Amplification of Nonlinear Waves in the Medium with Temporal Fluctuations

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The nonlinear wave propagation in random media seems to be very important physical phenomenon having a long and rich history. The scope of this paper is to highlight the main ideas of the new approach to this problem and to give an up-to-date view of the theory. The paper is organized in the following way. We start with the description and discussion of the so-called "mean field method" /1-3/. Then in Section 2 an exactly solvable model describing nonlinear wave propagation in the medium with fluctuating parameters /4/ is considered in order to demonstrate that the "mean field method" is correct. In Section 3 a specific asymptotic procedure of obtaining the equations for a mean wave form /5/ is presented. As an example, in Section 4 the description of the effect of nonlinear wave amplification in the medium with temporal fluctuations is given. In this case the amplification takes the form of an explosive instability.

1. THE MEAN FIELD METHOD

Let us consider briefly the mean field method (see /1-3/). We start with the following basic system taken in the operator form

$$Lu = \varepsilon\alpha(t)Mu + \varepsilon^2 N\{u\}, \quad (1)$$

where L and M are the linear deterministic operators, N is the nonlinear operator, α is the random function with a zero mean value ($\langle\alpha\rangle = 0$) and ε is the small parameter.

Let us present the wave field as the sum of the mean and fluctuating components

$$u = \bar{u} + u', \quad \langle u\rangle = \bar{u}. \quad (2)$$

Then the initial system (1) can be splitted into two different parts, Indeed: for the mean wave field

$$L\bar{u} = \varepsilon\langle\alpha Mu'\rangle + \varepsilon^2 N\{\bar{u}\} + \varepsilon^2\langle N\{u'\}\rangle, \quad (3)$$

and for the fluctuating component

$$Lu' = \varepsilon\alpha M\bar{u} + \varepsilon^2[\alpha Mu' - \langle\alpha Mu'\rangle] + \varepsilon^2[N\{u'\} - \langle N\{u'\}\rangle]. \quad (4)$$

As it follows from system (4), the scattered field is proportional to the small parameter ε. Therefore, two approximations are used:
- the first is the Bourret' approximation in which multiple scattering

is neglected (the terms in the first square brackets in (4) are neglected);
- the second is the Howe approximation in which nonlinearity of the scattered field is supposed to be neglible (the terms in the second square brackets in (4) and the last term in (3) are neglected).

After utilizing both these approximations (suppose for a while that they both are valid), system (4) becomes trivial and can be integrated easily. This approach permits to close the mean field Eq. (3). Finally, the governing equation for the mean field is as follows:

$$L\bar{u} = \varepsilon^2 \langle \alpha M L^{-1} \alpha M \rangle \bar{u} + \varepsilon^2 N\{\bar{u}\}. \tag{5}$$

The main feature of this equation is that it includes the nonlinear effects and linear scattering as the additive terms. So this equation can be obtained by the combination of the results of the linear random wave theory on the one hand and the nonlinear wave theory in deterministic media on the other hand. Equation (5) leads to the correct results in the linear theory. It was a common opinion that such an approach can be applied to the nonlinear as well as to linear problems.

2. EXACTLY SOLVABLE MODEL OF NONLINEAR WAVE SCATTERING

In fact, the situation is more complicated. In /4/ an exactly solvable nonlinear model for the wave scattering in random media has been analysed. This model is based on the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \alpha(t)\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{6}$$

where α is the random function. Substitution

$$y = x - \int \alpha(t)\,dt \tag{7}$$

permits to transform the stochastic equation (6) into the deterministic Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} + \frac{\partial^3 u}{\partial y^3} = 0. \tag{8}$$

The solutions of this equation are well known. In the result we can obtain the solution of the initial stochastic equation (6) in the form of the deterministic function with a stochastic argument

$$u(x,t) = U[x - \int \alpha(t)\,dt, t]. \tag{9}$$

On the basis of (9) all moments of function u, including the mean field, can be easily calculated.

However, the mean field obtained by such a method does not coincide with the solution of mean field equation (5) which follows from the

Howe approximation. Equation (5) in our case takes the form of the Korteweg-de Vries-Burgers equation (for details see /4/)

$$\frac{\partial\langle u\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial^3\langle u\rangle}{\partial x^3} - \delta\frac{\partial^2\langle u\rangle}{\partial x^2} = 0. \tag{10}$$

The reason of such differences between the solutions is associated with phase fluctuations of a wave in the medium with slowly fluctuating parameters (in "the statistical language" it means incorrect description of the "forward-scattering"). For the sake of simplicity let us consider field realizations with different (but constant within each realization) wave propagation velocity α. It is evident that the wave impulses run off in different realizations because of the difference between propagation velocities. So the mean field decays while the wave energy and the form of the pulse are preserved.

The important fact is that in such a case the magnitude of the scattered field appears not to be small in comparison with the mean field. Consequently, we must treat the scattered field as a nonlinear one. All this leads to the situation where the conditions of validity of the mean field equation (5) are not met.

Moreover, we consider that the mean field is not an adequate characteristic of the process. The form of an impulse is much more interesting and thus it is necessary to obtain the corresponding equations for the mean form of the wave and not for the mean field.

3. THE MEAN WAVE FORM METHOD

We use the asymptotic method to find the equation for the mean wave form /5/. Let us consider a model wave equation with quadratic nonlinearity and temporal fluctuation

$$\frac{\partial^2 u}{\partial t^2} - \quad 1 + \varepsilon\alpha(t) \quad {}^{2}\, \frac{\partial^2 u}{\partial x^2} = \varepsilon^2\frac{\partial^2 u^2}{\partial x^2} \,. \tag{11}$$

The main idea of our approach is to eliminate phase fluctuations by means of transition to the reference system, which moves with unfixed fluctuating speed $c(t)$

$$x' = x - \int c(t)\,dt, \quad t' = t. \tag{12}$$

In terms of new variables, Eq.(11) has the form

$$\frac{\partial^2 u}{\partial t^2} - 2c\frac{\partial^2 u}{\partial t\partial x} - \frac{dc}{dt}\frac{\partial u}{\partial x} + \quad c^2 - (1+\varepsilon\alpha)^2\, \frac{\partial^2 u}{\partial x^2} = 0 \tag{13}$$

where primes are omitted.

To analyse (13) we use the asymptotic method of multiple scales. After introducing a hierarchy of fast t and slow ($T=\varepsilon^2 t$, $T_1=\varepsilon^3 t$, etc.)

times, we present the solution of (13) in the form of an asymptotic expansion

$$u = u^{(0)}(x,T,\ldots) + \varepsilon u^{(1)}(x,t,T,\ldots) + \ldots ,$$

$$c = 1 + \varepsilon c^{(1)}(t,T,\ldots) + \ldots . \tag{14}$$

The main terms of expansion (14) describe the wave propagating to the right with a near-sonic speed. The zeroth order equation obtained by the perturbation theory is satisfied trivially. In the first order we obtain the equation for the scattered field

$$\frac{\partial^2 u^{(1)}}{\partial t^2} - 2\frac{\partial^2 u^{(1)}}{\partial t\partial x} = \frac{\partial c^{(1)}}{\partial t}\frac{\partial u^{(0)}}{\partial x} + 2(\alpha - c^{(1)})\frac{\partial^2 u^{(0)}}{\partial x^2}. \tag{15}$$

One can easily see that the scattered field is not increasing under the following condition

$$c^{(1)} = \alpha. \tag{16}$$

This condition eliminates the phase fluctuations. Then the integration of Eq.(15) yields

$$u^{(1)} = \int_0^\infty \frac{\partial u^{(0)}}{\partial x}(x+2\tau,T)\alpha(t-\tau)d\tau. \tag{17}$$

In the second order we have the inhomogeneous linear equation for $u^{(2)}$

$$\frac{\partial}{\partial t}\left[\frac{\partial u^{(2)}}{\partial t} - 2\frac{\partial u^{(2)}}{\partial x}\right] = F(x,t,T), \tag{18a}$$

$$F = 2\frac{\partial^2 u^{(0)}}{\partial T\partial x} + \frac{\partial^2 u^{(0)2}}{\partial x^2} + 2\alpha\frac{\partial^2 u^{(1)}}{\partial t\partial x} + \frac{\partial \alpha}{\partial t}\frac{\partial u^{(1)}}{\partial x} + 2c^{(2)}\frac{\partial^2 u^{(0)}}{\partial x^2} +$$

$$+ \frac{\partial c^{(2)}}{\partial t}\frac{\partial u^{(0)}}{\partial x} . \tag{18b}$$

One can see that $u^{(2)}$ is not increasing if the mean value of F is equal to zero

$$\langle F\rangle = \lim_{\Delta\to\infty}\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} F(x,t,T)dt = 0. \tag{19}$$

After substituting F to (19), we finally find the desired equation for the slow evolution of the nonlinear field in the random medium. It has the following form:

$$\frac{\partial u}{\partial T} + \frac{\sigma^2}{2}\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \int_0^\infty \frac{\partial^2 u}{\partial x^2}(x+2\tau,T)W(\tau)d\tau = 0, \tag{20}$$

$$W(\tau) = \langle\alpha(t)\alpha(t+\tau)\rangle, \quad \sigma^2 = W(0),$$

where index "zero" is omitted and W is the correlation function of the fluctuations.

Equation (20) is the main result of our approach. Its coefficients are determined quantities - this property is the main advantage of Eq. (20) in comparison with the initial equation. The field averaging is performed in the reference system moving at unfixed speed and following the fluctuations of the wave phase. Therefore, function u describes the mean form of a wave rather than the mean field.

Note also that the averaging over fast time arises as a natural condition of the absence of secular terms in asymptotic expansion. In this sense it differs considerably from the averaging over the ensemble of realizations usually used in this kind of problems. Besides this, the averaging over fast time fits better the experimental situation where just time averaging is typical.

4. AMPLIFICATION OF NONLINEAR WAVES IN MEDIUM WITH TEMPORAL FLUCTUATIONS

We apply this approach to investigate the effects of wave amplification in the medium with temporal fluctuations. In linear approximation we assume that u is a monochromatic wave which leads to the dispersion relation

$$\omega = \frac{\sigma^2}{2}k - \frac{ik^2}{2}\phi(2k), \tag{21}$$

where ϕ is the Fourier-spectrum of fluctuations. Now we can derive two important conclusions:

- if W decreases monotonically with the argument increasing (Re $\phi > 0$), the dispersion relation is of an active character, i.e the imaginary part of the frequency is negative (Im $\omega < 0$). This property corresponds to the parametrical energy transformation from random fluctuations to the coherent wave field;
- amplification of the wave depends upon the amplitude of the Fourier-component of fluctuations at double frequency. Such a situation is very similar to the Bragg scattering of waves over spatial inhomogeneities. The only difference is the sign Im ω, since the scattering over spatial irregularities always leads to damping. In our case of temporal fluctuations, scattering may lead to amplification.

Let us return now to the mean wave form equation and discuss the limiting cases of high- and low-frequency fluctuations. In the first case the mean wave form equation reduces to the Burgers equation with negative viscosity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \sigma\frac{\partial^2 u}{\partial x^2} = 0, \quad \delta = \int_0^\infty W(\tau)d\tau > 0. \tag{22}$$

This equation can be solved exactly by the means of the Hopf transformation. The main result of solving (22) is the "explosion" for the arbitrary Cauchy data during a finite time /6/. For example, a quasi-sinusoidal wave with a small Reynolds number ($Re = u_0/\delta k$)

$$u(x,t) = (u_0 e^{\delta k^2 t} \sin kx)(1 + \frac{u_0}{2\delta k} e^{\delta k^2 t} \cos kx)^{-1} \tag{23}$$

explodes during the finite time

$$T_{exp} = \frac{1}{\delta k^2} \ln \frac{u_0}{2\delta k} . \tag{24}$$

Let us consider a case of low-frequency fluctuations. Then the mean wave form equation reduces to the Ostrovsky equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + au = 0, \quad a = \frac{1}{8}\frac{d^2 W}{d\tau^2}(0). \tag{25}$$

This equation is conservative and corresponds to the adiabatic approximation. So the wave amplification is absent. Note that OSTROVSKY has obtained (25) for internal waves in a rotating fluid of a constant depth and he has found some interesting solutions of (25) including waves with pointed crests /7/.

In conclusion we should like to note that our approach gives the possibility to eliminate phase fluctuations, if the motion is one-dimensional. The cases of two- and three-dimensional fluctuations need special investigation.

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