Beta-Induced Translation of Strong Isolated Eddies

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ABSTRACT

Strong eddies on the β plane are considered within the framework of the one-layer primitive equations. The attention is focused on calculating the β-induced changes to the spatial structure of the eddy, as well as the speed of its translation. In contrast to the earlier studies, the results of this paper are valid for eddies with Ro $\sim$ 1 and are applicable to both lenses and eddies in a layer of nonzero average depth. It is demonstrated that no steady solution exists for eddies translating at a speed inside the Rossby-wave speed range, reflecting that such eddies must radiate and lose energy. In addition to the speed restriction, steadily translating eddies must satisfy a certain constraint imposed on their far-field asymptotics.

1. Introduction

Eddies play an important role in oceanic and atmospheric circulations. Examples range from several types of oceanic eddies (Gulf Stream rings, meddies, etc.) to tropical typhoons and the Great Red Spot of Jupiter. In addition to their obvious practical importance, eddies have a strong theoretical appeal: radial symmetry of an isolated vortex gave rise to some of the sleekest results in fluid dynamics.

Theoretical interest in oceanic eddies was mainly prompted by the pioneering work of Nof (1981), who used an ingeniously simple method to calculate the translation speed of a lens (i.e., eddy, the depth profile of which vanishes at a certain distance from the center) on the β plane. Nof's analysis, however, did not include a calculation of the β-induced correction to the velocity field and was inapplicable to eddies in a layer of nonzero average depth. The former shortcoming was partly overcome by Killworth (1983), who calculated the β-induced correction to a radially symmetric lens with quadratic depth profile; whereas the latter shortcoming turned out to be more of a problem. In contrast to the lens case, the layer of nonzero depth "supports" Rossby waves—hence, some eddies (those with translation speed lying inside the Rossby-wave speed range) must radiate and lose energy. However, the nonzero-depth analog of Nof's (1981) lens formula predicts *steady* translation for both radiating and nonradiating eddies!

It should also be mentioned that investigation of this problem was hampered by the complexity of the primitive equations. Accordingly, Nycander and Sutyrin (1992) considered eddies with Ro $\ll$ 1. Using a relatively simple geostrophic equation, they constructed a radially symmetric solution given by an arbitrary function inside a circle and the modified Bessel function on the outside. Remarkably, this steady solution is bounded in space only for nonradiating eddies, indicating that the spatial structure of the eddy is crucial.

The present paper generalizes and extends the aforementioned results, providing a complete description of the dynamics of eddies on the β plane. The only assumption we shall use is that the β effect is weak and the translation speed of the eddy is much less than the velocity in its core. No assumption about the Rossby number is implied. A simple asymptotic method is proposed for calculating the translation speed of the eddy and its spatial structure (sections 3 and 4). The method is valid where the velocity is much greater than the translation speed, so the periphery of the eddy (where the velocity is small) is considered separately in section 5.

2. Formulation

Consider a 1.5-layer fluid of average depth H_0 on the β plane. Introducing the Coriolis parameter f, the reduced acceleration due to gravity g', and the deformation radius $R_d = (g'H_0)^{1/2}/f$, one can define the following nondimensional variables:

$$x = x_*/R_d, \quad y = y_*/R_d, \quad t = ft_*,$$
$$u = u_*/(fR_d), \quad v = v_*/(fR_d), \quad h = h_*/H_0, \quad (2.1)$$

where x_*, y_*, t_*, u_*, v_*, and h_* are the dimensional spatial coordinates, time, velocities, and depth of the

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fluid, respectively. The standard primitive equations then become

$$\left.\begin{aligned} u_t + uu_x + vu_y + h_x &= (1 + \alpha y)v \\ v_t + uv_x + vv_y + h_y &= -(1 + \alpha y)u \\ h_t + (uh)_x + (vh)_y &= 0 \end{aligned}\right\}, \quad (2.2)$$

where

$$\alpha = \frac{R_d}{R_e}\cot\phi$$

represents the β effect (R_e is the earth's radius, ϕ is the latitude). Bearing in mind that we are concerned with solutions steadily translating in the zonal direction, let us change to a comoving reference frame

$$\tilde{x} = x - ct, \quad \tilde{y} = y, \quad \tilde{t} = t,$$

where c is the speed of translation. Substituting the new variables into (2.2) and putting $\partial/\partial\tilde{t} = 0$, we obtain (tildas omitted)

$$\left.\begin{aligned} -cu_x + uu_x + vu_y + h_x &= (1 + \alpha y)v \\ -cv_x + uv_x + vv_y + h_y &= -(1 + \alpha y)u \\ -ch_x + (uh)_x + (vh)_y &= 0 \end{aligned}\right\}. \quad (2.3)$$

Next we introduce polar coordinates (r, θ)

$$x = r\cos\theta, \quad y = r\sin\theta$$

and velocities (U, V)

$$u = U\cos\theta - V\sin\theta, \quad v = U\sin\theta + V\cos\theta.$$

Rewriting (2.3) in terms of (r, θ) and (U, V), we obtain

$$(U - c\cos\theta)U_r + \frac{1}{r}(V + c\sin\theta)(U_\theta - V) + h_r = (1 + \alpha r\sin\theta)V \quad (2.4a)$$

$$(U - c\cos\theta)V_r + \frac{1}{r}(V + c\sin\theta)(V_\theta + U) + \frac{1}{r}h_\theta = -(1 + \alpha r\sin\theta)U \quad (2.4b)$$

$$[(U - c\cos\theta)rh]_r + [(V + c\sin\theta)h]_\theta = 0. \quad (2.4c)$$

Regularity of the solution at the origin implies that

$$U_\theta - V = 0, \quad V_\theta + U = 0,$$
$$h_\theta = 0 \quad \text{at} \quad r = 0. \quad (2.5)$$

We are concerned with eddies in a layer of nonzero average depth, and hence require that

$$U, V \to 0, \quad h \to 1 \quad \text{as} \quad r \to \infty,$$

which dimensionally means that the average (equilibrium) depth of the fluid is H_0 [see (2.1)]. The lens analog of this condition

$$h \to 0 \quad \text{as} \quad r \to \infty$$

will not be considered in this paper. We shall only remark that all results obtained here for nonzero-depth eddies can be readily generalized for lenses.

Assume that the β effect is weak and the speed of β-induced translation of the eddy is much smaller than the typical velocity within its core (of radius r_c). In this case, the $O(\alpha, c)$ terms in (2.4) can be treated as perturbations. Although this approximation fails at the eddy's periphery (where the velocity is of the order of, or smaller than, the translation speed—see Fig. 1), it will enable us to calculate a number of *global* characteristics of the eddy. The peripheral region will be considered later in section 5. Note that, until then, the term "infinity" will mean "the intermediate region where $r \gg r_c$, $|V(r)| \gg |c|$" (see Fig. 1).

Omitting the formal scaling (which is straightforward here), we simply assume that, in the core and intermediate region,

$$\alpha \ll 1$$

and expand U, V, h and c as follows:

$$\left.\begin{aligned} U(r, \theta) &= \alpha U'(r, \theta) + O(\alpha^2) \\ V(r, \theta) &= V(r) + \alpha V'(r, \theta) + O(\alpha^2) \\ h(r, \theta) &= h(r) + \alpha h'(r, \theta) + O(\alpha^2) \\ c &= \alpha c' + O(\alpha^2) \end{aligned}\right\}. \quad (2.6)$$

The leading-order terms in the above solution describe a stationary cyclostrophically balanced eddy on the f plane:

$$U = 0, \quad c = 0;$$
$$h_r = V + \frac{1}{r}V^2. \quad (2.7)$$

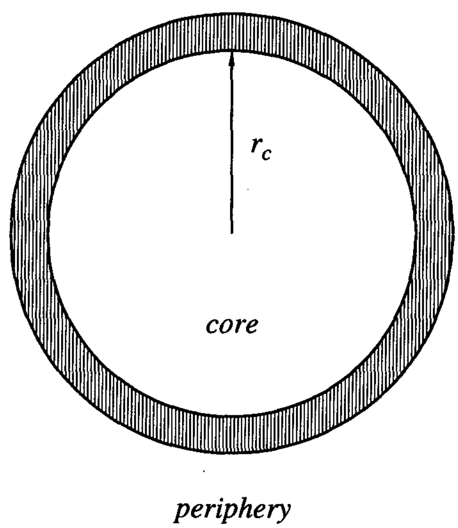

FIG. 1. The asymptotic "map" of the eddy: (i) the core ($r \lesssim r_c$, $|V| \gg |c|$); (ii) the intermediate region, shaded ($r \gg r_c$, $|V| \gg |c|$); and (iii) the periphery ($r \gg r_c$, $|V| \lesssim |c|$).

In what follows, we shall assume that $V(r)$ is analytic at $r = 0$, which implies that

$$h_r(0) = 0, \quad h_{rr}(0) \neq 0.$$

We shall also assume that $V(r)$ decays exponentially as $r \to \infty$, which will be justified in sections 4 and 5.

Substituting (2.6) into (2.4), we omit the $O(\alpha^2)$ terms

$$\frac{1}{r} V U'_\theta - \frac{1}{r}[(V' + c' \sin\theta)V + VV'] + h'_r = r \sin\theta V + V'$$

$$(U' - c' \cos\theta)V_r + \frac{1}{r} V V'_\theta + \frac{1}{r} V U' + \frac{1}{r} h'_\theta = -U'$$

$$[(U' - c' \cos\theta) rh]_r + [(V' + c' \sin\theta)h + Vh']_\theta = 0.$$

This system can be solved by the substitution

$$U' = \Phi(r)\cos\theta, \quad V' = \Psi(r)\sin\theta, \quad h' = \Gamma(r)\sin\theta, \tag{2.8}$$

which yields

$$-\frac{1}{r} V\Phi - \frac{1}{r}[(\Psi + c')V + V\Psi] + \Gamma_r = rV + \Psi \tag{2.9a}$$

$$(\Phi - c')V_r + \frac{1}{r} V\Psi + \frac{1}{r} V\Phi + \frac{1}{r}\Gamma = -\Phi \tag{2.9b}$$

$$[(\Phi - c')rh]_r + (\Psi + c')h + V\Gamma = 0. \tag{2.9c}$$

Equations (2.9) should be supplemented by the boundary conditions resulting from substitution of (2.5) into (2.8):

$$\Psi + \Phi = 0, \quad \Gamma = 0 \quad \text{at} \quad r = 0. \tag{2.10}$$

We shall also require that the energy of the eddy be finite:

$$\Gamma = o(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{2.11a}$$

It turns out that condition (2.11a) also restricts Φ. Indeed, taking the limit $r \to \infty$ in (2.9b),

$$-c'V_r + \frac{1}{r}\Gamma \to -\Phi \quad \text{as} \quad r \to \infty$$

and recalling that $V(r)$ is an exponentially decaying function, we obtain

$$\Phi = o(r^{-2}) \quad \text{as} \quad r \to \infty. \tag{2.11b}$$

Equations (2.9)–(2.11) form an eigenvalue problem for c'. The lens analog of (2.9)–(2.11) was solved by Killworth (1983) for the particular case

$$h = \begin{cases} 1 - (r/r_c)^2 & \text{if} \quad r \leq r_c \\ 0 & \text{if} \quad r > r_c. \end{cases}$$

In the next section of this paper, we shall present the exact solution for *arbitrary* $h(r)$ (nonmathematically minded readers are advised to jump to the beginning of section 4).

3. Solution of (2.9)–(2.11)

Our solution is based on the fact that the homogeneous equivalent of (2.9) is satisfied by

$$\Psi = V_r, \quad \Phi = -\frac{1}{r} V, \quad \Gamma = h_r,$$

which describes an infinitesimal shift of a stationary ($c' = 0$) eddy on the f plane along the x axis.

First, we combine equations (2.9a,b),

$$r[V \times (2.9\text{a}) + (r + 2V) \times (2.9\text{b})],$$

$$rh \times (2.9\text{b}) - V \times (2.9\text{c})$$

and eliminate Ψ:

$$[(r + 2V)(r + rV_r + V) - V^2]\Phi + (r + 2V)\Gamma + rV\Gamma_r = c'[r^2V_r + (rV^2)_r] + r^2V^2 \tag{3.1a}$$

$$h(r + rV_r + V)\Phi - V(hr\Phi)_r + (h - V^2)\Gamma = c'r(hV_r - Vh_r). \tag{3.1b}$$

Substituting $r = 0, \infty$ into (2.9) and combining the resulting equalities with (2.10)–(2.11), we eliminate Ψ from the boundary conditions:

$$\Gamma_r + (1 + V_r)\Phi = c'V_r, \quad \Gamma = 0 \quad \text{at} \quad r = 0; \tag{3.2a}$$

$$\Phi = o(r^{-2}), \quad \Gamma = o(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{3.2b}$$

Next, Φ and Γ will be replaced by $\tilde{\Phi}$ and $\tilde{\Gamma}$, such that

$$\Phi = -\frac{1}{r} V\tilde{\Phi}, \quad \Gamma = h_r\tilde{\Gamma}. \tag{3.3}$$

Substitution of (3.3) into (3.1) yields

$$V^2(r + V)\tilde{\Gamma}_r + A(\tilde{\Gamma} - \tilde{\Phi}) = F \tag{3.4a}$$

$$hV^2\tilde{\Phi}_r + B(\tilde{\Gamma} - \tilde{\Phi}) = G, \tag{3.4b}$$

where

$$\left.\begin{aligned} A &= V\left(r + 3V + \frac{1}{r}V^2 + rV_r + 2VV_r\right) \\ B &= (h - V^2)h_r \\ F &= c'[r^2V_r + (rV^2)_r] + r^2V^2 \\ G &= c'r(hV_r - Vh_r) \end{aligned}\right\}. \tag{3.5}$$

The boundary conditions for $\tilde{\Phi}$ and $\tilde{\Gamma}$ can be obtained through substitution of (3.3) into (3.2):

$$(1 + V_r)(\tilde{\Gamma} - \tilde{\Phi}) \to c', \quad r\tilde{\Gamma} \to 0 \quad \text{as} \quad r \to 0; \tag{3.6a}$$

$$V\tilde{\Phi} = o(r^{-1}), \quad V\tilde{\Gamma} \to o(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{3.6b}$$

It has been used in (3.6a) that

$$h_{rr}(0) = V_r(0)[1 + V_r(0)],$$

which, in turn, follows from (2.7).

The form of (3.4) indicates that we can reduce this system to a single equation for $\tilde{\Gamma} - \tilde{\Phi}$. Indeed, combining (3.4a) with (3.4b),

$$(3.4\text{a}) \times h - (3.4\text{b}) \times (r + V),$$

we obtain

$$rhh_r(\tilde{\Gamma} - \tilde{\Phi})_r + (rhh_r)_r(\tilde{\Gamma} - \tilde{\Phi}) = c'[(rhV)_r + r^2h_r] + r^2hV. \tag{3.7}$$

It is now convenient to introduce χ such that

$$\tilde{\Gamma} - \tilde{\Phi} = \frac{\chi}{rhh_r}. \tag{3.8}$$

Substitution of (3.8) into (3.6)–(3.7) yields

$$\chi_r = c'[(rhV)_r + r^2h_r] + r^2hV \tag{3.9}$$

$$\chi(0) = 0 \tag{3.10a}$$

$$\chi(\infty) = 0. \tag{3.10b}$$

Integrating (3.9) with respect to r over $(0, \infty)$, we see that (3.9) is consistent with (3.10) only if

$$c' = -\frac{\int_0^\infty r^2hVdr}{\int_0^\infty r^2h_rdr}. \tag{3.11}$$

Although this formula has been derived for eddies in a layer of nonzero average depth, it coincides exactly with its lens counterpart (Nof 1981, see also Killworth 1983). Formula (3.11) implies that the eddy has a nonzero net angular momentum:

$$\int_0^\infty r^2h_rdr \neq 0,$$

which is not a severe restriction, as the "no net angular momentum theorem" (Flierl et al. 1983) is valid only for fluids with rigid lid.

4. Spatial structure of the eddy

To illustrate the need of a closer look at the spatial structure of the eddy, we replace V in expression (3.11) by $(h_r - V^2/r)$ [see (2.7)] and integrate by parts:

$$c' = -1 - \frac{\frac{1}{2}\int_0^\infty r[(h-1)^2 + hV^2]dr}{\int_0^\infty r(h-1)dr}. \tag{4.1}$$

If the integral in the denominator of (4.1) (mass anomaly) is negative, c' may lie within $[-1, 0]$. Dimensionally, this means that the speed of an eddy (cyclone) may lie inside the Rossby-wave speed range. Clearly, such an eddy must radiate and lose energy, but formula (4.1) still predicts *steady* translation!

In order to resolve the paradox, we should examine the convergence of the integrals in (4.1). The question to ask here is: Whether (3.1) *guarantees* the *decay* of β-induced corrections to the *eddy's profile* as $r \to \infty$? Indeed, (4.1) has been derived as a condition necessary for the decay of a certain *linear combination* of the depth- and velocity-corrections, not the *corrections themselves*. To ensure that both h- and V-fields decay at infinity, we should return to the original functions Φ and Γ. Substituting (3.8) into (3.4):

$$\tilde{\Gamma}_r = \frac{F - A\dfrac{\chi}{rhh_r}}{V^2(r+V)}, \quad \tilde{\Phi}_r = \frac{G - B\dfrac{\chi}{rhh_r}}{hV^2},$$

we integrate these equalities and then rewrite them using (3.3) and (3.5):

$$\Phi = \frac{1}{r}V\int\left\{\frac{c'r(hV_r - Vh_r)}{hV^2} - \frac{(h - V^2)\left(V + \dfrac{1}{r}V^2\right)}{hV^2}\frac{\chi}{rhh_r}\right\}dr \tag{4.2a}$$

$$\Gamma = h_r\int\left\{\frac{c'[r^2V_r + (rV^2)_r] + r^2V^2}{V^2(r+V)} - \frac{V\left(r + 3V + \dfrac{1}{r}V^2 + rV_r + 2VV_r\right)}{V^2(r+V)}\frac{\chi}{rhh_r}\right\}dr. \tag{4.2b}$$

[Recall that $\chi(r)$ is determined by (3.9)–(3.10).] Taking now the limit $r \to \infty$ and bearing in mind that V and $(h - 1)$ decay exponentially, we expand (4.2) in powers of V and keep the leading-order terms only:

$$\Phi \to \frac{1}{r} V \int \frac{c' r^2 V_r - \chi}{r V^2} dr,$$

$$\Gamma \to V \int \frac{c' r^2 V_r - \chi}{r V^2} dr \quad \text{as} \quad r \to \infty.$$

Generally speaking, the first terms in the above integrals prevent Φ from decaying and even cause a linear growth in Γ—unless, of course, they are cancelled by χ:

$$\chi \to c' r^2 V_r \quad \text{as} \quad r \to \infty. \tag{4.3}$$

Differentiating (4.3) [smoothness of $\chi(r)$ and $V(r)$ implied] and comparing it with the leading-order terms of (3.9), we have

$$c'(r^2 V_r)_r \to c'[(rV)_r + r^2 V] + r^2 V \quad \text{as} \quad r \to \infty,$$

which can be reduced to Bessel's equations:

$$V_{rr} + \frac{1}{r} V_r - \left(\gamma^2 + \frac{1}{r^2}\right) V \to 0 \quad \text{as} \quad r \to \infty, \tag{4.4a}$$

where

$$\gamma^2 = \frac{c' + 1}{c'}. \tag{4.4b}$$

Thus, the boundedness condition of the β-induced correction constrains the far-field asymptotics of the eddy. It also restricts the allowable values of c': as V decays sufficiently fast only for positive γ^2, we must assume that

$$c' > 0 \quad \text{or} \quad c' < -1. \tag{4.5}$$

If $\gamma^2 < 0$ $(-1 \leqslant c' \leqslant 0)$, the solution to (4.4) is a slowly decaying function and has *infinite energy*. As V has, in this case, oscillatory asymptotics, it is very tempting to interpret it as a wave field radiated by the cyclone. However, one should bear in mind that the translation speed of an eddy with infinite energy must be infinite [see (4.1)]; hence, the solution of (4.4) with $\gamma^2 < 0$ violates the condition under which it was derived (we assumed that $-1 < c' < 0$). The only possible conclusion is that this solution does not exist.

5. The periphery of the eddy

As mentioned before, the asymptotic method used in sections 3 and 4 is not valid in the region where the velocity matches the eddy's translation speed (see Fig. 1). This can be seen formally by comparing the zero- and first-order solutions. Taking into account the asymptotics of the modified Bessel function:

$$V \sim r^{-1/2} \exp(-\gamma r) \quad \text{as} \quad r \to \infty$$

and expression (4.2b), one can show that

$$\frac{\alpha \Gamma}{h - 1} \sim \alpha r^2 \quad \text{as} \quad r \to \infty.$$

Thus, for $r \sim \alpha^{-1/2}$, the first-order correction begins to dominate the zero-order solution, and the expansion fails. Although this does not affect expression (4.1) (the translation speed depends on the *global* characteristics of the eddy), the results obtained for the eddy's spatial structure are not valid in the peripheral region. Even the important restriction (4.5) is under threat, as it was derived from equation (4.4) valid in the intermediate region: indeed, why can we not allow the solution to "misbehave" in this narrow strip?

In this section, (4.4) will be shown to correctly describe both the *intermediate region* of the eddy and its *periphery* as well.

We start from the exact equations (2.3). In the peripheral region, the nonlinear terms are small and must be scaled by α:

$$u = \alpha \hat{u}, \quad v = \alpha \hat{v}, \quad h = 1 + \alpha \eta, \quad c = \alpha c'. \tag{5.1}$$

Substitution of (5.1) into (2.3) yields (hats omitted)

$$\left.\begin{aligned} &\alpha(-c' u_x + u u_x + v u_y) + \eta_x = (1 + \alpha y) v \\ &\alpha(-c' v_x + u v_x + v v_y) + \eta_y = -(1 + \alpha y) u \\ &-c' \eta_x + [u(1 + \alpha \eta)]_x + [v(1 + \alpha \eta)]_y = 0 \end{aligned}\right\}. \tag{5.2}$$

The small parameter in these equations appears exactly in the same places as in the quasigeostrophic approximation (this should have been expected, as the velocity in the peripheral region is small, and so is the Rossby number). Following the standard quasigeostrophic approach, one can derive from (5.2)

$$-c'(\nabla^2 \eta - \eta)_x + J(\eta, \nabla^2 \eta) + \eta_x = O(\alpha), \tag{5.3}$$

where $J(\eta, \psi) = \eta_x \psi_y - \eta_y \psi_x$. Omitting small terms, (5.3) can be transformed into

$$J(\eta + c' y, \nabla^2 \eta - \eta + y) = 0 \tag{5.4}$$

and reduced to a Helmholtz-type equation:

$$\nabla^2 \eta - \eta + y = F_0(\eta + c' y), \tag{5.5}$$

where F_0 is an undetermined function. Similarly to the modon theory, F_0 can be fixed by taking the limit $y \to \infty$, after which (5.5) yields

$$F_0(y) = \frac{1}{c'} y$$

and then becomes

$$\nabla^2 \eta - \gamma^2 \eta = 0, \tag{5.6}$$

where γ is given by (4.4b). Given that the solution to (5.6) is to be matched to the *radially symmetric* solution of (4.4), we obtain

$$\eta_{rr} + \frac{1}{r} \eta_r - \gamma^2 \eta = 0,$$

which is equivalent to (4.4) (recall that, in the intermediate and peripheral regions, $\eta_r \approx V$). Thus, the region of applicability of (4.4) is larger than we expected: it is valid in both intermediate and peripheral regions of the eddy. This also validates restriction (4.5) for the translation speed, following from (4.4).

Finally, note for completeness that (5.5) does not represent the *general* solution of (5.4). The latter, in fact, is

$$\nabla^2\eta - \eta + y = F_0(\eta + c'y) \quad \text{for} \quad (x, y) \in D_0 \qquad (5.7a)$$

$$\nabla^2\eta - \eta + y = F_m(\eta + c'y) \quad \text{for} \quad (x, y) \in D_m, \qquad (5.7b)$$

where F_0 and F_m $(m = 1, 2, \cdots, M)$ are undetermined functions, D_m are nonoverlapping simply connected domains on the (x, y) plane and D_0 is the "remainder" of the plane including the infinity. At the boundaries ∂D_m of the domains, η should satisfy the smoothness conditions

$$\eta \text{ and } \frac{\partial \eta}{\partial \mathbf{n}}$$

are continuous for $(x, y) \in \partial D_m$. Solutions (5.7b) describe modonlike structures embedded in the periphery of the eddy and translating at the same speed.

6. Discussion

Thus, we have considered the β-induced translation of isolated eddies. The speed of translation is given by (4.1) or, dimensionally, by

$$c_* = c_0\left(1 + \frac{E}{g'H_0M}\right), \qquad (6.1)$$

where $c_0 = -\beta R_d^2$ is the long-Rossby-wave speed, E and M are the net energy and mass anomaly of the eddy. Given that $E > 0$ and $c_0 < 0$, formula (6.1) shows that all anticyclones ($M > 0$) drift to the west faster than any Rossby wave ($c_* < c_0$), while for cyclones ($M < 0$) (6.1) predicts either "slow speed" westward translation ($c_0 < c_* < 0$) or eastward translation ($c_* > 0$). It turns out, however, that west-translating cyclone solutions are not bounded spatially and, hence, are meaningless physically.

The solution, representing the eddy, consists of the leading-order radially symmetric circulation and a dipolar (β induced) correction. It was demonstrated that the latter is bounded at infinity only if the translations speed satisfies (4.5) and the asymptotics of the radially symmetric profile satisfy (4.4). Dimensionally, these constraints are

$$c_* > 0 \quad \text{or} \quad c_* < c_0 \qquad (6.2)$$

and

$$V_* \to \text{const}(g'H_0)^{1/2}K_1(\gamma_* r_*) \quad \text{as} \quad r_* \to \infty, \qquad (6.3a)$$

where

$$\gamma_*^2 = \frac{c_* - c_0}{c_* R_d^2} \qquad (6.3b)$$

and K_1 is the modified Bessel function. It should be noted that the geostrophic limit of (6.1)–(6.3) agrees with the results obtained by Nycander and Sutyrin (1992).

It is interesting to discuss what happens to eddies with far-field asymptotics not satisfying (6.3). It can be conjectured that eddies, for which formula (6.1) predicts $c_* > 0$ or $c_* < c_0$, "adjust" their far fields and become steady (which is usually the case with "almost" steady solutions of nonlinear PDEs). Eddies, for which (6.1) predicts $c_0 < c_* < 0$, do not have a steady solution to tend to, so they are destined to disintegrate. There are three possible scenarios of their evolution:

(i) a complete loss of radial symmetry and further evolution as a two-dimensional flow ("*violent death*");

(ii) strong restructuring causing a significant change in the mass of the eddy. If the translation speed, corresponding to the new value of the mass, fits into the "allowed" range, the eddy becomes steady ("*change of gender*");

(iii) drift towards the so-called "rest latitude" and slow decay there without losing (to the leading order) radial symmetry ("*death from old age*").

The third scenario seems to be the most popular in the oceanographic community (it has been mentioned to me by V. Larichev, G. Reznik, P. Rhines, and several others). Its appeal is based on a clear agreement between the potential vorticity conservation for fluid particles:

$$\frac{f + \beta y_* + v_{*x_*} - u_{*y_*}}{h_*} = \text{const} \qquad (6.4)$$

and energy losses due to radiation:

$$u_*, v_* \to 0, \quad h_* \to H_0 \quad \text{as} \quad t_* \to \infty.$$

As $t_* \to \infty$, the $(f + \beta y_*)$ term in (6.4) becomes dominant and must represent *all* of the initial potential vorticity. Thus, the particle may lose its energy only at

$$y_* = \frac{1}{\beta}\,(\text{initial potential vorticity} \times H_0 - f), \quad (6.5)$$

which is called the "rest latitude."

If, however, the eddy consists of particles with *different* values of potential vorticity, those tend to *different* rest latitudes separated by distances greater than the initial radius of the eddy [recall that the β effect is weak; hence β in formula (6.5) is small]. The eddy begins to stretch in the meridional direction and lose its radial symmetry, which seems to reduce the third scenario to the first one.

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