The Stability of Zonal Jets in a Rough-Bottomed Ocean on the Barotropic Beta Plane

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(Manuscript received 7 October 1998, in final form 22 June 1999)

ABSTRACT

The author considers the stability of a barotropic jet on the beta plane, using the model of a ''rough-bottomed ocean'' (i.e., assuming that the horizontal scale of bottom irregularities is much smaller than the width of the jet). An equation is derived, which governs disturbances in a sheared flow over one-dimensional bottom topography, such that the isobaths are parallel to the streamlines. Interestingly, this equation looks similar to the equation for internal waves in a vertically stratified current, with the density stratification term being the same as the topography term. It appears that the two effects work in a similar way, that is, to return the particle to the level (isobath) where it ''belongs'' (determined by its density or potential vorticity). Using the derived equation, the author obtains a criterion of stability based on comparison of the mean-square height of bottom irregularities with the maximum shear of the current. It is argued that the influence of topography is a stabilizing one, and it turns out that ''realistic'' currents can be stabilized by relatively low bottom irregularities (30–70 m). This conclusion is supported by numerical calculation of the growth rate of instability for jets with a Gaussian profile.

1. Introduction

Ninety percent of theoretical papers dealing with flows on the beta plane assume the oceanic bottom to be flat. The reason for that is obvious—bottom irregularities appear to hamper both analytical and numerical treatment. The situation is particularly drastic with theory of barotropic and baroclinic instabilities, where very few studies take into account bottom topography. The lack of attention to this important question looks even more surprising, as all necessary mathematical tools had been developed long time ago by Rhines and Bretherton (1973) in their pioneering work on the dynamics of Rossby waves over topography (in still water).

In order to illustrate how topography can affect instability, consider a barotropic zonal flow $U(y)$ on the beta plane (U is the velocity of the fluid, y is the northward Cartesian coordinate). In order to simplify the problem, we consider one-dimensional depth variation $D(y)$ such that the isobaths are parallel to the streamlines. The stability of $U(y)$ can be readily examined through the standard methods, resulting in the conclusion that the flow is stable, given the usual sufficient condition of monotonicity of potential vorticity:

$$\mathrm{PV}(y) = \beta y - U_y - \frac{f_0}{H_0}D,$$

where H_0 is the mean depth of the ocean, and f_0 and β are the Coriolis parameter and its meridional gradient, both depending on the latitude θ (in all examples considered in this paper we assume $\theta = 30°$). It turns out, however, that this criterion is unusable for many important oceanic applications.

Consider a relatively weak jet with a Gaussian profile

$$U(y) = U_0 \exp\left[-\left(\frac{y}{L_U}\right)^2\right], \tag{1}$$

$$U_0 = 0.15 \text{ m s}^{-1}, \qquad L_U = 100 \text{ km}, \tag{2}$$

where U_0 is the maximum velocity and L_U is the ''width'' of the current. For the case of flat bottom, the potential vorticity (PV) profile associated with this flow is monotonic (see Fig. 1a), which proves that the current is stable. Let us now include a small, short-scale depth variation, say

$$H(y) = H_0 + 50 \text{ m} \times \sin\left(\frac{2\pi y}{5 \text{ km}}\right), \tag{3}$$

$$H_0 = 5000 \text{ m}, \tag{4}$$

where $H(y)$ is the depth of the ocean [(3) appears to be qualitatively applicable everywhere except for the continental shelf and midocean ridges, as elsewhere in the ocean bottom irregularities are small and short—say, 10–300 m in height, and 1–10 km in horizontal scale]. Affected by the topography, the PV profile becomes strongly nonmonotonic (see Fig. 1b)—but does this suggest that the flow is potentially unstable? Of course not,

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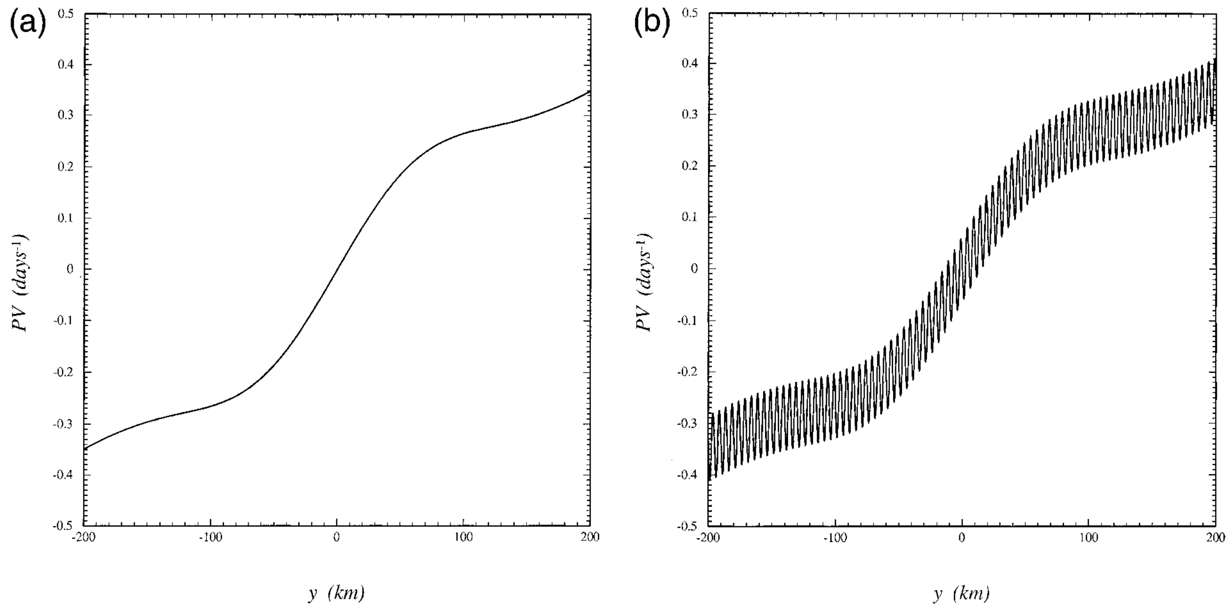

FIG. 1. Potential vorticity profile for flow (1)–(2) for a (a) flat bottom and (b) sinusoidal topography (3)–(4).

spatial inhomogeneities are unlikely to introduce instability into an otherwise stable system! Moreover, in some cases topography can stabilize the flows, which would be unstable without it. This renders the PV criterion virtually meaningless; it never holds for realistic oceanic topography.

The present paper studies the effect of topography on the instability of barotropic flows on the beta plane. Our main goal is to *derive a topography-modified stability criterion.*

Following Rhines and Bretherton (1973), we shall consider the (most interesting physically) case of ''rough-bottomed ocean,'' where

- the height of bottom irregularities is small compared to the mean depth of the ocean and
- the horizontal scale of topography is short compared to the width of the flow.

Observe that the latter assumption makes the problem difficult to simulate numerically, as one has to resolve the short-scale topography-induced component of the flow. As a result, we shall take a combined analytical–numerical approach: in section 2 of this paper, we shall employ the method of multiple scales to derive an asymptotic equation that ''averages out'' the effect of rough topography and ''parameterizes'' it in the form of a smooth term, depending on the mean-square height of bottom irregularities (all other characteristics of topography turn out to be nonessential). This equation generalizes the usual linear ODE that governs the stability of normal modes in a zonal jet over flat bottom, and it will be used to derive a stability theorem for flows over topography (section 3). In sections 4 and 5, we shall compute numerically the growth rate of the instability for jets with a Gaussian profile in a wide range of the parameters involved. It turns out that even weak bottom irregularities (less than $\pm 2\%$ of the mean depth) may stabilize a flow with ''realistic'' oceanic parameters.

2. Basic equations

Barotropic motion on the beta plane is governed by

$$\nabla^2\psi_t + J\left(\psi, \nabla^2\psi - \frac{f_0}{H_0}d\right) + \beta\psi_x = 0, \qquad (5)$$

where (x, y) and t are the Cartesian coordinates and time, $\psi(x, y, t)$ is the streamfunction, f_0 and β are the Coriolis parameter and its meridional gradient, and $d(x, y)$ is the deviation of the ocean depth $H(x, y)$ from its mean value H_0:

$$d = H - H_0.$$

Equation (5) implies that topography is small, $|d| \ll H_0$.

Consider small disturbances superposed on a zonal current:

$$\psi = -\int U(y)\, dy + \psi', \qquad (6)$$

where $U(y)$ is the velocity of the current and ψ' describes the disturbance. We shall examine the simplest case where the isobaths are straight lines. Although this model is too idealized to describe the ''real'' ocean, it has been found very helpful for investigation of fundamental properties of wave dynamics over topography (see Rhines and Bretherton 1973; Samelson 1992; Reznik and Tsybaneva 1998). In order to avoid lee waves, we shall further assume that isobaths are parallel to the velocity of the flow:

$$d = D(y). \tag{7}$$

Substituting (6) and (7) into (5), we linearize the resulting equation and omit the primes:

$$\nabla^2\psi_t - \psi_x\left(U_{yy} + \frac{f_0}{H_0}D_y\right) + U\nabla^2\psi_x + \beta\psi_x = 0.$$

Following the usual normal mode approach, we introduce the wavenumber k and phase speed c of the disturbance and put

$$\psi(x, y, t) = \psi(y)e^{ik(x-ct)},$$

which yields

$$(c - U)(\psi_{yy} - k^2\psi) + \psi\left(U_{yy} + \frac{f_0}{H_0}D_y - \beta\right) = 0. \tag{8}$$

It will be assumed that

$$D(y) = \sum_{n=1}^{N} A_n \sin(q_n y + \phi_n), \tag{9}$$

where the amplitudes A_n, frequencies q_n, and phases ϕ_n are arbitrary constants. Finally, we shall assume that the flow under consideration is a jet:

$$U \to 0 \quad \text{as } y \to \pm\infty,$$

and the disturbance should decay far away from it:

$$\psi \to 0 \quad \text{as } y \to \pm\infty. \tag{10}$$

In principle, Eq. (8) can be solved numerically for any given $U(y)$ and $D(y)$; however, this straightforward approach is difficult to realize in view of the boundary condition (10). In order to illustrate the difficulty, consider (8) for the case of a flat bottom and take the limit $y \to \pm\infty$:

$$c(\psi_{yy} - k^2\psi) - \beta\psi = 0 \quad \text{as } y \to \pm\infty. \tag{11}$$

Now, ψ can be found explicitly:

$$\psi \to \text{const}_1 \exp\left(-\sqrt{\frac{\beta}{c} + k^2}y\right) \quad \text{as } y \to \infty,$$

$$\psi \to \text{const}_2 \exp\left(\sqrt{\frac{\beta}{c} + k^2}y\right) \quad \text{as } y \to -\infty,$$

which can be used for "shooting" the solution from infinity. If, however, we include topography, the analogue of Eq. (11) will have variable coefficients [recall that, unlike $U(y)$, topography does not decay at infinity], and the asymptotic behavior of ψ cannot be determined.[1] This difficulty, together with the fact that the scale of topography is short (which necessitates a small spatial step), make the straightforward numerical solution of (8), (10) difficult.

In what follows, we shall demonstrate that our enemy [fast-changing, nondecaying coefficients of Eq. (8)] can be made our ally. This will be achieved through the asymptotic method of multiple scales, which replaces fast-changing coefficients of an ODE by a smooth term having approximately the same effect on the solution.

In the next subsection, we shall "prepare" Eq. (8) for the multiple-scale analysis.

a. *Scaling of Eq. (8)*

In order to scale Eq. (8), we introduce three spatial scales: the width L_U of the current, the horizontal scale L_D of bottom topography, and the wavelength L_k of the disturbance. Following Rhines and Bretherton (1973), we shall assume that

$$L_U \gg L_D,$$

which holds relatively well for those regions of the ocean located far away from coasts and midocean ridges. It should be expected that the wavelength of the most unstable disturbances is comparable to the width of the flow,

$$L_k \sim L_U,$$

which implies that the wavenumber should be scaled using L_U:

$$\tilde{k} = kL_U.$$

The spatial variable will be scaled by L_D:

$$\tilde{y} = y/L_D,$$

in addition to which we shall introduce the *slow* spatial variable

$$\tilde{Y} = y/L_U.$$

The spatial derivative then becomes

$$\frac{d\psi}{dy} = \frac{1}{L_D}\frac{\partial\psi}{\partial\tilde{y}} + \frac{1}{L_U}\frac{\partial\psi}{\partial\tilde{Y}}.$$

Here $U(y)$ will be scaled by the characteristic velocity U_0, and $D(y)$ will be scaled by the amplitude D_0 of the topography:

$$\tilde{U}(\tilde{Y}) = U(y)/U_0, \qquad \tilde{D}(\tilde{y}) = D(y)/D_0.$$

As in all problems of hydrodynamic stability, the phase speed scales with U_0:

$$\tilde{c} = c/U_0.$$

In terms of the new variables, Eq. (8) becomes (tildes omitted)

[1] An obvious solution to this difficulty would be solving the equation in a zonal channel; however, I have decided against this approach. If the walls of the channel are placed too close to the jet, they would change the flow's stability properties; and if the walls are placed too far from the jet, the channel calculation becomes just as difficult as that in the unbounded domain.

$$(c - U)(\psi_{yy} + 2\sqrt{\varepsilon_L}\psi_{yY} + \varepsilon_L\psi_{YY} - \varepsilon_L k^2\psi) + \psi(\varepsilon_L U_{YY} + \varepsilon_D D_y - \varepsilon_\beta) = 0, \quad (12)$$

where

$$\varepsilon_L = \frac{L_D^2}{L_U^2}, \qquad \varepsilon_D = \frac{f_0 L_D D_0}{H_0 U_0}, \qquad \varepsilon_\beta = \frac{\beta L_D^2}{U_0}.$$

The relationship between y and Y becomes

$$Y = \sqrt{\varepsilon_L} y. \quad (13)$$

In order to estimate ε_L, ε_D, and ε_β for the "real" ocean, we shall put

$$U_0 = 0.25 \text{ m s}^{-1}, \qquad L_U = 100 \text{ km}. \quad (14)$$

(The value for U_0 was chosen relatively small because we consider *barotropic* flows, for which even a weak velocity corresponds to a significant mass flux.) The parameters of topography are

$$D_0 = 100 \text{ m}, \qquad L_D = 5 \text{ km}.$$

Assuming the latitude and mean depth of the ocean to be

$$\theta = 30°, \qquad H_0 = 5000 \text{ m}, \quad (15)$$

we obtain

$$\varepsilon_L \approx 0.0025, \qquad \varepsilon_D \approx 0.058, \qquad \varepsilon_\beta \approx 0.0020.$$

Thus, all three parameters are small, but ε_D is greater than the other two—which will be our main assumption:

$$1 \gg \varepsilon_D \gg \max\{\varepsilon_L, \varepsilon_\beta\}. \quad (16)$$

As a matter of convenience, we shall assume that $\varepsilon_L \sim \varepsilon_\beta$, and also that $\varepsilon_\beta \sim \varepsilon_D^2$. Accordingly, we put

$$\varepsilon_L = \varepsilon^2, \qquad \varepsilon_D = \varepsilon, \qquad \varepsilon_\beta = \varepsilon^2\alpha, \quad (17)$$

where ε is the "formal" small parameter and α is a constant of the order of unity. Substituting (17) into (12) and (13), we obtain

$$(c - U)(\psi_{yy} + 2\varepsilon\psi_{yY} + \varepsilon^2\psi_{YY} - \varepsilon^2 k^2\psi) + \psi(\varepsilon^2 U_{YY} + \varepsilon D_y - \varepsilon^2\alpha) = 0, \quad (18)$$

$$Y = \varepsilon y.$$

In the limit $U \to 0$, Eq. (18) coincides with the corresponding equation of Rhines and Bretherton (1973).

In the next subsection, Eq. (18) will be examined asymptotically using the method of multiple scales. The technical side of the analysis will be similar to that of Rhines and Bretherton (1973), and nonmathematically minded readers are advised to jump to the beginning of section 2c.

b. *Asymptotic analysis of Eq. (18)*

According to the usual scheme of the multiple scales method, we shall seek a solution to (18) in the form of a series

$$\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \cdots, \qquad c = c^{(0)} + \varepsilon c^{(1)} + \cdots$$

The equation for $\psi^{(0)}$ is

$$(c^{(0)} - U)\psi_{yy}^{(0)} = 0.$$

As $y \to \pm\infty$, the only bounded solution is

$$\psi^{(0)}(y) = \Psi^{(0)}(Y), \quad (19)$$

where $\Psi^{(0)}$ is an undetermined function. The next order yields the following equation for $\psi^{(1)}$:

$$(c^{(0)} - U)(2\psi_{yY}^{(0)} + \psi_{yy}^{(1)}) + c^{(1)}\psi_{yy}^{(0)} + D_y\psi^{(0)} = 0. \quad (20)$$

Substituting (19) into (20), we obtain

$$\psi^{(1)}(y, Y) = -\frac{\Psi^{(0)}(Y)}{c^{(0)} - U(Y)}\int D(y)\, dy + \Psi^{(1)}(Y),$$

where $\Psi^{(1)}$ is the "constant of integration." Taking into account expression (9) for $D(y)$, we have

$$\psi^{(1)} = \frac{\Psi^{(0)}}{c^{(0)} - U}\sum_{n=1}^{N}\frac{A_n}{q_n}\cos(q_n y + \phi_n) + \Psi^{(1)}.$$

Finally, the second order yields

$$(c^{(0)} - U)(\psi_{YY}^{(0)} + 2\psi_{yY}^{(1)} + \psi_{yy}^{(2)} - k^2\psi^{(0)}) + c^{(1)}(2\psi_{yY}^{(0)} + \psi_{yy}^{(1)}) + c^{(2)}\psi_{yy}^{(0)} + (U_{YY} - \alpha)\psi^{(0)} + D_y\psi^{(1)} = 0.$$

Substituting the expressions for $\psi^{(0)}$ and $\psi^{(1)}$, we obtain

$$(c^{(0)} - U)\left[\psi_{yy}^{(2)} + \Psi_{YY}^{(0)} - k^2\Psi^{(0)} - 2\left(\frac{\Psi^{(0)}}{c^{(0)} - U}\right)_Y \sum_{n=1}^{N} A_n \sin(q_n y + \phi_n)\right] - c^{(1)}\frac{\Psi^{(0)}}{c^{(0)} - U}\sum_{n=1}^{N} A_n q_n \cos(q_n y + \phi_n)$$
$$+ (U_{YY} - \alpha)\Psi^{(0)} + \left[\frac{\Psi^{(0)}}{c^{(0)} - U}\sum_{n=1}^{N} A_n q_n \cos(q_n y + \phi_n) + \Psi^{(1)}\right]\sum_{n=1}^{N}\frac{A_n}{q_n}\cos(q_n y + \phi_n) = 0.$$

This equality should be treated as an equation for $\psi^{(2)}$:

$$\psi_{yy}^{(2)} = F(y, Y), \quad (21)$$

where

$$F = -\left[\Psi^{(0)}_{YY} - k^2\Psi^{(0)} - 2\left(\frac{\Psi^{(0)}}{c^{(0)} - U}\right)_Y \sum_{n=1}^{N} A_n \sin(q_n y + \phi_n)\right] - c^{(1)}\frac{\Psi^{(0)}}{(c^{(0)} - U)^2}\sum_{n=1}^{N} A_n q_n \cos(q_n y + \phi_n) - \frac{U_{YY} - \alpha}{c^{(0)} - U}\Psi^{(0)}$$
$$- \left[\frac{\Psi^{(0)}}{(c^{(0)} - U)^2}\sum_{n=1}^{N} A_n q_n \cos(q_n y + \phi_n) + \frac{\Psi^{(1)}}{c^{(0)} - U}\right]\sum_{n=1}^{N}\frac{A_n}{q_n}\cos(q_n y + \phi_n).$$

Clearly, (21) has a bounded (in y) solution for $\psi^{(2)}$ if, and only if, the slow-varying terms included in $F(y, Y)$ add up to zero, which yields

$$(c^{(0)} - U)(\Psi^{(0)}_{YY} - k^2\Psi^{(0)}) + (U_{YY} - \alpha)\Psi^{(0)} + \frac{\Psi^{(0)}}{c^{(0)} - U}\sum_{n=1}^{N}\frac{1}{2}A_n^2 = 0. \quad (22)$$

Equation (22) is the final product of our derivation.

c. Discussion

First, we shall rewrite (22) in terms of the original dimensional variables and omit the superscript $^{(0)}$:

$$\Psi_{yy} - \left[k^2 + \frac{\beta - U_{yy}}{c - U} - \frac{\gamma}{(c - U)^2}\right]\Psi = 0, \quad (23)$$

where

$$\gamma = \frac{f_0^2}{H_0^2}\sigma^2 \quad \text{and} \quad \sigma^2 = \frac{1}{2}\sum_{n=1}^{N} A_n^2 = \langle d^2\rangle$$

is, in fact, the standard deviation of the depth of the ocean from its mean. Interestingly, (23) looks similar to the equation for internal waves in a vertically stratified current, with the topography term being exactly as the density stratification term. It appears that the two effects work in a similar way, i.e., return the particle to the level where it ''belongs'' (determined by its density or potential vorticity). Given that stratification is, generally, a stabilizing influence on the flow, it should be expected that the influence of topography is also a stabilizing one. This conclusion can be interpreted using the following argument.

- Particles in the flow must preserve their values of potential vorticity;
- these PV values are ''linked''—through topography—to the isobaths where the particles were initially.

Thus, the requirement that the initial PV of a particle be conserved constrains the ability of unstable disturbances to move it away from its initial position and thus weakens instability. Generally, potential vorticity of flows over topography is similar to density of stratified flows. It should be noted, however, that the analogy between topographic flows on the beta plane and density stratified flows is limited by the fact that there is no beta effect in the latter case.

Finally, the boundary condition (10) becomes

$$\Psi \to 0 \quad \text{as } y \to \pm\infty. \quad (24)$$

Equations (23), (24) form an eigenvalue problem for c. If k Im$c > 0$, the jet is unstable.

3. A stability theorem for currents over topography

As mentioned before, the usual stability criterion for flows on the beta plane (monotonicity of PV) is never satisfied for the type of topography that we are looking at. Indeed, it can be readily demonstrated that, since $D(y)$ varies (oscillates) much faster than the other two terms in

$$PV = \beta y - U_y - \frac{f_0}{H_0}D,$$

PV cannot be monotonic. The nonmonotonicity of the PV profile, however, does not *guarantee* instability, and in what follows we shall argue that there exists a wide class of flows that *are* stable.

Since our eigenvalue problem is similar to that describing internal waves, the stability criterion should be similar to that derived by Miles (1961). Following his approach, we introduce a new variable Φ such that

$$\Psi = (c - U)^{1/2}\Phi. \quad (25)$$

Substitution of (25) into (23) yields

$$[(c - U)\Phi_y]_y - \left[k^2(c - U) + \frac{(U_y)^2/4 - \gamma}{c - U} - \frac{1}{2}U_{yy} + \beta\right]\Phi = 0.$$

Next, we multiply this equation by Φ^* (the asterisk denotes complex conjugate) and integrate it over $-\infty < y < \infty$. Integrating by parts, using the boundary condition (24), and taking the imaginary part of this identity we obtain

$$(\mathrm{Im}c)\int_{-\infty}^{\infty}\left\{|\Phi_y|^2 + \left[k^2 + \frac{\gamma - (U_y)^2/4}{|c - U|^2}\right]|\Phi|^2\right\}dy = 0. \quad (26)$$

One can see that, if

$$\gamma \geq \frac{1}{4}\max\{(U_y)^2\},$$

the second factor on the lhs of (26) is strictly positive; hence Im$c = 0$ (stability). This condition of stability can be rewritten in terms of the effective Richardson number:

the flow is stable if

$$\mathrm{Ri} \geq \frac{1}{4}, \tag{27}$$

where

$$\mathrm{Ri} = \left(\frac{\sigma}{H_0}\frac{f_0}{\max|U_y|}\right)^2. \tag{28}$$

Equation (27) can be further rewritten as:

the flow is stable if

$$\frac{2\sigma}{H_0} \geq \mathrm{Ro},$$

where $\mathrm{Ro} = \max|U_y|/f_0$.

It is interesting to estimate the height of bottom irregularities that would be sufficient to stabilize a current with realistic parameters. Calculating, for the Gaussian jet (1),

$$\max\{|U_y|\} = \sqrt{\frac{2}{e}}\frac{|U_0|}{L}$$

and assuming parameters (14), (15), we obtain that the flow becomes stable for relatively weak topography:

$$\sigma \gtrsim 74 \text{ m}.$$

Observe that this is a *sufficient,* not a *necessary,* condition of stability.

In the next section, it will be demonstrated that eastward jets can be made stable by a much lower σ than what our stability criterion predicts, whereas for westward jets the threshold values of σ are fairly close to its predictions.

4. Examples

We consider a Gaussian jet with the values of U_0, L_U, and H_0 given by (14) and (15). Several values of the mean-square height of bottom irregularities were considered in the range

$$0 \text{ m} \leq \sigma \leq 74 \text{ m}.$$

Equation (23) was solved using the Runge–Kutta method with variable step (a smaller step was used near the critical levels). Two solutions were ''shot'' from $y = \infty$ and $y = -\infty$ toward $y = 0$, and their Wronskian (computed at $y = 0$) was fed to the root-finding routine determining c. The growth rate

$$\omega = k\,\mathrm{Im}c$$

of the instability of the westward Gaussian jet is shown in Fig. 2. Two slightly overlapping unstable modes were found, with the eigenfunctions of the long-wave mode being much ''wider'' spatially than those of the short-wave mode (this occurs regardless of topography). As the average height σ of topography grows, the growth rate rapidly decreases, and no unstable disturbances were found for $\sigma \approx 72$ m. In terms of the effective Richardson number (28), this corresponds to $\mathrm{Ri} \approx 0.238$.

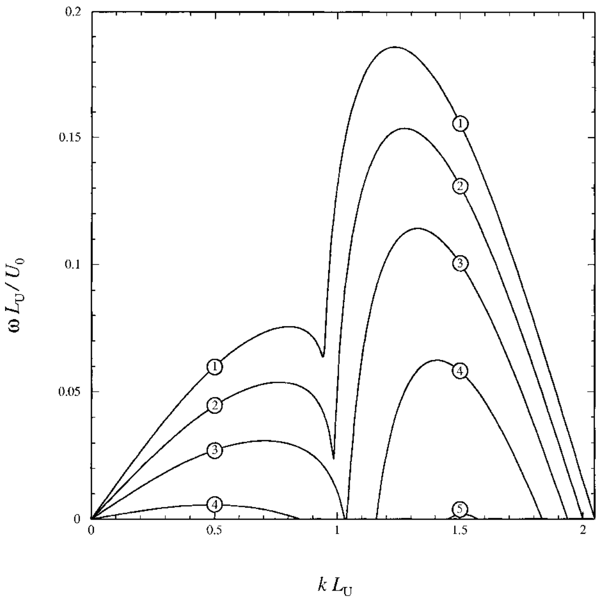

FIG. 2. Growth rate vs wavenumber for a westward Gaussian jet with (1) $\sigma = 0$ (flat bottom), (2) $\sigma = 28$ m, (3) $\sigma = 42$ m, (4) $\sigma = 56$ m, and (5) $\sigma = 70$ m: ω and k are scaled using the width $L_U = 100$ km and maximum velocity $U_0 = 0.25$ m s^{-1} of the jet.

An interesting difference has been observed in the spatial structure of disturbances with and without topography: in the former case, the peaks near the critical levels are much higher than those in the latter case—see Fig. 3. There are two reasons for that: first, the most ''singular'' term in Eq. (23),

$$\frac{\gamma}{(c - U)^2},$$

vanishes when $\gamma = 0$ (i.e., for the flat-bottom case). Second, Imc for topography-influenced disturbances is smaller than that for disturbances over flat bottom, which strengthens the effect of critical levels.

We have also considered the eastward Gaussian jet with the same parameters. It turned out that its instability over a flat bottom was noticeably weaker than that in the westward case and, correspondingly, could be stabilized by a weaker topography ($\sigma \approx 27$ m). The Richardson number for this case is fairly small, $\mathrm{Ri} \approx 0.034$.

5. Can topography destabilize an otherwise stable flow?

It should be noted that it is mathematically possible for a particular jet profile to satisfy the flat-bottom sta-

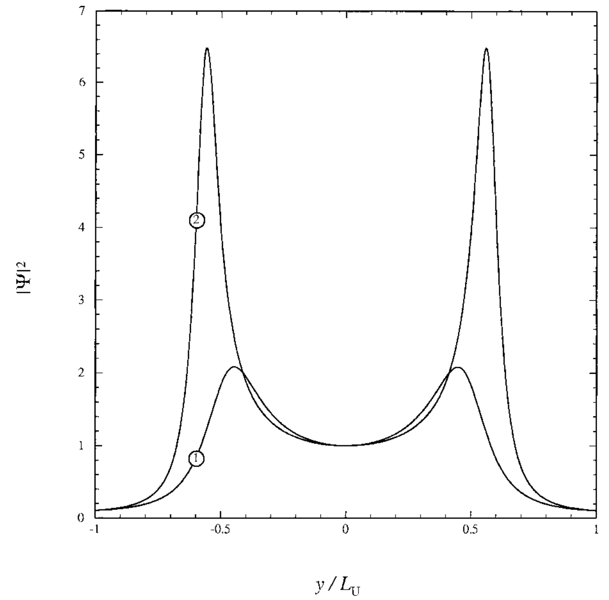

FIG. 3. Spatial structure of the eigenfunctions: (1) $\sigma = 0$ m (flat bottom) and (2) $\sigma = 42$ m. In both cases, $kL_U = 1.5$ and y is scaled by the width $L_U = 100$ km of the jet. The positions of the peaks of $|\Psi|^2$ coincide with the critical levels.

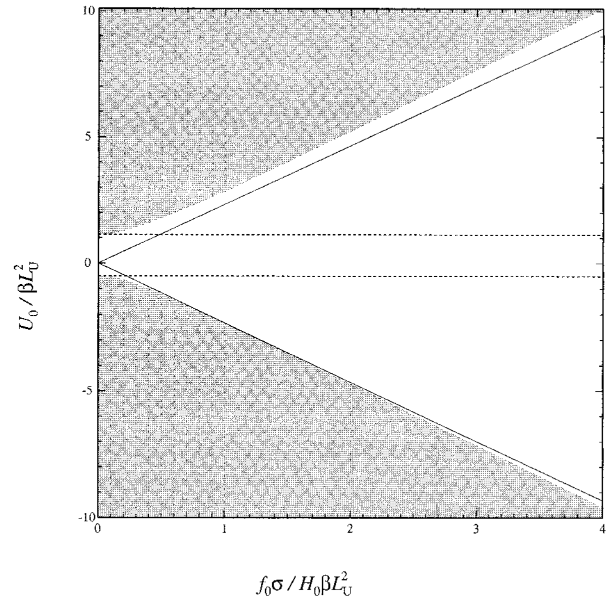

FIG. 4. Stability properties of Gaussian jets vs the height of topography and strength of the jet. The region of instability is shaded. U_0 is the maximum velocity, L_U is the width of the jet [see (1)], σ^2 is the mean-square height of topography, H_0 is the mean depth of the ocean, and β is the beta parameter.

bility criterion and not satisfy the topography-modified criterion. Indeed, if we rewrite the condition of monotonicity of a jet's PV profile in the form

$$\max\{U_{yy}\} \leq \beta \tag{29}$$

and then rewrite ''our'' condition (27) in the form

$$\max\{|U_y|\} \leq \frac{2f_0\sigma}{H_0}, \tag{30}$$

it becomes clear that they are ''independent''; that is, (29) can hold and, at the same time, (30) be violated. Should this be viewed as an indication that topography can destabilize an otherwise stable jet? In this section, it will be argued that it cannot.

The stability of the Gaussian jet has been examined in a broad range of the parameters involved, and the results are shown in Fig. 4. One can observe the following features.

- Clearly, the flow ''complies'' with both stability conditions—see the turns of the marginal stability curves when they ''leave'' criterion (30) (solid line) and begin to ''follow'' (29) (dashed line).
- Figure 4 suggests that (30) is a lot ''tighter'' for westward flows than for eastward flows (see the gap between the solid line and the instability region in the top half of Fig. 4).

This suggests that there might be a tighter stability condition, involving *both* stabilizing effects, topography and beta effect (the latter could have also explained the asymmetry of Fig. 4 with respect to the change of the sign of U_0). Unfortunately, despite several attempts, I was unable to find any such condition.

It should also be emphasized that our conclusion about the stabilizing influence of topography is applicable only to jets. It seems apparent that one can think of a flow *in a channel,* which would be stable over a flat bottom and unstable over topography. The important difference between jets in an unbounded ocean and flows in a channel is that, in the latter case, we can include into consideration flows like

$$U = U'y,$$

where U' is a constant. This flow always satisfies (29)—but, for sufficiently large U' (small Ri), should be unstable over topography. The mechanism of this (hypothetical) instability is not clear; bottom irregularities may somehow release the energy of the flow to disturbances. We shall not dwell on this instability in further detail, as it is unlikely to occur for jets and, therefore, is of limited relevance to the ocean.

6. Conclusions

We have considered the stability of barotropic flows on the beta plane in a rough-bottomed ocean. An equation was derived [see (23)] governing the disturbances in a sheared current over one-dimensional topography (such that the isobaths are straight lines parallel to the streamlines of the mean current). Using this equation, a criterion of stability has been obtained [see (27)], based on a comparison of the mean-square height σ of bottom irregularities with the maximum shear of the current. We argued that the influence of topography is a stabilizing one, which was supported by numerical calculation of the growth rate of barotropic instability of the Gaussian jet: it turned out that ''realistic'' currents can be stabilized by relatively low bottom irregularities ($\sigma \approx 30$–70 m).

Using the example of a Gaussian jet, it was also demonstrated that, although there is no mathematical reason for flows over topography to ''observe'' the flat-bottom stability criterion, they still do—*both* criteria need to be violated to make a jet potentially unstable. In other words, bottom irregularities appear to never destabilize an otherwise stable jet, which agrees with our conclusion about the stabilizing influence of topography.

This argument can also be interpreted using potential vorticity. As PV of a particle depends—through topography—on the spatial coordinates, it is more difficult for unstable disturbances to move it away from the isobath where it was located initially. This can be readily understood if one assumes, for the sake of argument, that the topography term in PV is much greater than the vorticity term, in which case the PV conservation would force the particles to remain within a very short distance from their initial isobaths, and the instability should simply disappear.

Our conclusions were derived using an asymptotic method based on assumption (16), the dimensional form of which is

$$1 \gg \frac{f_0 L_D D_0}{H_0 U_0} \gg \max\left\{\frac{L_D^2}{L_U^2}, \frac{\beta L_D^2}{U_0}\right\}, \quad (31)$$

where f_0 and β are the Coriolis parameter and its meridional gradient, H_0 is the mean depth of the ocean, U_0 and L_U are the amplitude and width of the current, and D_0 and L_D are the amplitude and horizontal scale of topography (effectively, $D_0 = \sigma$). Assumption (31) holds well for those regions of the ocean located far away from coasts and midocean ridges.

It should be emphasized that our stability criterion has been obtained using the *lowest* order of the perturbation method and, strictly speaking, may not eliminate higher-order (weaker) instabilities. One can argue, however, that such instabilities can exist only in those cases where the lowest-order approximation predicts that the flow is *marginally* stable.

Finally, we note that the main limitation of the present work is the assumption that the isobaths are straight lines. Although this simple model has been found helpful for investigation of the fundamental properties of flows over topography, it is still far too idealized to describe the ''real'' ocean. Thus, our next target should be the generalization of the results obtained here for two-dimensional topography.[2]

Another interesting and important development would be to find out if topography can stabilize *baroclinic* instability (which is much stronger than *barotropic* instability considered here). Preliminary results show that the effect of topography on currents localized in the near-surface layer (e.g., the subtropical front in the North Pacific) is relatively weak, whereas the instability of the Antarctic Circumpolar Current (which penetrates deep into the ocean) is noticeably weakened by topography.

REFERENCES

Miles, J. W., 1961: On the stability of heterogeneous shear flows. *J. Fluid Mech.,* **10,** 496–509.

Reznik, G. M., and T. B. Tsybaneva, 1999: Planetary waves in a stratified ocean of variable depth. Part 1: Two-layer model. *J. Fluid Mech.,* **388,** 115–145.

Rhines, P. B., and F. Bretherton, 1973: Topographic Rossby waves in a rough-bottomed ocean. *J. Fluid Mech.,* **61,** 583–607.

Samelson, R. M., 1992: Surface-intensified Rossby waves over rough topography. *J. Mar. Res.,* **50,** 367–384.

[2] The first step in this direction has been made by Samelson (1992) who considered a doubly periodic topography in still water. In contrast to the present work, he assumed that the horizontal scale of topography is comparable to the width of the current, $L_D \sim L_U$, which made the problem tractable numerically (in this case, one does not need to resolve small and large scales simultaneously).