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LETTER TO THE EDITOR

On the spectral singularity problem in the perturbation theory for systems close to the KdV equation

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Abstract. We discuss the well known problem of long-wave divergences in the perturbation theory based on the inverse scattering transform. We outline a feasible way to remedy this shortcoming.

The problem of divergences in perturbation theory (PT) for systems close to the Korteveg–de Vries (KdV) equation is well known (see, e.g., Karpman and Maslov 1977, Newell 1980). Let us consider it for the following perturbed equation:

$$u_t - 6uu_x + u_{xxx} = -\varepsilon[\alpha(u+2xu_x) + \beta(2u+xu_x)] \qquad (1)$$

where $|\varepsilon| \ll 1$ and α and β are auxiliary parameters ~ 1. Equation (1) has important applications in the particular case $\beta = -2\alpha$ when the resultant perturbation $3\alpha\varepsilon u$ in the RHS of (1) accounts for dissipation ($\alpha\varepsilon < 0$) or pumping ($\alpha\varepsilon > 0$) in a corresponding physical system. Traditional PT for nearly integrable systems (Karpman and Maslov 1977, Newell 1980) is based on constructing 'a perturbed (L, A) pair', i.e. an overdetermined system of equations for an axiliary function $\Psi(x, t)$, all the perturbation being included into the second (time-evolution) equation:

$$\hat{L}\Psi \equiv -\Psi_{xx} + (u-E)\Psi = 0 \qquad (2a)$$

$$\hat{A}\Psi \equiv \Psi_t - 2(u+2E)\Psi_x + u_x\Psi = \varepsilon F(u, \Psi) \qquad (2b)$$

where the functional F corresponds to the perturbation in (1). The direct and inverse scattering problems for the Schrödinger equation (2a) establish the one-to-one mapping of the 'potential' $u(x)$ into the scattering data consisting of a reflection coefficient $\tau(E)$ and certain characteristics of bound states (if any). Note that τ satisfies the inequality

$$|\tau(E)| < 1. \qquad (3)$$

The evolution of the scattering data in time is determined by the equation (2b) which can be solved asymptotically, using $|\varepsilon| \ll 1$.

The main difficulty of PT for systems close to the KdV equation is the presence of integrals over the spectral parameter E, divergent in the 'long-wave' region (at $E \to 0$). Nevertheless, some kinds of perturbations do not give rise to the divergence. In particular, in the case $\alpha = 0$ the divergences are absent at least in the first order of PT (Karpman 1978). Another remarkable property of this particular case is its exact integrability (Calogero and Degasperis 1982): equation (1) with $\alpha = 0$ has the exact (L,A) pair:

$$-\Psi_{xx} + [u - E\exp(-2\beta t)]\Psi = 0 \qquad (4a)$$

$$\Psi_t - 2[u - \tfrac{1}{2}\varepsilon\beta x + 2E\exp(-2\beta t)]\Psi_x + (u_x - \tfrac{1}{2}\varepsilon\beta)\Psi = 0. \qquad (4b)$$

There is another integrable case: $\beta = 0$† (Calogero and Degasperis 1982), the corresponding (L,A) pair being

$$-\Psi_{xx} + [u - \tfrac{1}{2}\varepsilon\alpha x - E\exp(2\varepsilon\alpha t)]\Psi = 0 \tag{5a}$$

$$\Psi_t - 2[u + 2E\exp(2\varepsilon\alpha t)]\Psi_x + u_x\Psi = 0. \tag{5b}$$

Nevertheless, formal application of the standard PT to the latter case entails appearance of the divergences.

The starting point of the present paper is the following observation: in the case $\beta = 0$ the form of the scattering data defined by the unperturbed L equation $(2a)$ is drastically distinct from that defined by the 'perturbed' one $(5a)$, namely, due to infinite increase of the effective potential $u_{\text{eff}} = u + \varepsilon\alpha x/2$ in $(5a)$ at $x \to \pm\infty$, a discrete spectrum is absent and the reflection coefficient satisfies the equality

$$|\tau(E)| \equiv 1 \tag{6}$$

(this follows from the quantum mechanical interpretation of the Schrödinger equation $(5a)$‡). On the other hand, treating the 'integrable' perturbation $\varepsilon\alpha(u + xu_x)$ in the spirit of usual PT, one would employ the unperturbed L equation and, consequently, quite inadequate approximation of the 'genuine' scattering data (compare (3) with (6)). We deem that this is the cause of appearance of the divergences in this case. Note also that, when $\alpha = 0$ (another integrable case), the effective potential in the 'perturbed' (L,A) pair $(4a)$ does not grow at $x \to \pm\infty$ and, accordingly, no divergences appear in PT.

Of course, in the integrable cases $\alpha = 0$ or $\beta = 0$, PT is not needed at all. Let us consider how to construct a divergence-free PT for a general (non-integrable) perturbation of the type (1). The above observation suggests that in this case, on a level with the A equation, the L equation should be properly modified as well. A natural way to realise this idea is to construct a (L,A) pair for (1) in the form of a power series in ε (such an approach is not new: something similar was discussed by Kodama (1986) and Menyuk (1986)):

$$-\Psi_{xx} + \left(u + \sum_{n=1}^{\infty} \varepsilon^n u_n - E\right)\Psi = 0 \tag{7a}$$

$$\Psi_t - 2\left(u + \sum_{n=1}^{\infty} \varepsilon^n w_n + 2E\right)\Psi_x + \left(u + \sum_{n=1}^{\infty} \varepsilon^n w_n\right)_x \Psi = 0. \tag{7b}$$

Straightforward calculations yield the following expressions for the two lowest terms of the series:

$$u_1 = 2E(\beta - \alpha)t - \tfrac{1}{2}\alpha x \tag{8a}$$

$$w_1 = 4E(\alpha - \beta)t - \tfrac{1}{2}\beta x \tag{8b}$$

$$u_2 = -2E(\beta - \alpha)^2 t^2 + \tfrac{3}{2}\alpha\beta t x + v \tag{9a}$$

$$w_2 = 4E(\alpha - \beta)^2 t^2 + \tfrac{3}{2}\alpha\beta t x + v \tag{9b}$$

where $v(x, t)$ satisfies the linear equation

$$v_t + 6(uv)_x + v_{xxx} = I\alpha\beta t(xu)_x. \tag{9c}$$

† In both the integrable cases $\alpha = 0$ or $\beta = 0$, equation (1) can be reduced to the unperturbed KdV equation by means of simple transforms of variables.

‡ All the information about $u(x)$ is mapped into the phase of the reflection coefficient arg $\tau(E)$.

It is easy to see that the effective potential in (7a) contains the term $\sim x$ already in the first order of PT (see (8a)). The only exception is the case $\alpha = 0$ (the same pertains to the second order; see (9a)). It is remarkable that, as mentioned above, this is actually the only case when divergences do not appear in the standard PT.

We remark that a correct version of PT can be based upon the truncation of the series in (7). However, due to the secular dependence of u_n and w_n on t (see (8) and (9)), such truncated expansions cease to be applicable at the times $t \gtrsim \varepsilon^{-1}$. On the other hand, comparison of the expressions (7)-(9) in the exactly integrable cases with the corresponding exact (L,A) pairs (4) and (5) demonstrates that one may hope to sum up all the secular terms into something like $\exp\{\varepsilon f(\alpha, t)t\}$. Another approach can be based on the fact that (1) does not depend explicitly upon the time t. Hence, we can solve it with the aid of the truncated (L,A) pair by steps, i.e. several times for the time interval $0 < t < \Delta$ ($\Delta \sim 1$), the initial data for each step being determined by the 'final' data of the preceding step:

$$u_m\big|_{t=0} = u_{m-1}\big|_{t=\Delta}.$$

The count of time in each step begins from zero, thus, the secular effects do not accumulate, the step's number playing the role of a slow time variable. Efforts to construct such a technique are in progress.

At the same time, there is a more simple approach to the problem. It is easy to see that the transformation

$$\begin{aligned} t \to \tilde{t} &= (3\beta\varepsilon)^{-1}[1 - \exp(-3\beta\varepsilon t)] \\ x \to \tilde{x} &= x \exp(-\beta\varepsilon t) \\ u \to \tilde{u} &= u \exp(-2\beta\varepsilon t) \end{aligned} \tag{10}$$

eliminates the second term in the RHS of (1):

$$\tilde{u}_{\tilde{t}} - 6\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = \gamma(\tilde{t})(\tilde{u} + 2\tilde{u}_{\tilde{x}}\tilde{x}) \tag{11a}$$

$$\gamma(\tilde{t}) = -\varepsilon\alpha/(1 - 3\varepsilon\beta\tilde{t}). \tag{11b}$$

Contrary to (1) with $\beta = 0$, equation (11) is not exactly integrable due to the presence of the time-dependent coefficient γ. Nevertheless, if a characteristic timescale of a solution that we are interested in is much smaller than a characteristic time during which the coefficient $\gamma(t)$ significantly alters, we may treat (11a) in the adiabatic approximation, i.e. first assume γ = constant and write a corresponding exact solution and then formally insert into it the function (11b) instead of the constant.

To illustrate this scheme, let us apply it to the following exact solution of (11a) with γ = constant (Calogero and Degasperis 1982):

$$\tilde{u} = (\tfrac{1}{2}\gamma)^{2/3} U((\tfrac{1}{2}\gamma)^{1/3}\tilde{x} - z(\tilde{t}), \rho(\tilde{t})) \tag{12a}$$

$$U(y, \rho) = 2\rho[2\mathrm{Ai}'(y)\mathrm{Ai}(y) + \rho(\mathrm{Ai}(y))^4 G(y)]G(y) \tag{12b}$$

$$G(y) = [1 + \rho(\mathrm{Ai}'(y))^2 - \rho y(\mathrm{Ai}(y))^2]^{-1} \tag{12c}$$

where $\mathrm{Ai}(y)$ is the Airy function, and $z(\tilde{t}) = z_0 \exp(-2\gamma\tilde{t})$, $\rho(\tilde{t}) = \rho_0 \exp(-2\gamma\tilde{t})$. Replacing in the latter expressions the constant γ by the function (11b), we obtain

$$\begin{aligned} z(\tilde{t}) &= z_0(1 - 3\varepsilon\beta\tilde{t})^{-2\alpha/3\beta} \\ \rho(\tilde{t}) &= \rho_0(1 - 3\varepsilon\beta\tilde{t})^{-2\alpha/3\beta}. \end{aligned} \tag{12d}$$

The formulae (12) give an approximate solution to (11a) provided $|\varepsilon|\tilde{t} \gg 1$. According to (10), this condition has sense only for the case $\beta\varepsilon < 0$, i.e. for the pumping type perturbation, when it means simply $|\varepsilon|t \gg 1$. So, on the contraty to the standard perturbative technique (Newell 1980) which is relevant for the early stage of evolution ($|\varepsilon|t \leqslant 1$), our approach works at the late stage and is meaningful only for the pumping case. In particular, the approximate solution (12) may be interpreted as that describing generation of an infinite number of new solitons (see a graph of the function $U(y, \rho)$ in Calogero and Degasperis (1982)) at the late stage of evolution of an initial profile $u_0(x)$. Note, however, that due to the property $\int_{-\infty}^{\infty} U(y, \rho)\, dy = 0$ this may pertain only to profiles restricted by the constraint $\int_{-\infty}^{\infty} u_0(x)\, dx = 0$. Earlier the singularity of PT was interpreted by Wright (1980) as just a manifestation of birth of new solitons.

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