilde{B}$, propagating along an ambient magnetic field, with polarization transverse to g. To establish the idea that a fluctuating transverse Alfvén field could be stabilizing, we assume there is no equilibrium transverse field. A shear is assumed in the wave amplitude $\tilde{B}$, which varies along the g direction.

Stabilization of interchange modes by radiofrequency fields in magnetic mirrors and in tokamaks has been studied in Refs. 4–6. The main difference between these studies and our work is the frequency domain; the work of these authors was to build an effective ponderomotive stabilizing force by using rf waves at the ion gyrofrequency scale. Our work explores the possibility of using low-frequency [ideal magnetohydrodynamic (MHD)] waves, in particular, shear Alfvén waves.

In this brief communication, we do not attempt to ascertain if such waves could in practice be coupled to tokamak-type plasmas with the desired shears, or if the power requirements can be met—our aim is to demonstrate that this effect is possible theoretically. If a theoretical calculation is encouraging, one can begin to explore if a layer of Alfvén waves could be excited to mitigate β limits in tokamaks or, with appropriate placing of power, using Alfvén wave "carpets" to modify rotational transform requirements for strong toroidal field fusion systems.

For our study, we shall use the "reduced MHD" equations,[7,8]

$$\frac{\partial \rho}{\partial t} + \{\phi, \rho\} = 0, \tag{1}$$

$$\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \{\phi, \nabla_\perp^2 \phi\} - \frac{g}{\bar{\rho}}\frac{\partial \rho}{\partial y} - \frac{1}{4\pi\bar{\rho}}\{\psi, \nabla_\perp^2 \psi\} - \frac{\bar{B}_z}{4\pi\bar{\rho}}\frac{\partial \nabla_\perp^2 \psi}{\partial z} = 0, \tag{2}$$

$$\frac{\partial \psi}{\partial t} + \{\phi, \psi\} - \bar{B}_z \frac{\partial \phi}{\partial z} = 0, \tag{3}$$

where $\bar{\rho}$ and ρ are the mean density and its variation, the acceleration due to gravity is $g = 2TR^{-1}$ (T is the temperature, R^{-1} is the magnetic field's curvature), $\bar{B}_z$ is the mean toroidal magnetic field, ψ is the toroidal component of the vector potential (so the transverse field is $\mathbf{B}_\perp = \mathbf{z} \times \nabla_\perp \psi$), and

$$\{\phi, \psi\} = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}.$$

Equations (1)–(3) are applicable if $\bar{B}_z$ is uniform and strong, such that the energy of the transverse flow (given by $\nabla_\perp \phi$)

and that of the transverse field $\mathbf{B}_\perp$ is much lower. The strong axial field forces motions to be "flute-like," i.e., the ordering $B_z\partial/\partial z \sim \mathbf{B}_\perp \cdot \boldsymbol{\nabla}_\perp$ is implied, in the context of a "slab" (Cartesian) geometry. For simplicity, we shall adopt the Boussinesq approximation, i.e., $\bar{\rho}$ is constant.

Equations (1)–(3) have an exact solution describing an Alfvén wave propagating in the positive direction of the z axis in a plasma with linear density stratification,

$$\rho = \bar{\rho}' x, \quad \psi = \bar{\psi}(x, z - v_A t), \quad \phi = -\psi, \tag{4}$$

where $\bar{\rho}'$ is a constant characterizing the density gradient and $v_A = \bar{B}_z/\sqrt{4\pi\bar{\rho}}$. This will represent the imposed ambient "carpet" of shear Alfvén waves, with the magnetic field amplitude polarized in the y direction and the wave amplitude stratified in the x direction.

We now examine stability about steady state (4). We assume

$$\rho = \bar{\rho}' x + \tilde{\rho}, \quad \psi = \bar{\psi}(x, z - v_A t) + \tilde{\psi},$$
$$\phi = -\bar{\psi}(x, z - v_A t) + \tilde{\phi}, \tag{5}$$

where the "tilded" variables describe the perturbation. Next, it is convenient to change to the reference frame comoving with the wave, i.e., introduce

$$x_{\text{new}} = x, \quad y_{\text{new}} = y, \quad z_{\text{new}} = z - v_A t, \quad t_{\text{new}} = t. \tag{6}$$

Substituting Eqs. (5) and (6) into Eqs. (1)–(3), linearizing the equations, and omitting the subscript "new," we obtain

$$\frac{\partial \tilde{\rho}}{\partial t} - \frac{\bar{B}_z}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \tilde{\rho}}{\partial z} - \frac{\bar{B}_y}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \tilde{\rho}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\bar{\rho}' = 0, \tag{7}$$

$$\bar{\rho}\left(\frac{\partial \boldsymbol{\nabla}_\perp^2 \tilde{\phi}}{\partial t} - \frac{\bar{B}_z}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \boldsymbol{\nabla}_\perp^2 \tilde{\phi}}{\partial z} - \frac{\bar{B}_y}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \boldsymbol{\nabla}_\perp^2 \tilde{\phi}}{\partial y} + \frac{\partial \tilde{\phi}}{\partial y}\frac{1}{\sqrt{4\pi\bar{\rho}}}\frac{\partial^2 \bar{B}_y}{\partial x^2}\right) - \frac{g}{\bar{\rho}}\frac{\partial \tilde{\rho}}{\partial y} - \frac{1}{4\pi}\left(\bar{B}_y \frac{\partial \boldsymbol{\nabla}_\perp^2 \tilde{\psi}}{\partial y} - \frac{\partial \tilde{\psi}}{\partial y}\frac{\partial^2 \bar{B}_y}{\partial x^2} + \bar{B}_z \frac{\partial \boldsymbol{\nabla}_\perp^2 \tilde{\psi}}{\partial z}\right) = 0, \tag{8}$$

$$\frac{\partial \tilde{\psi}}{\partial t} - \frac{\bar{B}_z}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \tilde{\psi}}{\partial z} - \frac{\bar{B}_y}{\sqrt{4\pi\bar{\rho}}}\frac{\partial \tilde{\psi}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\bar{B}_y - \bar{B}_z \frac{\partial \tilde{\phi}}{\partial z} = 0, \tag{9}$$

where

$$\bar{B}_y = \frac{\partial \bar{\psi}}{\partial x}$$

is the poloidal magnetic field induced by the wave. We are concerned with harmonic disturbances, i.e.,

$$\tilde{\rho} = \hat{\rho}(x,z)e^{i(k_y y - \omega t)}, \quad \tilde{\phi} = \hat{\phi}(x,z)e^{i(k_y y - \omega t)},$$
$$\tilde{\psi} = \hat{\phi}(x,z)e^{i(k_y y - \omega t)},$$

where ω and k_y are the frequency and poloidal wavenumber. Next, introduce the following nondimensional variables:

$$x_{\text{nd}} = k_y x, \quad y_{\text{nd}} = k_y y, \quad z_{\text{nd}} = k_z z,$$

$$\omega_{\text{nd}} = \sqrt{\frac{\bar{\rho}}{g\bar{\rho}'}}\omega, \quad \hat{\rho}_{\text{nd}} = \frac{1}{\bar{B}_z}\sqrt{\frac{4\pi g}{\bar{\rho}'}}\hat{\rho},$$

$$\hat{\phi}_{\text{nd}} = \frac{\sqrt{4\pi\bar{\rho}}k_y}{\bar{B}_z}\hat{\phi}, \quad \hat{\psi}_{\text{nd}} = \frac{k_y}{\bar{B}_z}\hat{\psi},$$

where k_z is the characteristic toroidal wavenumber of the Alfvén wave. We shall also introduce

$$\bar{B}_{y\ \text{nd}} = \frac{\bar{B}_y k_y}{\bar{B}_z k_z}.$$

Then, Eqs. (7)–(9) yield (the subscript "nd" and hats omitted)

$$\left(\frac{\partial}{\partial z} + i\bar{B}_y\right)\rho + i\alpha(\omega\rho - \phi) = 0, \tag{10}$$

$$\left(\frac{\partial}{\partial z} + i\bar{B}_y\right)\left(\frac{\partial^2}{\partial x^2} - 1\right)(\phi + \psi) - i\frac{\partial^2 \bar{B}_y}{\partial x^2}(\phi + \psi) + i\alpha\left[\omega\left(\frac{\partial^2}{\partial x^2} - 1\right)\phi - \rho\right] = 0, \tag{11}$$

$$\left(\frac{\partial}{\partial z} + i\bar{B}_y\right)(\phi + \psi) + i\alpha\omega\psi = 0, \tag{12}$$

where

$$\alpha = \frac{\sqrt{4\pi g\bar{\rho}'}}{\bar{B}_z k_z}.$$

Together with appropriate boundary conditions as $x, z \to \infty$, Eqs. (10)–(12) form an eigenvalue problem for ω. If $\text{Im}\ \omega > 0$, the plasma is unstable.

Mathematically, this brief communication is based on the following observation: if Eq. (12) is subtracted from Eq. (10), the resulting equation,

$$\left(\frac{\partial}{\partial z} + i\bar{B}_y + i\alpha\omega\right)\left(\rho - \frac{\phi + \psi}{\omega}\right) = 0, \tag{13}$$

can be used to eliminate ρ. Indeed, it follows from Eq. (13) that

$$\rho - \frac{\phi + \psi}{\omega} = f(x)\exp\left[-i\int \bar{B}_y(x,z)dz - i\alpha\omega z\right],$$

where $f(x)$ is an arbitrary function. Being concerned with unstable solutions ($\text{Im}\ \omega \neq 0$), we note that ρ, ϕ, and ψ are bounded as $z \to \pm\infty$ if and only if $f(x) = 0$—hence,

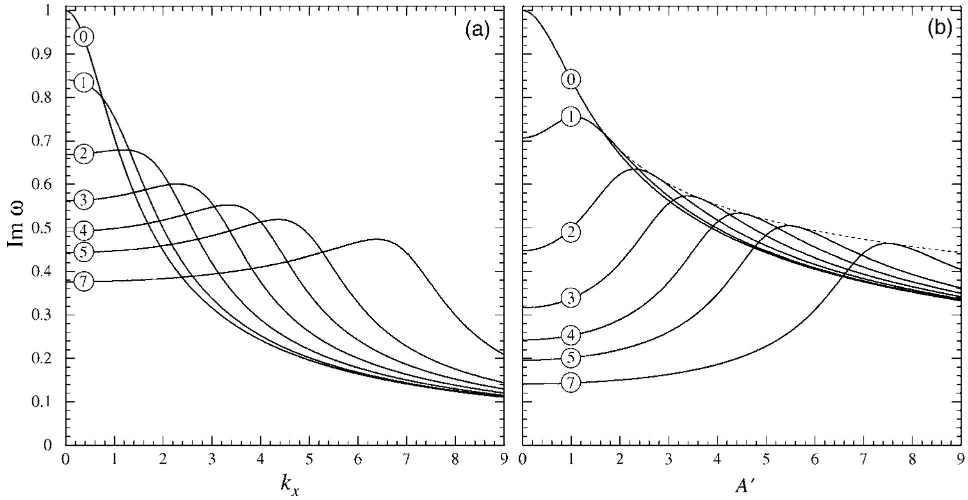

FIG. 1. Nondimensional growth rate Im ω as determined by formula (18). (a) Im ω vs the nondimensional radial wavenumber k_x (the curves are marked with the corresponding values of the amplitude shear A'). (b) Im ω vs A' (the curves are marked with the corresponding values of k_x). The dotted line shows the maximum growth rate (19).

$$\rho = \frac{\phi+\psi}{\omega}. \tag{14}$$

It is also convenient to apply the operator $\partial^2/\partial x^2 - 1$ to Eq. (12) and subtract it from Eq. (11),

$$2\frac{\partial}{\partial x}\left[\frac{\partial \bar{B}_y}{\partial x}(\phi+\psi)\right] + \alpha\left[\omega\left(\frac{\partial^2}{\partial x^2}-1\right)(\psi-\phi)+\rho\right] = 0. \tag{15}$$

Now, set Eqs. (12), (14), and (15) can be reduced to a single equation for $(\phi+\psi)$,

$$\left(\frac{\partial^2}{\partial x^2}-1\right)\left[2\left(\frac{\partial}{\partial z}+\mathrm{i}\bar{B}_y\right)+\mathrm{i}\alpha\omega\right](\phi+\psi) - 2\mathrm{i}\frac{\partial}{\partial x}\left[\frac{\partial \bar{B}_y}{\partial x}(\phi+\psi)\right] - \frac{\mathrm{i}\alpha(\phi+\psi)}{\omega} = 0. \tag{16}$$

Finally, introduce

$$\Phi = (\phi+\psi)\mathrm{e}^{-\mathrm{i}\bar{C}_y},$$

where $\bar{C}_y$ is such that

$$\frac{\partial \bar{C}_y}{\partial z} = -\bar{B}_y.$$

Then Eq. (16) becomes

$$\left[\frac{\partial^2}{\partial x^2} + 2\mathrm{i}\frac{\partial \bar{C}_y}{\partial x}\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial^2 \bar{C}_y}{\partial x^2} - \left(\frac{\partial \bar{C}_y}{\partial x}\right)^2 - 1\right]\left(2\frac{\partial \Phi}{\partial z} + \mathrm{i}\alpha\omega\Phi\right) - 2\left[\mathrm{i}\frac{\partial}{\partial x}\left(\frac{\partial \bar{B}_y}{\partial x}\Phi\right) - \frac{\partial \bar{C}_y}{\partial x}\frac{\partial \bar{B}_y}{\partial x}\Phi\right] - \frac{\mathrm{i}\alpha\Phi}{\omega} = 0. \tag{17}$$

Now, let

$$\bar{B}_y = A'x\sin z, \quad \bar{C}_y = A'x\cos z,$$

where the constant A' characterizes the amplitude shear of the Alfvén wave. Thus, the wave amplitude is stratified (though the waves are coherent). In this case, the coefficients of Eq. (17) are independent of x and we can seek a solution in the form

$$\Phi(x,z) = \Phi(z)\mathrm{e}^{\mathrm{i}k_x x},$$

where k_x is the radial wavenumber. Then, Eq. (17) reduces to a linear, first-order ordinary differential equation and can be immediately solved,

$$\Phi(z) = [1+(A'\cos z + k_x)^2]^{-1/2} \times \exp\left\{-\frac{\mathrm{i}\alpha}{2\omega}\int\left[\omega^2 + \frac{1}{1+(A'\cos z+k_x)^2}\right]\mathrm{d}z\right\}.$$

Assuming Im $\omega \neq 0$, one can see that $\Phi(z)$ is bounded as $z \to \infty$ if and only if

$$\omega^2 = -\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}z}{1+(A'\cos z + k_x)^2}. \tag{18}$$

The growth rate, Im ω, corresponding to Eq. (18) is shown in Fig. 1(a) as a function of k_x, and in Fig. 1(b) as a function of A'. The latter figure shows that a moderate increase in A' can actually strengthen the instability for $k_x \neq 0$. Note, however, that, for all curves in Fig. 1(b), intervals of transient growth are followed by decay—i.e., eventually, increasing A' does weaken the instability for all k_x. Furthermore, the most important characteristic, the maximum growth rate

$$\gamma(A') = \max_{-\infty<k_x<+\infty} \{\mathrm{Im}\,\omega\}, \tag{19}$$

is a monotonically decaying function of A'—see Fig. 1(b), where γ is shown as a dotted line. In other words, sheared Alfvén waves always make the instability weaker overall.

Observe also that, in the absence of a sheared wave, maximum growth occurs at *large* scales ($k_x=0$), which are the most dangerous ones, as they can disrupt the discharge. In the presence of an Alfvén wave, in turn, the wavelength of maximum growth is shorter [as shown in Fig. 1(a)].

Generally, to understand the mechanism of stabilization, one needs to visualize the field lines as "wavy" (modulated) curves, all traveling with the same speed, v_A. In the frame of the wave, this field structure is static. The magnetic field amplitude of the wave is in the direction transverse to **g** and thus, by the frozen-in condition, would presumably impede interchanges with wavenumbers in the transverse direction. The magnetic field structure, however, does not topologically disallow interchanges. These can still occur since the modulated field lines can slip past each other. Were the wave amplitude not stratified (and the wave carpet coherent, as we assume here), the interchange would be completely unimpeded. In the presence of amplitude stratification, on the other hand, interchanging field lines have to conform to local undulations—which takes extra energy, resulting in an increase in the effective inertia and, correspondingly, a decrease of the growth rate. Similar examples of increased inertia due to the necessity to conform to local conditions were reported before in Ref. 9 (for magnetic shear) and in Ref. 10 (for velocity shear). Note, however, that interchanges are still topologically allowable, which is why *complete* stabilization does not occur in such cases.

We have also examined various extensions of the present problem, e.g., the case in which the amplitude as well as the *phase* of the Alfvén wave are sheared, or where the effect of the wave is enhanced by a static magnetic/velocity shear. Unfortunately, all these problems are much more difficult technically, and this work is still in progress.

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