ELSEVIER

Physica D 77 (1994) 473–485

PHYSICA D

Solitary waves with damped oscillatory tails: an analysis of the fifth-order Korteweg–de Vries equation

R. Grimshaw[a], B. Malomed[b], E. Benilov[c]

[a]*Department of Mathematics, Monash University, Clayton, Victoria, 3168, Australia*
[b]*Department of Applied Mathematics, Tel Aviv University, Ramat Aviv 69978, Israel*
[c]*School of Mathematics, University of NSW, Kensington, New South Wales, 2033, Australia*

Received 17 June 1993; revised 11 March 1994; accepted 11 March 1994
Communicated by A.C. Newell

Abstract

We construct oscillatory solitary wave solutions of a fifth-order Korteweg–de Vries equation, where the oscillations decay at infinity. These waves arise as a bifurcation from the linear dispersion curve at that wavenumber where the linear phase speed and group velocity coincide. Our approach is a wave-packet analysis about this wavenumber which leads in the first instance to a higher-order nonlinear Schrödinger equation, from which we then obtain the steady solitary wave solution. We then describe a complementary normal-form analysis which leads to the same result. In addition we derive the nonlinear Schrödinger equation for all wavenumbers, and list all the various anomalous cases.

1. Introduction

The main purpose of this paper is to establish the existence of a new kind of solitary wave solution of the fifth-order Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} + u_{xxxxx} = 0\,. \tag{1.1}$$

These new solitary waves differ from classical solitary waves in that they are oscillatory, with oscillations that decay as $|x| \to \infty$. The possibility that such waves may exist as solutions of (1.1) was first suggested by the numerical results of Kawahara (1972). They are analogous to the new type of capillary-gravity solitary waves discovered theoretically by Iooss and Kirchgässner (1990) and Dias and Iooss (1993), and numerically by Hunter and Vanden Broeck (1983), Zufiria (1987), Longuet-Higgins (1989) and Vanden Broeck and Dias (1992) who obtained the most complete set of results. Indeed Hunter and Scheurle (1988) have proposed (1.1) as a model equation for capillary-gravity waves when the bond number is just less than the critical value of $\frac{1}{3}$, and hence our Study here can be regarded as a investigation of capillary-gravity solitary waves in this model context. Recently, Champneys and Toland (1993) have summarized the classes of solitary wave solutions known to occur as solutions of (1.1) using both theoretical and numerical methods. These new solitary waves can be expected to occur in other physical contexts and indeed Benjamin (1992) has established the existence of such waves on the interface of a

SSDI 0167-2789(94)00098-B

two-layer fluid where the lower layer is deep and the interface is subject to capillarity. His treatment is based on applying variational methods to a model equation which can be regarded as combined Korteweg–de Vries and Benjamin–Ono equation, and is different both from the approach of Iooss and Kirchgässner (1990) and Dias and Iooss (1993), and to our approach.

Complementary to the normal form analysis of Iooss and Kirchgässner (1990) and Dias and Iooss (1993), our contribution is to demonstrate that these oscillatory solitary waves can be regarded as wave-packet solitary waves. It is well known that to leading order wave packets are described by the nonlinear Schrödinger (NLS) equation (see (1.4a) below). The solitary wave solution of the NLS equation describes a solution of the original equation (here (1.1)) in which the oscillatory component propagates to leading order with the linear phase speed $c(k)$ (see (1.2b)) while the envelope propagates to leading order with the group velocity $c_g(k)$ (see (1.2c)) where k is the dominant wavenumber of the wave packet. In general $c \neq c_g$ so that this is not a steady travelling wave solution of the original equation. However, for values of k such that $c = c_g$ (i.e. $c(k)$ is stationary at this value of k) this becomes, at least to leading order, a steady solitary wave solution of the original equation. It transpires that to make this notion more precise requires consideration of higher-order correction terms to the NLS equation, and this analysis we carry out in Section 2. To demonstrate that our approach is equivalent to the normal form analysis of Iooss and Kirchgässner (1990) and Dias and Iooss (1993) in Section 3 we carry out a normal form analysis of Eq. (1.1) and show that the two approaches lead to the same solitary wave solution.

After the completion and submission of our work we became aware of the work of Akylas (1993) and Longuet-Higgins (1993) both of whom independently observed that for capillary-gravity solitary waves the NLS equation for the (unique) wavenumber k such that $c = c_g$ provides the lowest-order description of these newly-discovered waves. However, neither of these authors considered the higher-order corrections to the NLS equation which are needed to make this approach agree with the normal form analysis. It is pertinent to add here that this connection between the NLS equation (with $c = c_g$) and the existence of these oscillatory solitary waves, establishes that this new kind of solitary wave can generically be expected whenever the linear phase speed $c(k)$ has a minimum c_m (or maximum) for some finite non-zero value of $k = k_m$ (see Fig. 1). The bifurcation to the solitary waves then occurs for speeds less than (greater than) c_m.

To proceed, we first observe that the linearized version of Eq. (1.1) has sinusoidal travelling wave solutions of frequency ω and wavenumber k where

$$\omega = -k^3 + k^5 , \qquad (1.2a)$$

while

$$c = \omega/k = -k^2 + k^4 , \qquad (1.2b)$$

and

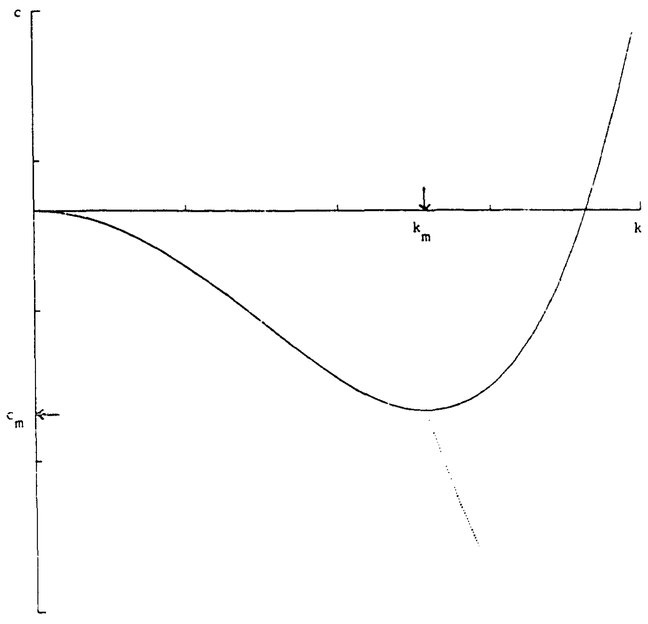

Fig. 1. The linear dispersion curve (1.2b) showing the schematic bifurcation at $k = k_m$.

$$c_g = \frac{d\omega}{dk} = -3k^2 + 5k^4 . \quad (1.2c)$$

Here $c(k)$ is the linear phase speed and $c_g(k)$ is the group velocity. The phase velocity has a minimum at $k = k_m = 2^{-1/2}$ where $c = c_g = c_m = -\frac{1}{4}$. We shall show that oscillatory solitary waves bifurcate from $k = k_m$. The situation is sketched schematically in Fig. 1.

Next, wave-packet analysis proceeds by seeking an asymptotic solution of (1.1) of the form

$$u = \alpha[A(X,T)\exp(i\theta) + \text{c.c.}] + \alpha^2[A_2(X,T)\exp(2i\theta) + \text{c.c.} + A_0] + \mathcal{O}(\alpha^3) , \quad (1.3a)$$

where

$$\theta = k(x - ct) \quad (1.3b)$$

and

$$X = \epsilon x , \quad T = \epsilon t . \quad (1.3c)$$

Here c.c. denotes the complex conjugate, while α and ϵ are small parameters measuring the wave amplitude and inverse of the packet length-scale respectively. The NLS equation generically requires the balance $\alpha = \epsilon$ and is given by

$$A_\tau + i\lambda A_{\xi\xi} + i\mu|A|^2 A = \mathcal{O}(\epsilon) , \quad (1.4a)$$

where

$$\tau = \alpha T , \quad \xi = X - c_g T . \quad (1.4b)$$

Here λ and μ are coefficients which are determined by the asymptotic analysis and are given here by (2.9a,c). The NLS equation (1.4a) has a solitary wave solution, but as noted above, for this to be a steady solitary wave solution of (1.1) requires $c = c_g$. From (1.2b,c) we see this occurs when $k = 0$ or $k = k_m$. This first case is not relevant here. There is a steady travelling wave solution which bifurates from $k = 0$, but it requires a different kind of analysis from that present here (see e.g. Pomeau et al., 1988, Boyd, 1991, and Grimshaw and Joshi, 1994). This is a non-local solitary wave, and consists of a solitary wave core described to leading order by the Korteweg–de Vries equation (i.e. (1.1) with u_{xxxxx} omitted) and co-propagating non-decaying oscillations of small amplitude. The second case $k = k_m$ is our concern here, but in order to make the notion of the matching of the envelope speed with the phase speed more precise than the leading order matching $c = c_g$ it transpires that is necessary to compute the higher-order $\mathcal{O}(\epsilon)$ terms in the NLS equation (1.4a).

In Section 2(i) we derive the higher-order NLS equation and the end result is (2.8). Our derivation uses a classical multi-scale asymptotic perturbation argument based on the expansion (1.3a). In Appendix A we outline an alternative derivation using a Zakharov spectral formulation. Then in Section 2(ii) we construct the steady oscillatory wave solution of (1.1) from the higher-order NLS equation (2.8). This is the main result of this study. However, as is well known, the derivation of the NLS equation can fail for certain values of k when various special resonances occur. These special cases require a different treatment and scaling and for completeness we describe these briefly in Section 2(iii). In Section 3 we describe the normal form approach of Iooss and Kirchgässner (1990) and Dias and Iooss (1993) when applied to (1.1) and verify that it leads to the same steady solitary wave solution. Some technical details of this analysis are described in Appendix B.

2. Approach through the NLS equation

(i) *Derivation*. As a preliminary step we allow u to depend on the phase variable θ (1.3b) and the slow variables X, T (1.3c) so that Eq. (1.1) becomes

$$-\omega u_\theta + k^3 u_{\theta\theta\theta} + k^5 u_{\theta\theta\theta\theta\theta} + \epsilon u_T + 6u(k u_\theta + \epsilon u_X) + 3\epsilon k^2 u_{\theta\theta X} + 3\epsilon^2 k u_{\theta XX} + \epsilon^3 u_{XXX} + 5\epsilon k^4 u_{\theta\theta\theta\theta X} + 10\epsilon^2 k^3 u_{\theta\theta\theta XX} + 10\epsilon^3 k^2 u_{\theta\theta XXX} + 5\epsilon^4 k u_{\theta XXXX} + \epsilon^5 u_{XXXXX} = 0 . \quad (2.1)$$

Next we substitute the expansion (1.3a) into

(2.1) and collect the terms proportional to $\exp(\mathrm{i}\theta)$. At leading order the dispersion relation (1.2a) is satisfied, while the remaining terms give

$$\begin{aligned}&\epsilon^2[A_T + (-3k^2 + 5k^4)A_X] \\ &\quad + \epsilon^3[(3\mathrm{i}k - 10\mathrm{i}k^3)A_{XX} + 6\mathrm{i}k(A_2\bar{A} + A_0A)] \\ &\quad + \epsilon^4[(1 - 10k^2)A_{XXX} + 6(A_2\bar{A} + AA_0)_X] \\ &\quad + \mathcal{O}(\epsilon^5) = 0 . \qquad (2.2)\end{aligned}$$

At the leading order ϵ^2-term in (2.2) we find that

$$A_T + c_g A_X = \mathcal{O}(\epsilon) , \qquad (2.3)$$

where we recall that the group velocity c_g is defined by (1.2c). This suggests the further changes of variables from X, T to ξ, τ (1.4b) which we shall exploit later.

Next we collect the terms proportional to $\exp(2\mathrm{i}\theta)$ and get

$$\begin{aligned}&\epsilon^2[(-2\mathrm{i}\omega - 8\mathrm{i}k^3 + 32\mathrm{i}k^5)A_2 + 6\mathrm{i}kA^2] \\ &\quad + \epsilon^3[A_{2T} + (-12k^2 + 80k^4)A_{2X} + 6AA_X] \\ &\quad + \mathcal{O}(\epsilon^4) = 0 . \qquad (2.4)\end{aligned}$$

Noting that A_2, like A, can be regarded as a function of the variables τ, ξ (1.4b) we can readily solve (2.4) to obtain

$$A_2 = \frac{A^2}{k^2(1 - 5k^2)} + \frac{2\mathrm{i}\epsilon(1 - 10k^2)}{k^3(1 - 5k^2)^2} AA_\xi + \mathcal{O}(\epsilon^2) . \qquad (2.5)$$

Note that this solution fails if $5k^2 = 1$ and we shall return to this special case later.

Finally we collect the mean terms (i.e. those independent of θ) and obtain

$$\epsilon^3(A_{0T} + 6|A|^2_X) + \mathcal{O}(\epsilon^5) = 0 . \qquad (2.6)$$

Again introducing the variables τ, ξ (1.4b) this equation can be readily solved for A_0 to give

$$\begin{aligned}A_0 &= \frac{6|A|^2}{k^2(-3 + 5k^2)} \\ &\quad + \frac{6\mathrm{i}\epsilon(10k^2 - 3)}{k^3(-3 + 5k^2)^2}(\bar{A}A_\xi - A\bar{A}_\xi) + \mathcal{O}(\epsilon^2) . \qquad (2.7)\end{aligned}$$

Note that this solution fails if $5k^2 = 3$ (i.e. $c_g = 0$) and we shall return to this special case later.

The final step in the derivation of the NLS is the substitution of (2.5) and (2.7) into (2.2). Omitting details we obtain, upon using the variables τ, ξ (1.4b)),

$$\begin{aligned}&A_\tau + \mathrm{i}\lambda A_{\xi\xi} + \mathrm{i}\mu|A|^2A \\ &\quad + \epsilon\Big(\lambda' A_{\xi\xi\xi} + \frac{\mu}{k}(|A|^2A)_\xi + \nu|A|^2A_\xi \\ &\quad + \nu'(|A|^2A_\xi - A^2\bar{A}_\xi)\Big) + \mathcal{O}(\epsilon^2) = 0 . \qquad (2.8)\end{aligned}$$

Here the coefficients λ, λ', μ, ν and ν' are defined by

$$\lambda = -\frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = k(3 - 10k^2) , \qquad (2.9a)$$

$$\lambda' = -\frac{1}{6}\frac{\mathrm{d}^3\omega}{\mathrm{d}k^3} = 1 - 10k^2 , \qquad (2.9b)$$

$$\mu = \frac{6(3 - 25k^2)}{k(1 - 5k^2)(-3 + 5k^2)} , \qquad (2.9c)$$

$$\nu = -\frac{12(1 - 10k^2)}{k^2(1 - 5k^2)^2} , \qquad (2.9d)$$

$$\nu' = \frac{36}{k^2}\frac{(3 - 10k^2)}{(-3 + 5k^2)^2} . \qquad (2.9e)$$

At leading order (2.8) reduces to the NLS equation (1.4a) but here we have retained the next-order terms as these will be needed in the sequel. However, unlike the NLS equation, this higher-order equation is not integrable in general. Indeed at present this higher-order NLS equation is known to be integrable only when it reduces to the derivative-NLS so that $\lambda' = 0$ and either $\nu + 3\nu' = 0$, or $\nu' = \mu/k$ (Ablowitz and Clarkson, 1991), or to the Hirota equation so that $\nu' = \mu/k$ and $\nu + 3\nu' = 3\mu\lambda'/\lambda$ (Newell, 1985), or when $\nu + 4\nu' = \mu/k$ and $2(\nu + 3\nu') = 3\mu\lambda'/\lambda$ (Sasa and Satsuma, 1991). It can be shown that the conditions for reduction to the derivative-NLS cannot be met for any value of k, but that for $k = 0.474$ and 0.724 (or 0.464 and

0.744) there is a reduction to the Hirota equation (Sasa–Satsuma equation).

(ii) *Steady solitary wave*. First we note that the NLS equation (1.4a) has the solitary wave solution.

$$A = a\,\mathrm{sech}[\gamma(\xi - v\tau)]\exp(\mathrm{i}l\xi - \mathrm{i}\sigma\tau)\,, \tag{2.10a}$$

where

$$v = -2\lambda l\,, \tag{2.10b}$$

$$\sigma = \lambda(\gamma^2 - l^2)\,, \tag{2.10c}$$

and

$$a^2 = 2\lambda\gamma^2/\mu\,. \tag{2.10d}$$

Of course, this solution requires that $\lambda\mu > 0$ (i.e. $\frac{3}{25} < k^2 < \frac{1}{5}$ or $\frac{3}{10} < k^2 < \frac{3}{5}$), which corresponds to the so-called focussing case of the NLS. Otherwise, when $\lambda\mu < 0$, the so-called defocussing case of the NLS, the corresponding solution is

$$A = a\tanh[\gamma(\xi - v\tau)]\exp(\mathrm{i}l\xi - \mathrm{i}\sigma\tau)\,, \tag{2.11a}$$

where

$$v = -2\lambda l\,, \tag{2.11b}$$

$$\sigma = \lambda(2\gamma^2 - l^2)\,, \tag{2.11c}$$

and

$$a^2 = -2\lambda\gamma^2/\mu\,. \tag{2.11d}$$

However, although each of these form a two-parameter family of solutions, in general they are not steady travelling wave solutions of the original equation (1.1). For this to occur we require that $c = c_g$ as noted in the Introduction and hence either $k = 0$ or $k = k_m$ $(=2^{-1/2})$. The case $k = 0$ requires a different kind of analysis from that considered here and has been discussed elsewhere. Here we consider the case when $k = k_m$. But now we note that for this solution to be exactly steady requires that both the amplitude and the phase have the same speed. Noting that $k = k_m$ belongs to the focussing case, it follows from (1.4) and (2.10a) that the total phase is $\theta + l\xi - \sigma\tau = (k_m + \epsilon l)(x - c_m t) - \epsilon^2\sigma t$ where the subscript m denotes quantities evaluated at $k = k_m$ so that $c_m = -\frac{1}{4}$. Further, the amplitude in (2.10a) is a function of $\xi - v\tau = \epsilon(x - c_m t) - \epsilon^2 v t$. It follows that for the amplitude and phase to be propagating at the same speed, $k_m v = \epsilon\sigma + \mathcal{O}(\epsilon^2)$. Thus v is $\mathcal{O}(\epsilon)$, and it then follows from (2.10b) that l is likewise $\mathcal{O}(\epsilon)$ and then from (2.10c) $\sigma = \lambda_m\gamma^2 + \mathcal{O}(\epsilon^2)$. Finally the relationship (2.10d) remains valid so that $a^2 = 2\lambda_m\gamma^2/\mu_m$, where here λ_m, μ_m denote the coefficients λ, μ (2.9a,c) evaluated at $k = k_m$. Although we shall see that these relations are valid, the fact that v is $\mathcal{O}(\epsilon)$ indicates that it is necessary to consider the $\mathcal{O}(\epsilon)$ terms in (2.8) when constructing a steady solitary wave solution. This we shall now proceed to do.

We set $k = k_m$ in (2.8) so that the coefficients λ, λ', μ, ν and ν' (2.9a–e) become respectively

$$\lambda_m = -2k_m\,,\quad \lambda'_m = -4\,,\quad \mu_m = -152k_m\,,$$
$$\nu_m = \tfrac{128}{3}\,,\quad \nu'_m = -576\,. \tag{2.12}$$

We then seek a solution of (2.8) of the form

$$A = R(\eta)\exp[\mathrm{i}\epsilon\phi(\eta) + \mathrm{i}\epsilon l\eta - \mathrm{i}\sigma\tau]\,, \tag{2.13a}$$

where

$$\eta = \xi - \epsilon v\tau\,. \tag{2.13b}$$

Note that in contrast to (2.10a) we have now replaced v and l by ϵv and ϵl respectively. From (1.4) the total phase is $\theta + \epsilon\phi(\eta) + \epsilon l(\eta) - \sigma\tau$. For a steady travelling wave this must be a function of η alone. Using (1.4b) and recalling that $c_g = c_m$ here it follows that

$$\sigma = k_m v\,. \tag{2.14}$$

This balance confirms that our scaling of v (and l) is correct. Note that the speed of this solitary wave relative to the group velocity c_g is $\epsilon^2 v$.

Next, substitution of (2.13a) into (2.8) and equating to zero the real and imaginary parts of the resulting equation gives, on omitting terms of $\mathcal{O}(\epsilon^2)$,

$$\lambda_m R_{\eta\eta} - \sigma R + \mu_m R^3 = 0\,, \tag{2.15a}$$

$$[-v - 2\lambda_{\mathrm{m}}(l + \phi_\eta)]R_\eta - \lambda_{\mathrm{m}}\phi_{\eta\eta}R + \lambda'_{\mathrm{m}}R_{\eta\eta\eta} + \left(\frac{3\mu_{\mathrm{m}}}{k_{\mathrm{m}}} + \nu_{\mathrm{m}}\right)R^2R_\eta = 0\,. \tag{2.15b}$$

Eq. (2.15a) has the solitary wave solution

$$R = a\,\mathrm{sech}(\gamma\eta)\,, \tag{2.16a}$$

where

$$\sigma = \lambda_{\mathrm{m}}\gamma^2\,, \tag{2.16b}$$

and

$$a^2 = 2\lambda_{\mathrm{m}}\gamma^2/\mu_{\mathrm{m}}\,. \tag{2.16c}$$

In fact this is just the same as the amplitude part of the NLS solitary wave (2.10a–d). Combining (2.14) and (2.16b) we see that

$$v = -2\gamma^2 \tag{2.17}$$

and hence is always negative. This result is to be expected as the speed of this solitary wave should be less than $c_{\mathrm{g}} = c_{\mathrm{m}} = -\frac{1}{4}$. Note also that if R is a solution of (2.15a) so is $-R$, without prejudice to the solution of (2.15b).

With R given by (2.16a), Eq. (2.15b) becomes an equation for the phase ϕ alone. Since ϕ must be bounded for all η, we get on integrating (2.15b) once,

$$2\lambda_{\mathrm{m}}\phi_\eta = (-v - 2\lambda_{\mathrm{m}}l + 4k_{\mathrm{m}}\sigma) + \left(-\frac{3\mu_{\mathrm{m}}}{2k_{\mathrm{m}}} + \frac{\nu_{\mathrm{m}}}{2}\right)R^2\,. \tag{2.18}$$

Further, without loss of generality we can assume that $\phi_\eta \to 0$ as $|\eta| \to \infty$, since we have already extracted the term $\epsilon l\eta$ in the phase of (2.13a). It follows that

$$-v - 2\lambda_{\mathrm{m}}l + 4k_{\mathrm{m}}\sigma = 0 \quad \text{or} \quad l = k_{\mathrm{m}}\gamma^2\,, \tag{2.19}$$

where we have used (2.12). It follows that the wavenumber correction $\epsilon^2 l$ is always positive. Next we use (2.12) and (2.19) to determine ϕ_η from (2.18),

$$\phi_\eta = -\tfrac{374}{3}\,k_{\mathrm{m}}R^2\,. \tag{2.20}$$

Finally using the expressions (2.12) in (2.16c) we get

$$a = \gamma/\sqrt{38}\,. \tag{2.21}$$

Altogether the relations (2.14), (2.17), (2.19), (2.20) and (2.21) completely determine the steady solitary wave (2.13a) in terms of the single parameter γ. Our results for the speed and wavenumber corrections are consistent with the schematic bifurcation shown in Fig. 1.

It is important to note here that $c = c_{\mathrm{m}}$ is an absolute minimum on the linear dispersion curve (Fig. 1), and hence the speed of this steady solitary wave is less than the speed of a linear oscillatory wave for all wavenumbers k. This excludes the possibility of a resonance between the solitary wave and a linear oscillatory wave which would lead to a non-local solitary wave where the solitary wave core is accompanied by a co-propagating non-decaying oscillatory tail. Such non-local solitary waves occur for (1.1) in the bifurcation from $k = 0$ (Pomeau et al., 1988, Boyd, 1991 and Grimshaw and Joshi, 1994, Karpman, 1993), and in other physical contexts. Indeed, Wai et al. (1990) have shown that the higher-order NLS (2.8) with the terms whose coefficients are μ/k, ν and ν' omitted (i.e. only the higher-order term $\epsilon\lambda' A_{\xi\xi\xi}$ is retained) has non-local solitary wave solutions due to the perturbing effect of the higher-order term. However, it is important to recognise that their result is not applicable here because the underlying resonance does not exist for the full equation (1.1). To assist in demonstrating this we carry out in Section 3 an alternative construction of the steady solitary wave solution using normal form analysis based on the concepts developed by Iooss and Kirchgässner (1990) and Dias and Iooss (1993) for capillary-gravity solitary waves. Further we note there that the steady solitary wave constructed here is oscillatory, due primarily to the phase factor $\exp(i\theta)$ in (1.3a), but the oscillations decay as $|\eta| \to \infty$ due to the localized form of the envelope $R(\eta)$ (2.16a). A plot of the steady solitary wave described to leading order by $2\,\mathrm{Re}[A\exp(\mathrm{i}\theta)]$ with A given by (2.13a) is shown in Fig. 2 for $\epsilon a = 1/\sqrt{38}$, and $1/2\sqrt{38}$. Here we have chosen the arbitrary constant of integration in (2.20) so that the solution is symmetric. Finally in this subsection we note the recent theoretical and numerical work of Champneys and Toland (1993) which suggests that the solitary wave solution (2.16a–c) of (1.1) may be only the first member of a complicated family of solitary waves, where the other members appear only when the speed $c < c_m$ by some finite amount.

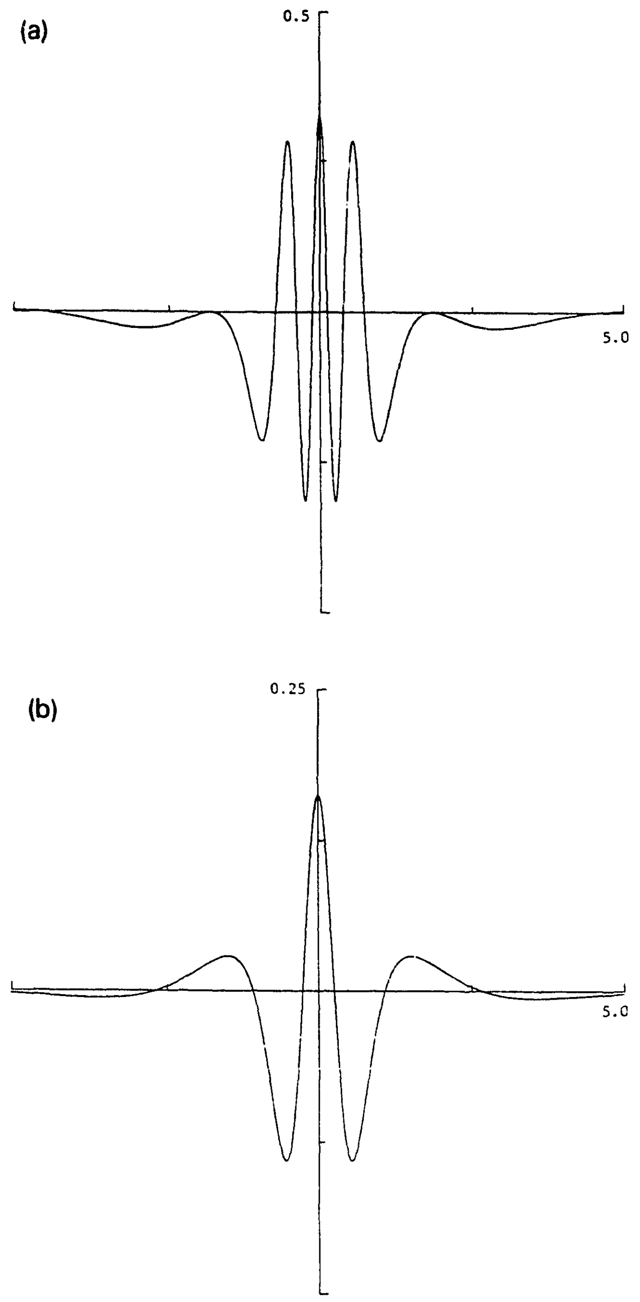

Fig. 2. Plots of the steady solitary wave solution for (a) $\epsilon a = 1/\sqrt{38}$, (b) $\epsilon = 1/2\sqrt{38}$.

(iii) *Special cases.* We have already noted that the solution (2.5) for A_2 fails when $5k^2 = 1$, while that for A_0 (2.7) fails when $5k^2 = 3$. These, and other special cases are the subject of this sub-section.

(a) *Second harmonic resonance, $5k^2 = 1$.* This arises whenever $c(k) = c(2k)$ where we recall from (1.2b) that $c(k)$ is the phase speed for wavenumbers k. The second harmonic now has the same order of magnitude as the primary harmonic, and the expansion (1.3a) is replaced by

$$u = \alpha[A(X, T)\exp(\mathrm{i}\theta) + A_2(X, T)\exp(2\mathrm{i}\theta) + \mathrm{c.c}] + \mathcal{O}(\alpha^2)\,, \tag{2.22}$$

where θ, X and T are again defined by (1.3b,c). We put

$$5k^2 = 1 + \epsilon\Delta \tag{2.23}$$

and let $\alpha = \epsilon$. Proceeding in a similar manner to the general case (i) we find that

$$A_T - \tfrac{2}{5}A_x + 6\mathrm{i}k_2 A_2 \bar{A} = 0\,, \tag{2.24a}$$

$$A_{2T} + \tfrac{4}{5}A_{2x} + \tfrac{6}{5}\mathrm{i}k_2\Delta A_2 + 6\mathrm{i}k_2 A^2 = 0\,. \tag{2.24b}$$

Here $k_2 = 5^{-1/2}$, and we note that $c_g = -\frac{2}{5}$ when $k = k_2$. Of course, the general form of these equations is typical for a second harmonic resonance (e.g. Craik, 1985).

(b) *Mean flow resonance, $5k^2 = 3$.* This arises whenever $c_g(k) = 0$ where we recall from (1.2c) that $c_g(k)$ is the group velocity for wavenumber k. In this case we replace the expansion (1.3a) with

$$u = \alpha[A(X, T)\exp(\mathrm{i}\theta) + \mathrm{c.c.}] + \alpha^{4/3}A_0(X, T) + \mathcal{O}(\alpha^{5/3})\,, \tag{2.25a}$$

$$5k^2 = 3 + \epsilon\Delta \tag{2.25b}$$

and let $\alpha = \epsilon^{3/2}$. Proceeding in a similar manner to the general case (i) we find that

$$A_T - 3ik_0 A_{\xi\xi} + 6ik_0 A_0 A = 0\,, \tag{2.26a}$$

$$A_{0\tau} - \tfrac{3}{5}\Delta A_{0\xi} + 6|A|^2_\xi = 0\,, \tag{2.26b}$$

where

$$\xi = X - \tfrac{3}{5}\Delta\tau\,. \tag{2.26c}$$

Here τ and ξ are again defined by (1.4b) where we note that $c_g = \frac{3}{5}\epsilon\Delta$. The general form of these equations is typical for a mean flow resonance (e.g. Grimshaw, 1977).

(c) *Higher-order dispersion, $10k^2 = 3$.* This arises whenever $\lambda(k) = -\frac{1}{2}d^2\omega/dk^2 = 0$ (see (2.9a)). Here we still use the expansion (1.3a) but put $\alpha = \epsilon^{3/2}$, and replace the definition of τ in (1.4b) with

$$\tau = \alpha^{4/3}T\,, \tag{2.27a}$$

while

$$10k^2 = 3 + \epsilon\Delta\,. \tag{2.27b}$$

We then proceed as in the general case (i), and obtain

$$A_\tau - ik_3\Delta A_{\xi\xi} - 2A_{\xi\xi\xi} - 120ik_3|A|^2A = 0\,. \tag{2.28}$$

Indeed this result can be easily obtained from (2.8) by noting that when $\lambda \approx 0$, the remedy is to retain the third-derivative dispersive term, and rescale accordingly. Here $k_3 = (\frac{3}{10})^{1/2}$, ξ is again defined by (1.4b) while the second harmonic term A_2 and mean term A_0 are again given by (2.5) and (2.7) respectively.

(d) *Higher-order nonlinearity, $25k^2 = 3$.* This arises whenever $\mu = 0$ (*see* (2.9c)). Here we still use the expansion (1.3a), but put $\alpha = \epsilon^{1/2}$, and replace the definition of τ in (1.4b) with

$$\tau = \alpha^2 T\,, \tag{2.29a}$$

while

$$25k^2 = 3 + \epsilon\Delta\,. \tag{2.29b}$$

We then proceed as in the general case (i) but note that it is necessary now to compute the third harmonic component A_3 as well as A_2 and A_0. Omitting details the result is

$$A_\tau + \tfrac{9}{5}ik_1A_{\xi\xi} + \tfrac{625}{12}ik_1\Delta|A|^2A + 125|A|^2A_\xi + \tfrac{375}{4}(|A|^2A_\xi - A^2\bar{A}_\xi) - \tfrac{55}{72}ik_1|25A|^4A = 0\,. \tag{2.30}$$

Apart form the new term proportional to $|A|^4A$ which arises from the third harmonic term, this result follows from (2.8) with $\mu \approx 0$ followed by appropriate rescaling. Here $k_1 = (\frac{3}{25})^{1/2}$, and ξ is again defined by (1.4b). Eq. (2.30) is a derivative NLS but is not integrable for the particular coefficients which occur here (Ablowitz and Clarkson, 1991).

3. Approach through normal form analysis

In Section 2(i) we used a standard multi-scale perturbation method to derive a higher-order (NLS equation (2.8), and then in Section 2(ii) we used this equation to construct a steady solitary wave solution of (1.1), which bifurcates from $k = k_m$ on the linear dispersion curve (see Fig. 1). Here we use a more rigorous analysis based on the reduction to a normal form to obtain the same result. This analysis is based on concepts used by Iooss and Kirchgässner (1990) and Dias and Iooss (1993) for capillary-gravity waves, and also described by Iooss and Pérouème (1993) and Iooss and Adelmeyer (1992) in a more general setting. The first step is to reduce (1.1) to an ordinary differential equation by seeking steady travelling wave solutions of the form $u = u(x - ct)$. Substitution into (1.1), and one integration gives

$$-cu + 3u^2 + u_{xx} + u_{xxxx} = 0\,. \tag{3.1}$$

Here we have set a constant of integration equal to zero, since for solitary waves we require $u \to 0$ as $|x| \to \infty$. The essence of the normal form analysis is the observation that when $k = k_m$ and

$c=c_{\rm m}$ the linearized part of (3.1) has two double eigenvalues which are non semi-simple.

The first step is to replace (3.1) by a first-order system

$$\boldsymbol{u}_x = M\boldsymbol{u} + N(\boldsymbol{u})\,. \tag{3.2}$$

Here the 4-vector $\boldsymbol{u}$ is defined by

$$\boldsymbol{u}^{\rm T} = [u_1, u_2, u_3, u_4]\,, \tag{3.3a}$$

where

$$u_1 = u\,,\quad u_2 = u_x\,,\quad u_3 = u_{xx}\,,\quad u_4 = u_{xxx}\,, \tag{3.3b}$$

while the matrix M and the nonlinear term N are given by

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -1 & 0 \end{pmatrix}. \tag{3.4a}$$

and

$$N(\boldsymbol{u})^{\rm T} = (0,0,0,-3u_1^2)\,. \tag{3.4b}$$

Then for $c=c_{\rm m}=-\frac{1}{4}$, the matrix M has the eigenvalues $\pm \mathrm{i}k_{\rm m}$, each being a double eigenvalue. For the eigenvalue $\pm\mathrm{i}k_{\rm m}$ there is just one eigenvector $\boldsymbol{v}(\bar{\boldsymbol{v}})$ where

$$\boldsymbol{v}^{\rm T} = (1, \mathrm{i}k_{\rm m}, -\tfrac{1}{2}, -\tfrac{1}{2}\mathrm{i}k_{\rm m})\,. \tag{3.5}$$

To complete the eigenspace we must find a generalised eigenvector $\boldsymbol{w}$ such that $(M-\mathrm{i}k_{\rm m}I)\boldsymbol{w}=\boldsymbol{v}$. We find that

$$\boldsymbol{w}^{\rm T} = (2\mathrm{i}k_{\rm m}, 0, \mathrm{i}k_{\rm m}, -1)\,. \tag{3.6}$$

It is useful to introduce the adjoint eigenvector $\boldsymbol{v}^*$ and generalised adjoint eigenvector $\boldsymbol{w}^*$ where $(M^{\rm T}+\mathrm{i}k_{\rm m}I)\boldsymbol{v}^*=0$ and $(M^{\rm T}+\mathrm{i}k_{\rm m}I)\boldsymbol{w}^*=\boldsymbol{v}^*$. We find that

$$\boldsymbol{v}^{*\rm T} = (1, 2\mathrm{i}k_{\rm m}, 2, 4\mathrm{i}k_{\rm m})\,, \tag{3.7a}$$

and

$$\boldsymbol{w}^{*\rm T} = (-2\mathrm{i}k_{\rm m}, 4, 4\mathrm{i}k_{\rm m}, 0)\,. \tag{3.7b}$$

In terms of the inner product $\langle \boldsymbol{a},\boldsymbol{b}\rangle = \bar{\boldsymbol{a}}^{\rm T}\cdot\boldsymbol{b}$, the adjoint eigenvector $\boldsymbol{v}^*$ is orthogonal to $\boldsymbol{v}$, $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{w}}$ while $\langle \boldsymbol{v}^*,\boldsymbol{w}\rangle = 8\mathrm{i}k_{\rm m}$, and the generalised adjoint eigenvector $\boldsymbol{w}^*$ is orthogonal to $\boldsymbol{w}$, $\bar{\boldsymbol{w}}$ and $\bar{\boldsymbol{v}}$ while $\langle \boldsymbol{w}^*,\boldsymbol{v}\rangle = 8\mathrm{i}k_{\rm m}$.

With these preliminaries, we now change variables from $\boldsymbol{u}$ to $(A,\bar{A},B,\bar{B})$ by the transformation

$$\boldsymbol{u} = A(x)\,\boldsymbol{v} + \bar{A}(x)\,\bar{\boldsymbol{v}} + B(x)\,\boldsymbol{w} + \bar{B}(x)\,\bar{\boldsymbol{w}} + \boldsymbol{\Phi}(\mu; A,\bar{A},B,\bar{B})\,, \tag{3.8a}$$

where

$$\mu = c - c_{\rm m}\,. \tag{3.8b}$$

Note that here $A(x)$ is not the same as $A(X,T)$ introduced in (1.4a) and used in Section 2, but we have preferred to use here the same notation as Iooss and Pérouème (1993) and Dias and Iooss (1993). Here $\boldsymbol{\Phi}(\mu;A,\bar{A},B,\bar{B})$ contains the higher-order terms in $A,\bar{A},B,\bar{B}$ and is chosen to produce the desired normal form. Substituting (3.8a) into (3.2) and taking the inner product with $\boldsymbol{w}^*$ and $\boldsymbol{v}^*$ gives respectively

$$A_x = \mathrm{i}k_{\rm m}A + B + f(\mu;A,\bar{A},B,\bar{B})\,, \tag{3.9a}$$

$$B_x = \mathrm{i}k_{\rm m}B + g(\mu;A,\bar{A},B,\bar{B})\,. \tag{3.9b}$$

Here f and g contain the higher-order terms and are given by

$$f(\mu;A,\bar{A},B,\bar{B}) = \langle \boldsymbol{w}^*, M\boldsymbol{\Phi}-\boldsymbol{\Phi}_x\rangle/8\mathrm{i}k_{\rm m}\,, \tag{3.10a}$$

$$g(\mu;A,\bar{A},B,\bar{B}) = \langle \boldsymbol{v}^*, M\boldsymbol{\Phi}-\boldsymbol{\Phi}_x\rangle/8\mathrm{i}k_{\rm m} - \tfrac{1}{2}\mu(A+\bar{A}+2\mathrm{i}k_{\rm m}(B-\bar{B})) + \tfrac{3}{2}(A+\bar{A}+2\mathrm{i}k_{\rm m}(B-\bar{B})+\Phi_1)^2\,. \tag{3.10b}$$

Note that the linear part of (3.9a,b) has eigenvalues λ where $\lambda^2 = -\frac{1}{2}\pm\sqrt{\mu}$ which are just the eigenvalues of the matrix M. Following Iooss and Adelmeyer (1992) (see also Iooss and Pérouème, 1993 and Dias and Iooss, 1993) we next observe that $\Phi(\mu;A,\bar{A},B,\bar{B})$ can be chosen to reduce (3.9a,b) to the normal form

$$A_x = \mathrm{i}k_{\rm m}A + B + \mathrm{i}AP(\mu;|A|^2,\tfrac{1}{2}\mathrm{i}(A\bar{B}-\bar{A}B))\,, \tag{3.11a}$$

$$B_x = \mathrm{i}k_{\rm m}B + \mathrm{i}BP(\mu;|A|^2,\tfrac{1}{2}\mathrm{i}(A\bar{B}-\bar{A}B)) + AQ(\mu;|A|^2,\tfrac{1}{2}\mathrm{i}(A\bar{B}-\bar{A}B))\,, \tag{3.11b}$$

where P and Q are real-valued functions, and are given to leading order by

$$P(\mu; |A|^2, \tfrac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B)) = p_0 u + p_1|A|^2 + \tfrac{1}{2}p_2\mathrm{i}(A\bar{B} - \bar{A}B) + \cdots, \tag{3.12a}$$

$$Q(\mu; |A|^2, \tfrac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B)) = q_0\mu + q_1|A|^2 + \tfrac{1}{2}q_2\mathrm{i}(A\bar{B} - \bar{A}B) + \cdots. \tag{3.12b}$$

It remains to calculate the coefficients p_0, p_1, p_2 and q_0, q_1, q_2. The coefficients p_0, q_0 are readily found by requiring that the eigenvalues of the linearization of (3.11a,b) must agree with the eigenvalues of M. Hence we find that

$$p_0 = -\tfrac{1}{2}k_\mathrm{m}, \quad q_0 = -\tfrac{1}{2}. \tag{3.13}$$

The coefficients p_1, p_2 and q_1, q_2 of the non-linear terms require a lengthy calculation which is described in Appendix B. The result is

$$p_1 = -\tfrac{146}{3}k_\mathrm{m}, \quad p_2 = \tfrac{832}{9}, \quad q_1 = -76, \quad q_2 = \tfrac{1204}{3}k_\mathrm{m}. \tag{3.14}$$

The system of equations (3.11a,b) has been discussed in detail by Iooss and Pérouème (1993). It is not Hamiltonian in general, but is integrable, possessing two integrals, one of these being $\frac{1}{2}\mathrm{i}(A\bar{B} - \bar{A}B) = C$. Here our interest is only in the reversible homoclinic orbits which correspond to $C = 0$. We put

$$A = R\exp(\mathrm{i}k_\mathrm{m}x + \mathrm{i}\phi), \tag{3.15a}$$

$$B = S\exp(\mathrm{i}k_\mathrm{m}x + \mathrm{i}\psi) \tag{3.15b}$$

and find that $C = -RS\sin(\phi - \psi)$. Thus $C = 0$ corresponds to $\phi - \psi = 0$, or π. Without loss of generality we choose $\phi - \psi = 0$ since the case $\phi - \psi = \pi$ can be covered by allowing S to be negative. Substituting (3.15a,b) into (3.11a,b) with $C = 0$ then gives

$$R_x = S, \tag{3.16a}$$

$$\phi_x = P(\mu; R^2, 0), \tag{3.16b}$$

$$S_x = RQ(\mu; R^2, 0). \tag{3.16c}$$

Eqs. (3.16a,c) can be combined to give

$$R_{xx} = RQ(\mu; R^2, 0), \tag{3.17a}$$

where

$$Q(\mu; R^2, 0) = q_0\mu + q_1R^2 + \cdots. \tag{3.17b}$$

Here (3.17b) follows from (3.12b) and the omitted terms are higher order in μ and R^2. Since here $q_0 < 0$ (3.13) and $q_1 < 0$ it follows from (3.17a,b) that there is a homoclinic orbit for $\mu < 0$. Retaining only the leading order terms in Q (i.e. Q is given by (3.17)) we find that

$$R = a\,\mathrm{sech}\,\gamma x, \tag{3.18a}$$

where

$$q_0\mu = \gamma^2, \tag{3.18b}$$

and

$$q_1a^2 = -2\gamma^2. \tag{3.18c}$$

Note that if R is a solution of (3.17a) so is $-R$. With R found from (3.17a) it is straightforward to obtain ϕ from (3.16b). Using the leading order expression (3.12a) for P we find that

$$\phi_x = p_0\mu + p_1R^2, \tag{3.19}$$

while S is readily found from (3.16a). Finally returning to (3.8a) and recalling (3.5) and (3.6) for $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively we find that to leading order as $\mu, a \to 0$

$$u \approx R\exp\left[\mathrm{i}k_\mathrm{m}x + \mathrm{i}\left(\phi + 2k_\mathrm{m}\frac{S}{R}\right)\right] + \text{c.c.}. \tag{3.20}$$

Here we recall that in this section x is a replacement for $x - ct$, where $c \approx c_\mathrm{m}$. Noting that the values of p_0, p_1, p_2 and q_0, q_1, q_2 are given by (3.13) and (3.14) we can now compare this result with that described by (1.4a) and (2.13a). It is readily established that there is exact agreement, where we note in particular that

$$\left(\phi + 2k_\mathrm{m}\frac{S}{R}\right)_x = p_0\mu + (p_1 + k_\mathrm{m}q_1)R^2. \tag{3.21}$$

This agreement confirms that the derivation of the steady solitary wave by the multi-scale perturbation method of Section 2(ii) is correct, and hence our interpretation of this solitary wave as a bifurcation of the NLS soliton at $k = k_m$ is valid.

Appendix A

In this appendix we give an alternative derivation of the higher-order NLS equation (2.8) using the Zakharov spectral formulation. Let us introduce the Fourier transform

$$U(\kappa, t) = \int_{-\infty}^{\infty} u(x, t) \exp(-i\kappa x)\, dx\,, \tag{A.1a}$$

so that

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\kappa, t) \exp(i\kappa x)\, d\kappa\,. \tag{A.1b}$$

Then it follows from (1.1) that

$$U_t + i\omega(\kappa)U + 3i\kappa \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\kappa_1, t)\, U(\kappa_2, t) \times \delta(\kappa - \kappa_1 - \kappa_2)\, d\kappa_1\, d\kappa_2 = 0\,. \tag{A.2}$$

Here $\omega(\kappa)$ is defined by (1.2a). Next we introduce $a(\kappa, t)$ by the transformation

$$U(\kappa, t) = a(\kappa, t) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\kappa, \kappa_1, \kappa_2)\, a(\kappa_1, t)\, a(\kappa_2, t) \times \delta(\kappa - \kappa_1 - \kappa_2)\, d\kappa_1\, d\kappa_2\,, \tag{A.3}$$

where $f(\kappa, \kappa_1, \kappa_2)$ is chosen to eliminate quadratic terms in the evolution equation for $a(\kappa, t)$. This procedure is analogous to the normal form analysis of Section 3. We find that to cubic order, on anticipating the cancellation of the quadratic terms when calculating the cubic terms,

$$\begin{aligned} &a_t + i\omega(\kappa)a \\ &-i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\kappa, \kappa_1, \kappa_2)\, [\omega(\kappa_1) + \omega(\kappa_2) - \omega(\kappa)] \\ &\times a(\kappa_1)\, a(\kappa_2)\, \delta(\kappa - \kappa_1 - \kappa_2)\, d\kappa_1\, d\kappa_2 \\ &+ 3i\kappa \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\kappa_1)\, a(\kappa_2)\, \delta(\kappa - \kappa_1 - \kappa_2)\, d\kappa_1\, d\kappa_2 \\ &+ 6i\kappa \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\kappa_1)\, a(\kappa_3)\, a(\kappa_4) \\ &\times f(\kappa_2, \kappa_3, \kappa_4)\delta(\kappa - \kappa_1 - \kappa_2) \\ &\times \delta(\kappa_2 - \kappa_3 - \kappa_4)\, d\kappa_1\, d\kappa_2\, d\kappa_3\, d\kappa_4 = 0\,, \end{aligned} \tag{A.4}$$

where here and henceforth we suppress the dependence of $a(\kappa, t)$ on t. To eliminate the quadratic terms in (A.4) we must choose

$$f(\kappa, \kappa_1, \kappa_2) = \frac{3\kappa}{\omega(\kappa_1) + \omega(\kappa_2) - \omega(\kappa)}\,, \tag{A.5}$$

and it then follows that

$$\begin{aligned} &a_t + i\omega(\kappa)a \\ &+ i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(\kappa, \kappa_1, \kappa_2, \kappa_3)\, a(\kappa_1)\, a(\kappa_2)\, a(\kappa_3) \\ &\times \delta(\kappa - \kappa_1 - \kappa_2 - \kappa_3)\, d\kappa_1\, d\kappa_2\, d\kappa_3 = 0\,. \end{aligned} \tag{A.6}$$

Here the nonlinear interaction coefficient $T(\kappa, \kappa_1, \kappa_2, \kappa_3)$ is given by

$$\begin{aligned} T(\kappa, \kappa_1, \kappa_2, \kappa_3) = 2\kappa[&f(\kappa_2 + \kappa_3, \kappa_2, \kappa_3) \\ &+ f(\kappa_3 + \kappa_1, \kappa_3, \kappa_1) \\ &+ f(\kappa_1 + \kappa_2, \kappa_1, \kappa_2)]\,, \end{aligned} \tag{A.7}$$

where we recall that $f(\kappa, \kappa_1, \kappa_2)$ is defined by (A.5)

To derive a NLS-type equation from (A.6) we assume that $a(\kappa)$ has a narrow spectrum centred around k. Letting $\kappa \to k$ in the linear terms of (A.6) it is readily seen that, with appropriate rescaling we recover exactly the linear terms of the higher-order NLS (2.8) where we note that λ, λ' are defined by (2.9a,b) respectively. To obtain the nonlinear terms we let $\kappa \to k$ in the nonlinear term in (A.6). The presence of the δ-function then requires $\kappa_1 \to -k$, $\kappa_2 \to k$, $\kappa_3 \to k$ (and the corresponding limits with a cyclic interchange of indices). The result is that the coefficients of the nonlinear terms in (2.8) are given by

$$\mu = 3 \lim T \,, \tag{A.8a}$$

$$\nu' = -3 \lim \frac{\partial T}{\partial \kappa_1} \,, \tag{A.8b}$$

$$\nu + \nu' = 3 \lim \left(\frac{\partial T}{\partial \kappa_2} + \frac{\partial T}{\partial \kappa_3} \right) , \tag{A.8c}$$

where in each case we take the limit $\kappa \to k$, $\kappa_1 \to -k$, $\kappa_2 \to k$, $\kappa_3 \to k$. Thus, for instance, we obtain

$$\mu = 18k \left(\frac{1}{c(k) - c(2k)} + \frac{2}{c_g(k)} \right) , \tag{A.9}$$

where we recall that $c(k)$ and $c_g(k)$ are defined by (1.2b,c) respectively. It is then readily shown that (A.9) agrees with (2.9c). Similarly evaluation of (A.8b,c) shows agreement with (2.9d,e).

Appendix B. Derivation of the nonlinear terms in the normal form

In this appendix we describe the determination of the coefficients p_1, p_2 and q_1, q_2 in (3.12a,b). This in turn involves the calculation of $\boldsymbol{\Phi}(\mu; A, \bar{A}, B, \bar{B})$ in the transformation (3.8a). It is sufficient for our purposes to put $\mu = 0$ and consider only the quadratic and cubic nonlinear terms in $\boldsymbol{\Phi}$. Thus we put

$$\begin{aligned}\boldsymbol{\Phi} = {} & (\boldsymbol{\alpha}_0 A^2 + \text{c.c.}) + \boldsymbol{\alpha}_1 |A|^2 + (\boldsymbol{\beta}_0 B^2 + \text{c.c.}) \\ & + \boldsymbol{\beta}_1 |B|^2 + (\boldsymbol{\gamma}_0 AB + \text{c.c.}) + (\boldsymbol{\gamma}_1 \bar{A} B + \text{c.c.}) \\ & + (\boldsymbol{\alpha}_2 A^3 + \boldsymbol{\alpha}_3 |A|^2 A + \boldsymbol{\beta}_2 B^3 + \boldsymbol{\beta}_3 |B|^2 B + \text{c.c.}) \\ & + (\boldsymbol{\gamma}_2 A^2 B + \boldsymbol{\gamma}_3 |A|^2 B + \boldsymbol{\gamma}_4 A^2 \bar{B} + \text{c.c.}) \\ & + (\boldsymbol{\gamma}_5 B^2 A + \boldsymbol{\gamma}_6 |B|^2 A + \boldsymbol{\gamma}_7 B^2 \bar{A} + \text{c.c.}) + \cdots .\end{aligned} \tag{B.1}$$

This expression is then substituted into (3.10a,b) and then we require that equations (3.9a,b) adopt the normal form (3.11a,b). Since there are no quadratic terms in the normal form, elimination of the coefficients of all quadratic terms in (3.9a,b) gives

$$M\boldsymbol{\alpha}_0 - 2\mathrm{i}k_\mathrm{m}\boldsymbol{\alpha}_0 = 3\boldsymbol{e}_4 \,, \tag{B.2a}$$

$$M\boldsymbol{\alpha}_1 = 6\boldsymbol{e}_4 \,, \tag{B.2b}$$

$$M\boldsymbol{\beta}_0 - 2\mathrm{i}k_\mathrm{m}\boldsymbol{\beta}_0 = -6\boldsymbol{e}_4 + \boldsymbol{\gamma}_0 \,, \tag{B.2c}$$

$$M\boldsymbol{\beta}_1 = 12\boldsymbol{e}_4 + \boldsymbol{\gamma}_1 + \bar{\boldsymbol{\gamma}}_1 \,, \tag{B.2d}$$

$$M\boldsymbol{\gamma}_0 - 2\mathrm{i}k_\mathrm{m}\boldsymbol{\gamma}_0 = 12\mathrm{i}k_\mathrm{m}\boldsymbol{e}_4 + 2\boldsymbol{\alpha}_0 \,, \tag{B.2e}$$

$$M\boldsymbol{\gamma}_1 = 12\mathrm{i}k_\mathrm{m}\boldsymbol{e}_4 + \boldsymbol{\alpha}_1 \,. \tag{B.2f}$$

Here $\boldsymbol{e}_4^\mathrm{T} = (0,0,0,1)$, and M is given by (3.4a) with $\mu = 0$, or $c = c_\mathrm{m} = -\frac{1}{4}$. The solution of (B.2a–f) is

$$\boldsymbol{\alpha}_0^\mathrm{T} = \tfrac{1}{3}(-4, -8\mathrm{i}k_\mathrm{m}, 8, 16\mathrm{i}k_\mathrm{m}) \,, \tag{B.3a}$$

$$\boldsymbol{\alpha}_1^\mathrm{T} = (-24, 0, 0, 0) \,, \tag{B.3b}$$

$$\boldsymbol{\beta}_0^\mathrm{T} = \tfrac{1}{9}(328, 480, -32, -400) \,, \tag{B.3c}$$

$$\boldsymbol{\beta}_1^\mathrm{T} = (144, 0, -48, 0) \,, \tag{B.3d}$$

$$\boldsymbol{\gamma}_0^\mathrm{T} = \tfrac{1}{9}(-176\mathrm{i}k_\mathrm{m}, 152, 256\mathrm{i}k_\mathrm{m}, -208) \,, \tag{B.3e}$$

$$\boldsymbol{\gamma}_1^\mathrm{T} = (-48\mathrm{i}k_\mathrm{m}, -24, 0, 0) \,. \tag{B.3f}$$

A similar procedure is followed for the cubic terms and leads to the following equations

$$q_1 = 3\langle \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1, \boldsymbol{e}_1 \rangle \,, \tag{B.4a}$$

$$8k_\mathrm{m} p_1 + \langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle = 0 \,, \tag{B.4b}$$

$$\begin{aligned} & 2k_\mathrm{m}(q_2 - 2p_1) + \langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle \\ & \quad = 12\mathrm{i}k_\mathrm{m} \langle -2\mathrm{i}k_\mathrm{m}\boldsymbol{\alpha}_1 + \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1, \boldsymbol{e}_1 \rangle \,, \end{aligned} \tag{B.4c}$$

$$4\mathrm{i}k_\mathrm{m} p_2 + 2\langle \boldsymbol{w}^*, \boldsymbol{\alpha}_3 \rangle - \langle \boldsymbol{v}^*, \boldsymbol{\gamma}_3 \rangle = 0 \,, \tag{B.4d}$$

$$-4k_\mathrm{m} q_2 + \langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle = 24\mathrm{i}k_\mathrm{m} \langle -2\mathrm{i}k_\mathrm{m}\boldsymbol{\alpha}_0 + \bar{\boldsymbol{\gamma}}_1, \boldsymbol{e}_1 \rangle \,, \tag{B.4e}$$

$$-4\mathrm{i}k_\mathrm{m} p_2 + \langle \boldsymbol{w}^*, \boldsymbol{\alpha}_3 \rangle - \langle \boldsymbol{v}^*, \boldsymbol{\gamma}_4 \rangle = 0 \,, \tag{B.4f}$$

$$\begin{aligned} & 4\mathrm{i}k_\mathrm{m} p_2 + \langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle \\ & \quad = 24\mathrm{i}k_\mathrm{m} \langle -2\mathrm{i}k_\mathrm{m}\boldsymbol{\alpha}_1 + \bar{\boldsymbol{\gamma}}_1 + \boldsymbol{\beta}_0, \boldsymbol{e}_1 \rangle \,, \end{aligned} \tag{B.4g}$$

$$-4ik_m p_2 + \langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle + 2\langle \boldsymbol{v}^*, \boldsymbol{\gamma}_4 \rangle$$
$$= 24ik_m \langle 2ik_m \boldsymbol{\gamma}_0 - 2ik_m \bar{\boldsymbol{\gamma}}_1 + \boldsymbol{\beta}_1, \boldsymbol{e}_1 \rangle . \qquad \text{(B.4h)}$$

Here $\boldsymbol{e}_1^{\mathrm{T}} = (1,0,0,0)$. These form eight equations for p_1, p_2, q_1, q_2 and $\langle \boldsymbol{v}^*, \boldsymbol{\alpha}_3 \rangle$, $\langle \boldsymbol{w}^*, \boldsymbol{\alpha}_3 \rangle$, $\langle \boldsymbol{v}^*, \boldsymbol{\gamma}_3 \rangle$ and $\langle \boldsymbol{v}^*, \boldsymbol{\gamma}_4 \rangle$. Using (B.3a–f) to evaluate the right-hand sides, they are then solved to give (3.14).

References

Ablowitz M.J. and P.A. Clarkson, 1991, Solitons, nonlinear evolution equations and inverse scattering, London Mathematical Society Lecture Notes, 149 (Cambridge Univ. Press) 516 pp.

Akylas, T.R., 1993, Envelope solitons with stationary crests, Phys. Fluids 5A, 789–791.

Benjamin, T.B., 1992, A new kind of solitary wave, J. Fluid Mech. 245, 401–411.

Boyd, J.P., 1991, Weakly non-local solitons for capillary-gravity waves: fifth-degree Korteweg–de Vries equation, Physica D 48, 129–146.

Champneys A.R. and J.F. Toland, 1993, Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems, Nonlinearity 6, 665–721.

Craik, A.D.D., 1985, Wave interactions and fluid flows (Cambridge Univ. Press) 322 pp.

Dias F. and G. Iooss, 1993, Capillary-gravity solitary waves with damped oscillations, Physica D 65, 399–423.

Grimshaw, R.H.J., 1977, The modulation of an internal gravity-wave packet and the resonance with the mean motion, Stud. Appl. Math. 56, 241–266.

Grimshaw R. and N. Joshi, 1994, Weakly non-local solitary waves in a singularly perturbed Korteweg–de Vries equation, SIAM J. Appl. Maths., submitted.

Hunter J.K. and J. Scheurle, 1988, Existence of perturbed solitary wave solutions to a model equation for water waves, Physica D 32, 253–268.

Hunter J.K. and J.-M. Vanden Broeck, 1983, Solitary and periodic gravity-capillary waves of finite amplitude, J. Fluid Mech. 134, 205–219.

Iooss G. and M. Adelmeyer, 1992, Topics in bifurcation theory and application, Advanced Series in Dynamical Systems, 3 (World Scientific, Singapore).

Iooss G. and K. Kirchgässner, 1990, Bifurcation d'ondes solitaires en présence d'une faible tension superficielle, C.R. Acad. Sci. Paris, 311, 265–268.

Iooss G and M.-C. Pérouème, 1993, Perturbed homoclinic solutions in reversible 1:1 resonance vector fields, J. Diff. Eq. 102, 62–88.

Kawahara, T., 1972, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan 33, 260–264.

Karpman, V.I., 1993, Radiation by solitons due to higher-order dispersion, Phys. Rev. E 4, 2073–2082.

Longuet-Higgins, M.S., 1989, Capillary-gravity waves of solitary type on deep water, J. Fluid Mech. 200, 451–470.

Longuet-Higgins, M.S., 1993, Capillary-gravity waves of solitary type and envelope solitons on deep water, J. Fluid Mech. 252, 703–711.

Newell, A.C., 1985, Solitons in mathematics and physics, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 244 pp.

Pomeau, Y., A. Ramani and B. Grammaticos, 1988, Structural stability of the Korteweg–de Vries solitons under perturbation, Physica D 3, 127–134.

Sasa N. and J. Satsuma, 1993, New-type of soliton solutions for a higher-order nonlinear Schrödinger equation, J. Phys. Soc. Japan 60, 409–417.

Vanden Broeck J.-M. and F. Dias, 1992, Gravity-capillary solitary waves in water of infinite depth and related free-surface flows, J. Fluid Mech. 240, 549–557.

Wai, P.K.A., H.H. Chen and Y.C. Lee, 1990, Radiations by "solitons" at the zero group-dispersion wavelength of single-mode optical fibers, Phys. Rev. A. 41, 426–439.

Zufiria, J.A., 1987, Symmetry breaking in periodic and solitary gravity-capillary waves on water of finite depth, J. Fluid Mech. 184, 183–206.