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LETTER TO THE EDITOR

Perturbation theory for two-dimensional solitons

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Abstract. The perturbed Kadomtsev-Petviashvili equation is analysed for media with a split dispersion law. For arbitrary perturbations the method of asymptotic description of the equation solutions is constructed. Applications of this method are given for problems in hydrodynamics and plasma physics.

Let us consider the following equation:

$$\partial u/\partial t+\tfrac{1}{4}\partial^3 u/\partial x^3+\tfrac{3}{2}u\partial u/\partial x-\tfrac{3}{4}\int_{-\infty}^{x}\partial^2 u/\partial y^2(x',y,t)\,\mathrm{d}x'=\varepsilon P(u). \tag{1}$$

At $\varepsilon=0$ equation (1) is the Kadomtsev-Petviashvili (KP) equation for media with split dispersion law. The KP equation is used to describe magnetic-acoustic waves in plasma, gravity-capillary waves on the surface of the fluid, phonons in liquid helium, etc. It was shown (see Dryuma 1973, Zakharov and Shabat 1974, Manakov 1981, Ablowitz and Fokas 1983) that the KP equation is integrable via the inverse scattering transform and has wave solutions localised in space, namely two-dimensional solitons. If $\varepsilon\neq 0$ equation (1) includes such important effects as dissipation in wave media, scattering of waves on spatial inhomogeneity, etc.

In this letter we construct a perturbation theory for equation (1) for $0<\varepsilon\ll 1$ and arbitrary perturbations $P[u]$. We note that asymptotic methods have been proposed so far only for one-dimensional nearly integrable systems (see, for instance, Kaup 1976, Karpman and Maslov 1977).

Let us consider the initial value problem for equation (1). If the initial field $u(x,y,0)$ decreases when $x^2+y^2\to\infty$ and satisfies $\int_{-\infty}^{\infty}u(x,y,0)\,\mathrm{d}x=0$, then for $t>0$ we can link the solution of equation (1) with the so-called 'scattering data' by the following formulae (see Ablowitz and Fokas 1983):

$$u(x,y,t)=2(\partial/\partial x)\Big(\varphi^+(x,y,t)+\varphi^-(x,y,t)+(1/2\mathrm{i}\pi)\iint_{-\infty}^{\infty}f(k_1,k_2,t)\mathrm{e}^{\mathrm{i}\theta}\chi(k_2,x,y,t)\,\mathrm{d}k_1\,\mathrm{d}k_2\Big). \tag{2}$$

Here $\theta=(k_2-k_1)x-(k_2^2-k_1^2)y$. Functions $\varphi^\pm$ and χ satisfy the following equations:

$$\chi(k,x,y,t)=1+\frac{\mathrm{i}\varphi^+(x,y,t)}{k-k^+(t)}+\frac{\mathrm{i}\varphi^-(x,y,t)}{k-k^-(t)}+\frac{1}{2\mathrm{i}\pi}\iint_{-\infty}^{\infty}\frac{f(k_1,k_2,t)}{k_1-k+\mathrm{i}0}\mathrm{e}^{\mathrm{i}\theta}\chi(k_2,x,y,t)\,\mathrm{d}k_1\,\mathrm{d}k_2 \tag{3}$$

$$[x-2k^{\pm}(t)y-\gamma^{\pm}(t)]\varphi^{\pm}(x,y,t)-\frac{\mathrm{i}\varphi^{\mp}(x,y,t)}{k^{\pm}(t)-k^{\mp}(t)}$$
$$=1+\frac{1}{2\mathrm{i}\pi}\iint_{-\infty}^{\infty}\frac{f(k_1,k_2,t)}{k_1-k^{\pm}(t)}\,\mathrm{e}^{\mathrm{i}\theta}\chi(k_2,x,y,t)\,\mathrm{d}k_1\mathrm{d}k_2$$

where $k^+=\bar{k}^-$, $\gamma^+=\bar{\gamma}^-$. Thus, the set of scattering data $\{k^-(t),\gamma^-(t),f(k_1,k_2,t)\}$† completely defines the field $u(x,y,t)$. To close formulae (2) and (3) we need evolutionary equations for scattering data. We present them, omitting intermediate calculations:

$$\mathrm{d}k^-/\mathrm{d}t=-(\varepsilon/2\mathrm{i}\pi)\iint_{-\infty}^{\infty}\bar{\varphi}^+(x,y,t)P(u(x,y,t))\varphi^-(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \tag{4a}$$

$$\mathrm{d}\gamma^-/\mathrm{d}t-3(k^-)^2=\varepsilon\Gamma(t) \tag{4b}$$

$$(\partial/\partial t)f(k_1,k_2,t)-\mathrm{i}(k_2^3-k_1^3)f(k_1,k_2,t)=-\varepsilon\pi S(k_1,k_1,k_2,t). \tag{4c}$$

Here

$$\Gamma=-\mathrm{i}B_1^-/2+\gamma^-B_0^-/2+\left[\left(\frac{\mathrm{i}B_0^++\gamma^-B_{-1}^+}{k^--k^+}\right)\left(\frac{\varphi^+(0,0,t)}{\varphi^-(0,0,t)}\right)\right]$$
$$+1/2\varphi^-(0,0,t)\iint_{-\infty}^{\infty}\frac{-\mathrm{i}S_0+\gamma^-S_{-1}}{k^--k_1}\chi(k_2,0,0,t)\,\mathrm{d}k_1\,\mathrm{d}k_2$$

where $B_j^{\pm}$ and $S_j^{\pm}$ are coefficients of the Loran series for functions $B^{\pm}(k,t)$ and $S(k,k_1,k_2,t)$ at $k\to k^-$:

$$B^{\pm}(k,t)=\pm\pi^{-1}\iint_{-\infty}^{\infty}\bar{\varphi}^{\pm}(x,y,t)P[u(x,y,t)]\chi(k,x,y,t)\,\mathrm{d}x\,\mathrm{d}y.$$

The function S is the solution of the equation

$$S(k,k_1,k_2,t)+\tfrac{1}{2}\mathrm{sgn}(k_2-k_1)\int_{-\infty}^{\infty}\mathrm{sgn}(k_1-m)f(m,k_2,t)S(k,k_1,m,t)\,\mathrm{d}m$$
$$=-(1/2\mathrm{i}\pi^2)\,\mathrm{sgn}(k_2-k_1)\iint_{-\infty}^{\infty}\bar{\chi}(k_2,x,y,t)$$
$$\times\chi(k,x,y,t)\,\mathrm{e}^{\mathrm{i}\theta}P[u(x,y,t)]\,\mathrm{d}x\,\mathrm{d}y.$$

Equations (4) are precise and at $\varepsilon=0$ (i.e. the KP equation) they can be trivially integrated (it is this very fact that is the basis of the inverse scattering transform). At $\varepsilon\neq 0$ equations (4) are not integrable, but for $0<\varepsilon\ll 1$ they can be analysed asymptotically via the method of successive approximations. In particular the first (adiabatic) approximation equations can be directly obtained by substituting the unperturbed functions $\varphi^{\pm}(x,y,t)$ and $\chi(k,x,y,t)$ into the right-hand sides of equations (4). These functions depend parametrically on $k^-(t)$ and $\gamma^-(t)$.

Let us now use the method proposed to solve two interesting physical problems. As the first example we consider the Kadomtsev–Petviashvili–Burgers equation (equation (1) for $P[u]=\partial^2u/\partial x^2$). This equation describes the propagation of waves in an 'almost' collisionless plasma and on the surface of an 'almost' ideal fluid. Suppose at

† To simplify the calculations we consider the case of one two-dimensional soliton with amplitude $a\sim 16(\mathrm{Im}\,k^-)^2$ and arbitrary 'non-soliton part' (described by a function $f(k_1,k_2,t)$).

the initial moment ($t=0$) we have only one two-dimensional soliton $u\ (x, y, 0)$:

$$u(x, y, 0)=4(\tfrac{1}{4}\eta_0^2+4\eta_0^2y^2-x^2)(\tfrac{1}{4}\eta_0^2+4\eta_0^2y^2+x^2)^{-2} \tag{5a}$$

i.e.

$$k^-(0)=-\mathrm{i}\eta_0 \qquad \gamma^-(0)=f(k_1, k_2, 0)=0. \tag{5b}$$

Then

$$\chi(k, x, y, 0)=1+(\mathrm{i}\varphi^+)/(k-\mathrm{i}\eta_0)+(\mathrm{i}\varphi^-)/(k+\mathrm{i}\eta_0) \tag{5c}$$

$$\varphi^\pm=(\pm\tfrac{1}{2}\eta_0\pm 2\mathrm{i}\eta_0 y+x)/(\tfrac{1}{4}\eta_0^2+4\eta_0^2y^2+x^2).$$

Substituting (5) into (4a) we get the first approximation equation for the amplitude of the propagating soliton:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t}=-16\varepsilon\eta^3 \qquad a(t)=16\eta^2(t)=a_0(1+2\varepsilon a_0 t)^{-1}.$$

As the second example, we consider the evolution of a gravity-capillary soliton in a fluid of variable depth or a plasma soliton in an inhomogeneous field. In these cases the perturbation may be represented in a general form: $\varepsilon P[u]=\varepsilon_1 u+\varepsilon_2(x-t)\partial u/\partial x$. We get:

$$\mathrm{d}\eta/\mathrm{d}t=(2\varepsilon_1-\varepsilon_2)\eta \qquad a(t)=a_0\exp[(4\varepsilon_1-2\varepsilon_2)t]$$

using general formulae (4) and (5). For $\varepsilon_1=\frac{1}{2}\varepsilon_2$ the soliton amplitude is unchanged in the first approximation. To define changes in soliton form and velocity one has to solve the evolutionary equations (4b) and (4c) for functions $f(k_1, k_2, t)$ and $\gamma^-(t)$ and reconstruct $u(x, y, t)$ via formulae (2) and (3) of the inverse scattering transform (we present the results in a more detailed paper).

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