The height of a static liquid column pulled out of an infinite pool

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We consider a solid cone whose vertex points down and dips in an infinite pool of liquid. If the cone is slowly lifted, a liquid column with its top attached to the cone is pulled out of the pool. In this paper, we compute the maximum height of the cone before the column ruptures. Two reasons for rupturing are identified. In some cases, no solution for a higher position of the cone exists. In other cases, a solution does exist, but is unstable. © *2010 American Institute of Physics*.
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I. INTRODUCTION

Despite an extensive literature on pendant and sessile drops, liquid bridges, and other static configurations supported by surface tension (e.g., Refs. 1–4), one simple yet important problem has been overlooked. Consider a solid object dipped initially in an infinite pool of liquid. If the object is slowly pulled upwards, a liquid column with its top attached to the object will be pulled out of the pool (we shall refer to this setting as the "pull-up problem"). As the object moves higher, the column should become thinner and, eventually, one would expect the column's radius to vanish at some cross-section between the base and the top causing a rupture. The most interesting characteristic of this process is the maximum height of the column just before the rupture occurs.

For a solid sphere, a related problem has been considered in Ref. 5 under an additional assumption that the sphere's radius is much smaller than the liquid's intrinsic capillary lengthscale. Reference 5, however, is devoted to the question as to how deep one needs to push the sphere down before it is completely covered with liquid, which is somewhat opposite to the pull-up problem.

In the present paper, the pull-up problem is studied for a solid cone with its vertex pointing down (see Fig. 1). Solutions describing a static liquid column with its top attached to the cone are presented in Secs. II–IV. These solutions are examined for stability in Sec. V.

II. FORMULATION OF THE PROBLEM

Consider a cone suspended in a vertex-down position above an infinite pool of liquid, and a liquid column rising from the pool with its top attached to the cone (see Fig. 1). The column is assumed to be axisymmetric and, thus, fully characterized by the dependence of its cross-sectional radius r on the vertical coordinate z, both of which are nondimensionalized by the capillary scale $\sqrt{\sigma/\rho g}$ (ρ and σ are the liquid's density and surface tension, g is the acceleration due to gravity). If the column is in a state of static equilibrium, $r(z)$ satisfies an equation reflecting the balance of capillary forces and gravity[2]

$$\left\{\frac{1+(r')^2-rr''}{r[1+(r')^2]^{3/2}}\right\}' = -1, \tag{1}$$

where the primes denote differentiation with respect to z. The expression in the curly brackets is the curvature of the liquid's surface, while the unity on the right-hand side represents the nondimensional acceleration due to gravity. The column's base, where it joins the pool, is described by the following boundary condition:

$$r \to +\infty, \quad r' \to -\infty \quad \text{as} \quad z \to 0. \tag{2}$$

At the column's top, the liquid's surface should approach the cone at a given contact angle θ,

$$r'(H) = \tan(\phi - \theta), \tag{3}$$

where H is the height of the contact line above the pool and ϕ is the angle between the cone's surface and the vertical (see Fig. 1). We should also require that the contact line lies on the cone's surface,

$$r(H) = (H - H_V)\tan \phi, \tag{4}$$

where H_V is the height of the cone's vertex. Equation (1) can be integrated with respect to z, and the resulting equation is compatible with the boundary condition (2) only if the constant of integration is zero (see Appendix A), which yields

$$r'' = z[1+(r')^2]^{3/2} + \frac{1}{r}[1+(r')^2]. \tag{5}$$

Equations (2)–(5) form a boundary-value problem for $r(z)$, where H_V, ϕ, and θ are parameters. Observe that the upper boundary H of the z-interval is unknown, which is the reason to have *three* boundary conditions for a *second*-order equation.

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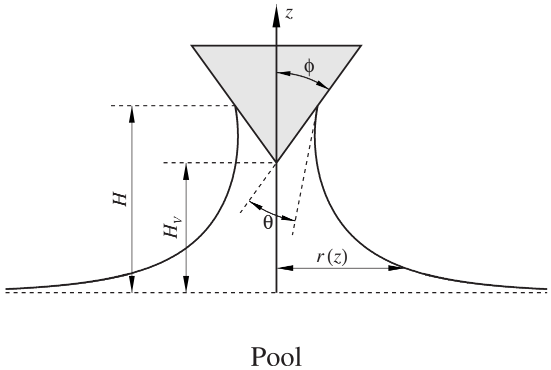

FIG. 1. The setting: a liquid column rising from an infinite pool with its top attached to a cone suspended above the pool.

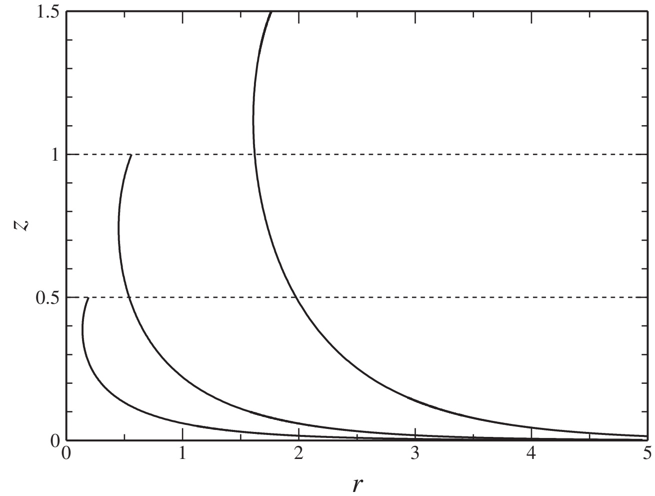

FIG. 2. Numerical solution of the boundary-value problem (2), Eqs. (5) and (6) for $\theta=45°$, $H=0.5,1,1.5$.

III. THE CASE OF A HORIZONTAL PLANE

Let $\phi=90°$, i.e., the cone becomes a horizontal plane suspended at $H=H_V$. This particular case has been examined numerically and asymptotically in Refs. 1 and 6. In this section, we shall briefly reexamine it, as the results obtained turn out to be useful for the general case as well.

Upon substitution of $\phi=90°$, the boundary condition (3) becomes

$$r'(H) = \cot\,\theta. \tag{6}$$

The boundary condition (4), in turn, can no longer be used (as its right-hand side is a product of zero and infinity), but it is not needed anyway as H is now a given parameter.

A. Numerical results

In the boundary value problem (2), Eqs. (5) and (6) were solved numerically (the numerical method is described in Appendix B). Typical solutions are shown in Fig. 2. Quite paradoxically, when the plane moves up, the liquid column becomes thicker!

More details are revealed in Fig. 3 that show the dependence of the radius of the column's top, $R=r(H)$, on its height H. One can see that, for a given value of the contact angle θ, there exists a maximum height $H_{\max}(\theta)$ such that the column's thickness tends to infinity. No solution exists for $H \geq H_{\max}$.

B. Discussion

The above results might come as a surprise, as the capillary force (pulling the column up) is proportional to the circumference of the column's top and, thus, to its radius, while the column's weight is proportional to its volume and, thus, to the radius squared. As the column becomes thicker, the former cannot possibly match the latter, and it is unclear what can keep the column suspended. Furthermore, for a perfectly wetting liquid ($\theta=0$), the capillary force has zero vertical component and, thus, cannot balance the weight of the column regardless of its radius.

To resolve the paradox, multiply Eq. (1) by πr^2 and integrate from $z=0$ to $z=H$. Integrating the left-hand side of the resulting equality by parts, one obtains

$$\pi\left\{ r\frac{rr''-1-(r')^2}{[1+(r')^2]^{3/2}}\right\}_{z=0}^{z=H} - 2\pi\int_0^H r'\frac{rr''-1-(r')^2}{[1+(r')^2]^{3/2}}\mathrm{d}z = \pi\int_0^H r^2\mathrm{d}z.$$

The first term on the left-hand side of this identity can be simplified by replacing r'' with the right-hand side of Eq. (5), and the integrand in the second term is the full derivative of $-r/\sqrt{1+(r')^2}$, i.e., this integral can be evaluated. Finally, using the boundary conditions (2) and (6) (where the former implies that, as $z\to 0$, r' tends to infinity faster than r, hence, $r/r'\to 0$), we obtain

$$\pi H r^2(H) + 2\pi r(H)\sin\,\theta = \pi\int_0^H r^2\mathrm{d}z. \tag{7}$$

The right-hand side of this identity is the nondimensional weight of the liquid column, while the second term on the left-hand side represents the vertical component of the capillary force (as anticipated, the former scales with r^2 and the latter with r).

To clarify the physical meaning of the first term on the left-hand side of identity (7), observe that the pressure at the base of the column should be the same as that at the pool's surface (located at the same level), i.e., zero. Then, hydrostatically, the nondimensional pressure at the column's top is $-H$. Multiplying this value by the area of the column's top, we obtain an upward force of $H\times\pi r^2(H)$.

Thus, the first term on the left-hand side of equality (7) represents the negative pressure applied to the top of the column (which, essentially, acts as a "suction cup"). Most importantly, this term scales with r^2, just like the column's weight [represented by the right-hand side of Eq. (7)]; hence, the two effects can remain in balance as $r\to\infty$. They can also be in balance if $\theta=0$ even though the second term in Eq. (7) vanishes in this case (i.e., the capillary force has no vertical component).

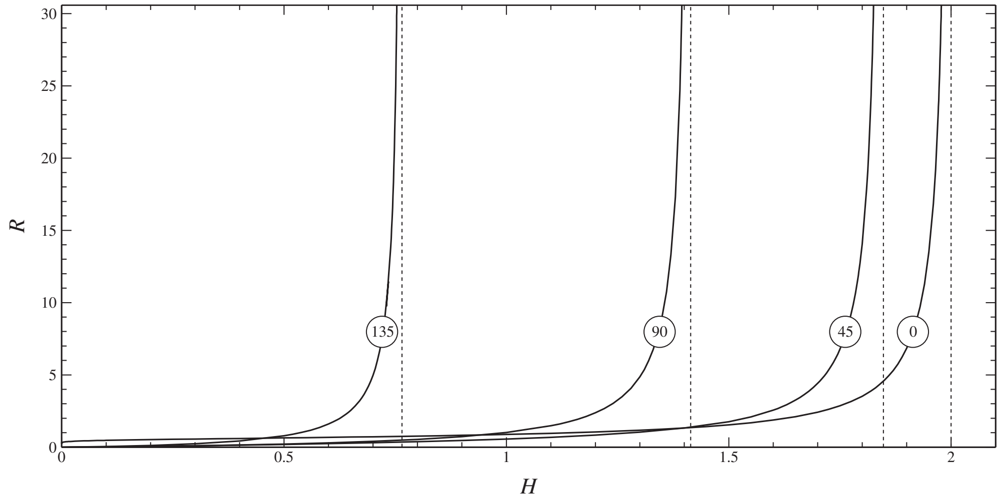

FIG. 3. The radius R of the column's top vs its height H. The curves are marked with the corresponding values of the contact angle θ. Vertical dotted lines show the column's maximum height for the corresponding value of θ as predicted by formula (8).

The above argument removes the apparent contradiction associated with the existence of solutions with large radii or zero contact angles.

C. Analytical results

In the boundary-value problem (2), Eqs. (5) and (6) do not have an obvious analytic solution, but the most important characteristic—the maximum height H_{max} above which no solution exists—can be readily found analytically. This calculation is based on the following numerical observation: when H approaches H_{max}, the solution splits into a z-dependent part tending to a finite limit and a constant part growing infinitely as $H \to H_{max}$, i.e.,

$$r \to r_0 + r_1(z) \quad \text{as} \quad H \to H_{max},$$

where

$$r_0 \to \infty \quad \text{as} \quad H \to H_{max}.$$

Thus, if $H \approx H_{max}$, the last term in Eq. (5) can be omitted, yielding a separable equation

$$\frac{d(r')}{[1+(r')^2]^{3/2}} \approx z dz.$$

Integrating it and using the boundary conditions (2) and (6), we obtain

$$\frac{\cot\theta}{(1+\cot^2\theta)^{1/2}} + 1 \approx \frac{1}{2}H^2.$$

Strictly speaking, this is an asymptotic equality, but in the limit $H \to H_{max}$ it becomes exact and yields

$$H_{max} = 2\cos\left(\tfrac{1}{2}\theta\right). \qquad (8)$$

Examples of H_{max} for various values of the contact angle θ are depicted in Fig. 3, which shows that formula (8) and our numerical results agree very well.

Interestingly, formula (8) also predicts that liquids with $\theta = 180°$ cannot be "pulled up" at all, as the maximum height of the column in this case is zero.

IV. THE GENERAL CASE

Having examined the particular case $\phi = 90°$, we shall first extend our results to

$$\theta - \phi \leq 90°. \qquad (9)$$

The rest of the parameter space, where

$$\theta - \phi > 90°, \qquad (10)$$

will be discussed in Sec. IV B.

A. Range (9)

In what follows, it is convenient to include θ and ϕ in the formal list of arguments, e.g., $r(z;\theta,\phi)$. Then, observe that Eqs. (2), (3), and (5) do not depend on θ and ϕ separately, but only as a combination $\phi - \theta$. As a result, solutions for range (9) can be deduced from the horizontal-plane particular case, $\phi = 90°$. For example, the solution for the contact angle $\theta = 35°$ and the cone with $\phi = 80°$ coincides with that for $\theta = 45°$ and $\phi = 90°$.

Generally, the following equality holds: $r(z;\theta,\phi) = r(z;\theta + 90° - \phi, 90°)$, but we need it only for $z = H$, where H is the height of the contact line, i.e.,

$$r(H;\theta,\phi) = r(H;\theta - \phi + 90°, 90°). \qquad (11)$$

This condition guarantees that the column's surface touches the cone at the correct contact angle, but we should also make sure that the contact line is indeed located at the cone's surface, i.e., the boundary condition (4) is satisfied.

Equations (4) and (11) determine H and $R = r(H;\theta,\phi)$. As illustrated in Fig. 4, they can be solved graphically on the (H,R)-plane. One can see that a solution exists only if the height H_V of the cone's vertex does not exceed a threshold value, $H_{V\,max}$, which is the desired maximum height after which the column ruptures.

If, however, the cone has not reached the maximum height, i.e., $0 \leq H_V < H_{V\,max}$, *two* solutions exist, as illustrated in Fig. 4.

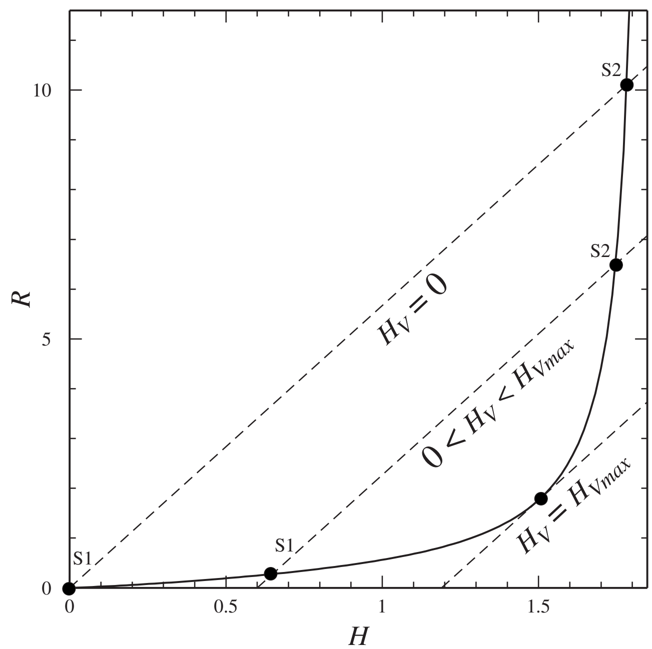

FIG. 4. An illustration of graphical solution of Eqs. (4) and (11). The solid curve represents Eq. (11) with its right-hand side computed for the contact angle θ=45° and the horizontal wall (ϕ=90°). The dashed lines represent Eq. (4) for the cone with ϕ=80° and its vertex located at various heights H_V. The intersections of the dashed lines and the curve correspond to the solutions for the cone and a liquid with θ=35°.

(1) For H_V=0, one of the two solutions corresponds to a column of zero radius and height, $R=H=0$ (see Fig. 4). As H_V increases, R and H grow, i.e., the column becomes thicker and taller while being pulled up. This family of solutions (found for various H_V) will be referred to as S1; examples of such solutions are shown in Fig. 5.

(2) Consider again the case H_V=0, but this time "pick" the solution with $R>0$ (see Fig. 4). Then, if H_V slowly increases, R and H both decrease, i.e., the column becomes thinner and shorter as it is pulled up. This family of solutions will be referred to as S2 (examples are shown in Fig. 5).

Thus, two solutions may exist for the same position of the cone, and it is unclear so far which one occurs in reality. Furthermore, both solutions behave in a counterintuitive way. The column corresponding to S1 becomes thicker while being pulled up, whereas the one corresponding to S2 becomes shorter while being pulled up. Interestingly, two solutions have been found for a solid sphere half-dipped in a pool in Ref. 5, but the question on which among the two is physically relevant was not discussed.

The paradox associated with the emergence of multiple solutions of the pull-up problem will be resolved in Sec. V.

B. Range (10)

To understand the difference between ranges (9) and (10), one only needs to imagine an equivalent of Fig. 1 for, say, ϕ=10° and θ=170° (which belong to the latter range). Then it becomes clear that solutions from range (10) cannot be sought as functions $r(z)$; the column's shape in this case should rather be represented by $h(r)$, where h is the vertical displacement of the liquid's surface and r is the horizontal radial coordinate. In terms of $h(r)$, the balance of gravity and surface tension has the form

$$h - \frac{(rh')'}{r[1+(h')^2]^{3/2}} = 0. \qquad (12)$$

Equation (12) is to be solved in the interval $R<r<\infty$ with an unknown boundary R, and the boundary conditions should be

$$h(R) = H_V + \frac{R}{\tan\phi}, \qquad h'(R) = \cot(\phi-\theta), \qquad (13)$$

$$h \to 0 \quad \text{as} \quad r \to \infty. \qquad (14)$$

The boundary-value problem (12)–(14) is very similar to problem (2)–(5) examined previously and, therefore, will not be discussed in detail. We shall only mention that solutions for all values of ϕ and θ from range (10) can be derived from the problem where the cone is replaced by a cylinder [the same way solutions for range (9) could be derived from the problem where the cone is replaced by a horizontal plane].

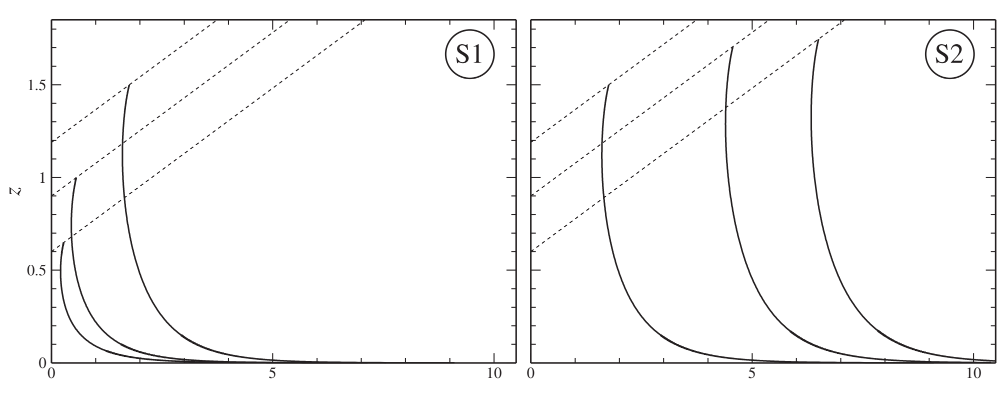

FIG. 5. Numerical solutions of the boundary-value problem (2)–(5) for the contact angle θ=35° and a cone with ϕ=80°. The curves in both panels are computed for the same heights of the cone, H_V=0.6, 0.9, 1.18916. The last value approximates $H_{V\,max}$, accordingly, the corresponding S1 and S2 solutions are virtually identical.

Note that, since the equilibrium contact angles for most liquids are $\theta < 90°$, range (10) does not have many applications.

V. STABILITY OF THE SOLUTIONS FOUND

A. Variational approach to stability

The simplest way to explore the stability of a static liquid column is to examine its energy (see, for example, Refs. 2–4). We shall introduce

$$E = P + S_{gs} + S_{sl} + S_{lg}, \tag{15}$$

where P is the potential energy associated with gravity, and S_{lg}, S_{gs}, and S_{sl} represent the surface energy associated with the liquid/gas, gas/solid, and solid/liquid interfaces, respectively. Considering for simplicity range (9), one can write

$$P = \int_0^{H_V} z(\pi r^2)\mathrm{d}z + \int_{H_V}^{H} z[\pi r^2 - \pi(z-H_V)^2\tan^2\phi]\mathrm{d}z. \tag{16}$$

To avoid the divergence associated with the cone's surface being infinite, let S_{gs} represent the energy difference between the current (partially wet) and initial (all dry) states. It is also convenient to combine S_{gs} with S_{sl}, which yields

$$S_{gs} + S_{sl} = (\sigma_{sl} - \sigma_{gs})\int_{H_V}^{H} \frac{2\pi(z-H_V)\tan\phi}{\cos\phi}\mathrm{d}z, \tag{17}$$

where σ_{sl} and σ_{gs} are the solid/liquid and gas/solid surface-tension coefficients scaled by the liquid/gas coefficient.

To avoid the divergency associated with the liquid/gas interface being infinite, we shall first truncate the column at $z = z_0 > 0$ and calculate its surface energy, then subtract the energy of a flat circle of radius $r(z_0)$ (which would be in place of the column if the liquid was unperturbed). Eventually we shall take the limit $z_0 \to 0$ and obtain

$$S_{lg} = \lim_{z_0\to 0^+}\left\{\int_{z_0}^{H} 2\pi r\sqrt{1+(r')^2}\mathrm{d}z - \pi r^2(z_0)\right\}. \tag{18}$$

Substituting Eqs. (16)–(18) into Eq. (15), we obtain

$$E = \pi\int_0^H zr^2\mathrm{d}z - \frac{1}{4}\pi(H-H_V)^3\left(H + \frac{1}{3}H_V\right)\tan^2\phi - \frac{\pi\cos\theta}{\sin\phi}r^2(H) + \pi\lim_{z_0\to 0^+}\left\{2\int_{z_0}^{H} r\sqrt{1+(r')^2}\mathrm{d}z - r^2(z_0)\right\},$$

where

$$\cos\theta = \sigma_{gs} - \sigma_{sl},$$

is effectively, Young's Law for the equilibrium contact angle.

The functional E is fully determined by $r(z)$; it also appears to depend on the column's height H, which, however, is related to $r(z)$ by condition (4). Accordingly, a "transversality condition" can be derived from Eq. (4),

$$\delta H = \frac{\delta r(H)}{\tan\phi - r'(H)}. \tag{19}$$

In what follows, we shall also need the following identities:

$$\delta[r(H)] = \frac{\tan\phi\,\delta r(H)}{\tan\phi - r'(H)},$$

$$\delta[r'(H)] = \delta r'(H) + \frac{r''(H)\delta r(H)}{\tan\phi - r'(H)},$$

which can be obtained using the transversality condition (19).

Following the usual scheme of the variational approach to stability, one should first demonstrate that the solution of the boundary-value problem (5), Eqs. (2) and (3) follows from the requirement that $\delta E = 0$, i.e., the steady state represents the steady point of E.

To examine the column's stability, the second variation, $\delta^2 E$, needs to be examined. Note that, in its derivation, one cannot use any of the steady-state conditions (5), Eqs. (2) and (3) as the perturbation $\delta r(z)$ is not necessarily steady. Condition (4), on the other hand, is valid for any perturbations, hence, it *can* be used.

Omitting relatively straightforward, but still cumbersome calculations, we shall present the final result only,

$$\delta^2 E = 2\pi\frac{r(H)r''(H)\cos^3(\phi-\theta)}{\tan\phi - r'(H)}[\delta r(H)]^2 + 2\pi\int_0^H \frac{-[1+(r')^2](\delta r)^2 + r^2(\delta r')^2}{r[1+(r')^2]^{3/2}}\mathrm{d}z. \tag{20}$$

Generally, if there exists a perturbation $\delta r(z)$ such that $\delta^2 E < 0$, the steady state is energetically unfavorable and, thus, unstable. If, on the other hand, $\delta^2 E > 0$ for all admissible $\delta r(z)$, we conclude that the steady state corresponds to an energy minimum and, thus, is stable.

One should keep in mind, however, that only those perturbations should be considered that preserve the column's net volume

$$V=\pi\int_0^H r^2 \mathrm{d}z-\frac{1}{3}\pi(H-H_V)^3\tan^2\phi.$$

Calculating δV and equating it zero, we obtain

$$\int_0^H r\delta r \mathrm{d}z=0.$$

Accordingly, the general form of a volume-preserving perturbation is

$$\delta r=\frac{1}{r}\sum_{n=1}^{\infty}a_n\sin(nkz+p_n), \tag{21}$$

where $a_n \in [0,\infty)$ and $p_n \in [0,2\pi)$ are arbitrary coefficients, and

$$k=\frac{2\pi}{H}.$$

B. The limit of small H_V

Unlike the steady state [which depends on $\phi-\theta$, see Eqs. (2), (3), and (5)], expression (20) for $\delta^2 E$ depends on θ and ϕ *separately*. As a result, the parameter space (ϕ,θ,H_V) of the stability problem is three-dimensional, which makes it difficult to explore numerically.

A great deal of physically important information, however, can be extracted from a study of S1 solutions with small H_V. In this case, the height and radius of the column are both small, and we can use the results of Ref. 6 obtained for a "pinhole" in a liquid film on a horizontal plane, which is mathematically equivalent to a small liquid column.

Adopting the results of Ref. 6 for our setting, one can write the solution for small H_V in the form

$$r\approx\begin{cases}\varepsilon\cosh\dfrac{z-z_{\min}}{\varepsilon} & \text{in zone 1 } [z=H-\mathrm{O}(\varepsilon)],\\[2mm] \dfrac{1}{2}\varepsilon\exp\left(\dfrac{z}{\varepsilon}\right) & \text{in zone 2 } [z=\mathrm{O}(H)],\\[2mm] \mathrm{Ki}_0\left(\dfrac{z}{\varepsilon}\right) & \text{in zone 3 } [z=\mathrm{O}(\varepsilon)],\end{cases} \tag{22}$$

where

$$\varepsilon=\frac{H_V}{-\ln H_V},\quad H\approx H_V+\frac{\varepsilon}{\cos(\phi-\theta)\tan\phi}, \tag{23}$$

$$z_{\min}=H-\varepsilon\ln\left[\tan(\phi-\theta)+\frac{1}{|\cos(\phi-\theta)|}\right],$$

and Ki_0 is the inverse function to the modified Bessel function of the zeroth order.

Now, consider perturbation (21) with

$$a_1=1,\quad p_1=0;\quad a_n=0 \quad \text{for} \quad n\geq 2, \tag{24}$$

and substitute Eq. (21), and Eq. (24) into expression (20). After straightforward algebra, one obtains

$$\delta^2 E=\int_0^H\left\{\frac{4\pi^2}{H^2}\cos^2 kz-4\left(\frac{r'}{r}\right)^2\sin^2 kz-\frac{z[1+(r')^2]^{1/2}}{r}[2(r')^2-1]\sin^2 kz\right\}\frac{\mathrm{d}z}{r[1+(r')^2]^{3/2}}. \tag{25}$$

To estimate the sign of $\delta^2 E$, observe that, as follows from solution (22) and (23):

$$\frac{1}{H^2}=\mathrm{O}\left[\frac{1}{(H_V)^2}\right],\quad \left(\frac{r'}{r}\right)^2=\mathrm{O}\left[\frac{\ln^2 H_V}{(H_V)^2}\right]\quad \text{in zones } 1-3,$$

$$\frac{z[1+(r')^2]^{1/2}[2(r')^2-1]}{r}=\begin{cases}\mathrm{O}(-\ln H_V) & \text{in zones } 1-2,\\ \mathrm{O}\left[\dfrac{\ln^2 H_V}{(H_V)^2}\right] & \text{in zone 3}.\end{cases}$$

It can now be shown that the main contribution to expression (25) comes from the second term in the curly brackets. Indeed, the first term is always smaller, whereas the third is comparable to the second only in a narrow region (zone 3), hence, its contribution to the integral is negligible.

Given the sign of the largest (second) term in expression (25), one can see that $\delta^2 E<0$, i.e., the steady state is unstable.

C. Discussion

The conclusion that columns with small H are unstable is of crucial importance, as it discriminates between solutions S1 and S2, making one of them irrelevant physically. Indeed, consider a liquid column immediately after it has been "picked up," i.e., after the cone has touched the liquid, then was pulled a little up. Since H_V is still very small, the corresponding S1 solution describes a column of a small height, while the S2 solution describes a column of an order-one height (see Fig. 4). Thus, in accordance with the above results, the former is unstable, which effectively means that the column can be picked up only through solutions of type S2. Then, provided the cone is pulled up slowly enough, one can safely assume that the solution cannot suddenly switch to the S1 type.

D. Numerical results

To see how our asymptotic small-H results extend to the region of finite H, we have examined a generalization of particular case (24),

$$\delta r=\frac{1}{r}\sin(kz+p), \tag{26}$$

numerically. The corresponding curve of marginal stability on the (H,θ)-plane for the case of horizontal plane ($\phi=90°$) is shown in Fig. 6. One can see that the region of instability, corresponding to $\delta^2 E<0$, reaches order-one values of H.

Note that even though solutions with sufficiently large H turned out to be stable with respect to perturbations (26),

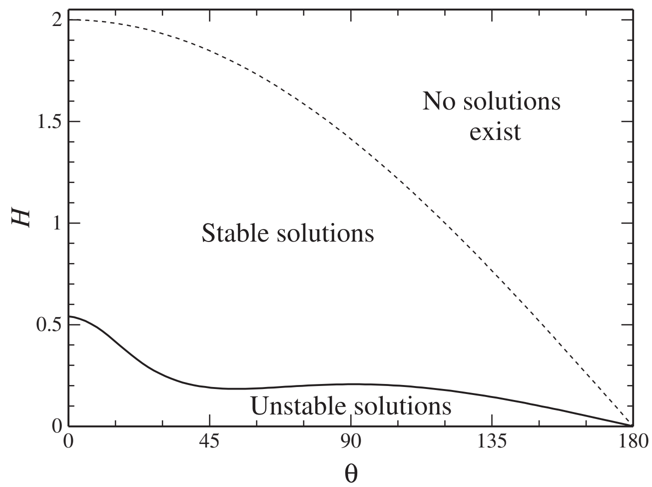

FIG. 6. The curve of neutral stability for perturbation (25) and horizontal plane, on the (θ, H)-plane (θ is the contact angle and H is column's height). The dotted line corresponds to the maximum height given by formula (8).

they can still be unstable with respect to general perturbations (21). To clarify this issue, sample computations were carried out for perturbations of the form

$$\delta r = \frac{1}{r}[\sin(kz + p_1) + a_2 \sin(2kz + p_2)]. \tag{27}$$

It turns out that the inclusion of the second harmonic noticeably expands the region of instability, as illustrated by the following example computed for $\phi = 90°$, $\theta = 45°$. In this case, the *single*-harmonic perturbation (26) corresponds to the marginally stable column with $H_{ns} \approx 0.191$ (as in Fig. 6), whereas, for the *two*-harmonic perturbation (27), $H_{ns} \approx 0.538$. Since this issue has no qualitatively important implications, it will not be discussed in further detail.

Note also that the observed instability is likely to be caused by the Plateau–Rayleigh mechanism, which often destabilizes axisymmetric configuration of capillary liquids. This conclusion agrees also with the fact that thick columns are stable, as the Plateau–Rayleigh mechanism affects only those configuration whose length-to-radius ratios exceed 2π.

Finally, the liquid column instability can impose an extra restriction on the cone's maximum height (in addition to the one imposed by the nonexistence of static solutions for $H_V > H_{V\,\max}$). To understand why, recall that there is a "strip" of unstable solutions with small R, i.e., located along the H-axis of Fig. 4. If this strip is wide enough, the S2 solutions for increasing values of H_V become unstable *before* they disappear at $H_V = H_{V\,\max}$.

VI. SUMMARY AND CONCLUDING REMARKS

We have examined the pull-up problem, where a liquid column is pulled out of an infinite pool by a solid cone, as illustrated in Fig. 1. It has been shown that, if the height H_V of the cone's vertex is sufficiently small, two solutions exists, S1 and S2 (the column corresponding to the former is the shortest of the two, see Figs. 4 and 5). When H_V grows, the behavior of the S1 solution is quite counterintuitive: the column becomes thicker while being pulled up, whereas the column described by the S2 solution becomes thinner. At some threshold value $H_V = H_{V\,\max}$, the two solutions coalesce and disappear at $H = H_{V\,\max}$.

It has been shown, however, that if the cone's vertex is sufficiently low, the S1 solution is unstable, which means that the liquid column can be picked up only through solution S2. Then, provided the cone is pulled up slowly enough, the solution cannot suddenly switch to the S1 type.

Note that even though S2 corresponds to a column becoming thinner, while being pulled up (which agrees with one's intuition), the column's height decreases (that is still counterintuitive). One would also expect the column to rupture only if its radius vanishes at some cross-section, which, however, turns out to be incorrect, as the critical solution (the one with $H_V = H_{V\,\max}$) has nonzero radius for all z.

Finally, we emphasize that all our results are based on the assumption that the column is axisymmetric and the cone is withdrawn slowly. In a general situation, however, the column may be destabilized by nonaxisymmetric disturbances and/or detachment of the contact line from the cone's surface. These processes depend on the rate of the cone's withdrawal, the liquid's viscosity, and the dynamics of the contact line, none of which were accounted for in the present work.

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APPENDIX A: THE SMALL-*z* ASYMPTOTICS OF THE SOLUTION OF EQ. (1)

Integration of Eq. (1) with respect to z yields

$$1 + (r')^2 - rr'' = (B - z)r[1 + (r')^2]^{3/2}, \tag{A1}$$

where B is a constant. Observe also that the boundary condition (2) implies that

$$[1 + (r')^2]^{3/2} \to -(r')^3 \quad \text{as} \quad z \to 0. \tag{A2}$$

Equation (A2) can be used to simplify Eq. (A1), but the final form of Eq. (A1) depends on whether or not $B = 0$.

First we shall consider the case $B \neq 0$. Taking into account Eqs. (2) and (A2), one can reduce Eq. (A1) to

$$r'' \to B(r')^3 \quad \text{as} \quad z \to 0. \tag{A3}$$

This equation can be readily solved

$$r \to \frac{\sqrt{2(C - Bz)}}{B} + D \quad \text{as} \quad z \to 0, \tag{A4}$$

where C and D are constants of integration. Clearly, the limiting solution (A4) does not satisfy the boundary condition (2), hence, the case $B \neq 0$ should be discarded.

Next assume that $B = 0$, in which case (A1) coincides with Eq. (5) used in the paper. As before, taking into account Eqs. (2) and (A2), one can reduce Eq. (5) to

$$(r')^2 - rr'' = zr(r')^3 \quad \text{as} \quad z \to 0.$$

Rewrite this equation in terms of $x = \ln z$,

$$(r')^2 - r(r'' - r') = r(r')^3 \quad \text{as} \quad x \to -\infty, \tag{A5}$$

and change the variables (x, r) to (r, q) where $q = r'$, after which Eq. (A5) becomes

$$\frac{\mathrm{d}q}{\mathrm{d}r} = \frac{q}{r} + 1 - q^2 \quad \text{as} \quad r \to +\infty. \tag{A6}$$

Seeking the solution of Eq. (A6) in the form $q = \Sigma_{n=0}^{\infty} q_n r^{-n}$ and recalling that $q = r'$, one can obtain

$$r' = -1 + \tfrac{1}{2} r^{-1} - \tfrac{3}{8} r^{-2} + \mathrm{O}(r^{-3}) \quad \text{as} \quad x \to -\infty.$$

Finally, solving this equation by perturbations and recalling that $x = \ln z$, we obtain

$$r = -\ln az - \frac{\ln|\ln az|}{2} - \frac{\ln|\ln az|}{4 \ln az} + \frac{1}{8 \ln az} + \mathrm{O}\left[\frac{\ln^2|\ln az|}{(\ln az)^2}\right] \quad \text{as} \quad z \to 0, \tag{A7}$$

where a is an arbitrary positive constant.

APPENDIX B: THE NUMERICAL METHOD FOR THE BOUNDARY-VALUE PROBLEM (2)–(5)

To solve problem (2)–(5) numerically, we used expansion (A7) to "shoot" $r(z)$ from a small positive value of z toward $z = H$, and adjusted the parameters a and H (using a root-finding routine based on Newton's method) until $r(H)$ and $r'(H)$ satisfied the upper boundary conditions (3) and (4). This approach works for all parameter values except

$$\phi - \theta = 90°, \tag{B1}$$

in which case the boundary condition (3) amounts to

$$r'(H) = -\infty, \tag{B2}$$

i.e., shooting the solution into a singular point.

In case (B1), we expanded the solution in powers of $(H-z)^{1/2}$,

$$r = r_0 + r_1(H-z)^{1/2} + r_2(H-z) + r_3(H-z)^{3/2} + \mathrm{O}[(H-z)^2] \quad \text{as} \quad z \to H, \tag{B3}$$

which, clearly, satisfies the boundary condition (B2). Substituting Eq. (B3) into Eq. (5), one can obtain

$$r_1 = -\sqrt{\frac{2}{H}}, \quad r_2 = \frac{1}{3Hr_0}, \tag{B4}$$

$$r_3 = -\frac{1}{25}\sqrt{\frac{2}{H}}\left(\frac{1}{Hr_0^2} + \frac{2}{H} - 6H\right). \tag{B5}$$

Substitution of Eq. (B3) into the boundary condition (4) yields

$$r_0 = (H - H_V)\tan\phi. \tag{B6}$$

Expansions (A7) and Eqs. (B3)–(B6) were used to shoot the solution from $z = 0$ and $z = H$ respectively, and the parameters a and H were adjusted until the two solutions and their derivatives matched at an intermediate point.

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