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Dynamics of ideal fluid flows over an uneven bottom

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Two problems of the stability of ideal fluid flows over an uneven bottom are considered. The first is the study of stratified flow with a 'rigid lid'. We use the method of multiple scales to derive an equation describing the evolution of internal waves corresponding to different modes and wave vectors. For the case of sinusoidal bottom irregularities we have constructed a solution describing the increase in time of the internal wave field – this proves the instability of the basic flow. The phenomenon is interpreted as a result of interaction (mutual generation) of internal waves with energies of opposite signs. Our consideration is based on the Hamiltonian approach which enables us to prove in the most simple way the existence of waves carrying negative energy. The case of random (not sinusoidal) bottom irregularities is also studied. Using the kinetic equation for the amplitudes of internal waves derived in the paper, we have established that the basic flow remains unstable as well. In the second part of the paper we consider the homogeneous flows with a free upper boundary. It is shown that this problem can be reduced to the previous one, with the only difference being that the role of unstable perturbations is now played by the surface (not internal) gravity waves. The Hamiltonian approach is consistently applied and allows us to take into account the nonlinearity of waves.

1. Introduction

The study of non-viscous fluid flows over an uneven bottom is a classic problem of hydrodynamics. The simplest case is calculation of two-dimensional potential motion in a region with uneven boundaries. The solution of this problem is given by an appropriate conformal mapping and can be found in any hydrodynamics textbook. However, the first natural complication of the problem, i.e. consideration of flows with a free upper boundary, meets certain difficulties and naive attempts to generalize the results obtained fail. In fact, in this case the system acquires a new degree of freedom, namely, surface gravity waves, and the mathematical statement of the problem becomes essentially nonlinear. We note also, that since the problem of flows over bottom topography is of great importance to physical oceanography, we are forced to give up consideration of two-dimensional problems. Besides, owing to the density stratification and rotation of the Earth, water motion becomes predominantly vertical. All these factors cause two other types of wave motion – internal gravity and Rossby waves.

The study of the influence of an uneven boundary upon the dynamics of an ideal fluid flow can be split into two problems – the calculation of the 'topographic' (steady) flow and the investigation of stability as a whole. The first problem has been widely discussed in various papers (cf. Krauss 1966 and Scorer 1978 and references therein), while a relatively small number of papers have considered the second problem, the most important being those by Yih (1976) and Charney & Flierl (1981). Yih studied the stability of topographic† surface and internal waves in the homogeneous two-layer fluid over sinusoidal topography. The instability discovered was interpreted as a type of Benjamin–Feir (1967) instability. Charney & Flierl studied the mathematically simpler case of homogeneous zonal flow on the β-plane also over sinusoidal topography. It appeared that in a certain range of parameters such flow is unstable to the increase of small-amplitude wave perturbations (barotropic Rossby waves). Note that both papers considered flows without any velocity shear.

In our previous papers, we have made an attempt to explain from a single point of view the results obtained earlier and to generalize them (Benilov & Chernyak 1985; Benilov 1985). The first paper considered stratified shear flow over sinusoidal topography. Within the bounds of linear two-dimensional equations we managed to construct the asymptotic solution describing an exponentially growing internal-wave perturbation, so that the instability of the flow was proved. We have also proposed an interpretation of the instability based on an analogy with the explosive instability of waves in nonlinear media (on the explosive instability see, e.g. Craik 1985 and Zakharov 1974). In fact, the spatial inhomogeneity of a parameter (the depth of fluid) results in wave interaction, and in this sense the influence of the inhomogeneity is quite similar to that of the wave medium nonlinearity. Further, we interpreted the discovered instability as the interaction (mutual generation) of waves with energies of opposite signs.‡ Note that nonlinear effects can result in the interaction of internal waves with energies of opposite signs, i.e. the explosive instability proper (Voronovich & Rybak 1978). However, this type of instability takes place only in a rather narrow range of parameters, while the instability caused by bottom topography develops, as it turns out, in nearly any stratified flow. Simple physical interpretation enables us to predict the development of the instability in all moving isotropic media with spatial stationary inhomogeneity of parameters, since, as we shall see later, the Doppler shift of wave frequency is sufficient for waves with energies of different signs to occur. Somewhat more complicated is the case of anisotropic systems. Thus, for example, the problem of stability of zonal flow on the β-plane over sinusoidal topography has unstable solutions only for eastward flow (Charney & Flierl 1981).

We should note that sinusoidal topography is an idealization that has nothing to do with real ocean conditions. That is why it is of great importance to study the influence of random (not sinusoidal) topography upon the dynamics of fluid flows. This problem was studied for the example of zonal flow on the β-plane by Benilov (1985). It was found that instability of the eastward flow develops in this case as well, but its growth rate is considerably smaller.

As was noted above, the influence of spatial inhomogeneity of the media parameters on the wave propagation is in a sense similar to the influence of nonlinear effects. That is why we use the Hamiltonian approach to generalize and extend the results obtained earlier. This approach was worked out in general by Zakharov (1968, 1974) for problems of nonlinear wave dynamics.

We shall discuss two problems concerning the stability of flows over topography:

† Forced waves beyond the obstacle are also called lee waves.

‡ Let us from the very beginning elucidate the mechanism causing the appearance of waves with negative energies. If the internal-wave vector is directed against the flow and its phase velocity with the flow, then, as we shall see below, the wave slows the flow down, the energy of the whole system decreases and we can say that the wave carries negative energy.

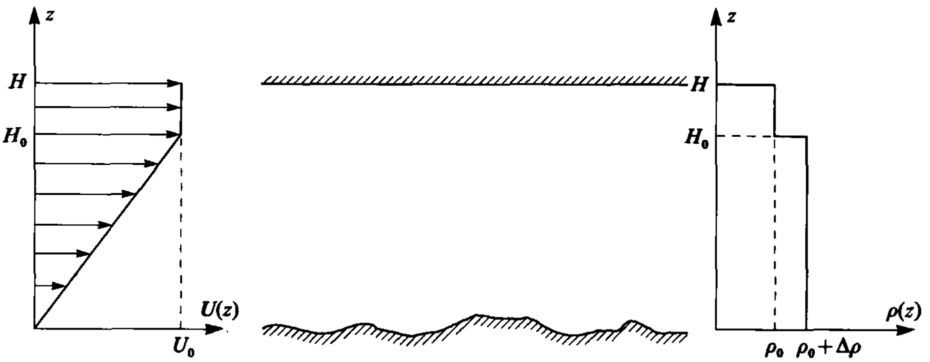

FIGURE 1. An example of a stratified flow (the model (33)).

the stratified flow with a 'rigid lid' approximation, and the flow of homogeneous fluid with a free boundary. The latter problem is used to study the influence of near-bottom flow (topographic surface waves) and nonlinear effects, which are especially easy to take into account when using the Hamiltonian formulation of the problem. Another advantage of the Hamiltonian approach is that the concept of waves with negative energies, essential to the paper, obtains a natural mathematical expression.

2. Statement of the problems and the governing equations

Consider the parallel shear flow of an ideal stratified fluid (cf. figure 1) with two functional parameters: the mean velocity field $U = U(z)$ and the Brunt–Väisälä frequency field

$$N = N(z) = \left(-\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{\frac{1}{2}},$$

where $\rho = \rho(z)$ is the density of the fluid, g is the gravitational acceleration, and z is the vertical coordinate (the z-axis has an upward direction). We study the flow in the region $H \geqslant z \geqslant h(\boldsymbol{r})$, where H is the mean depth of the basin and $h(\boldsymbol{r})$ describes the bottom topography ($\boldsymbol{r} = (x, y)$ is the horizontal coordinate). We discuss first the simple case when there is no near-bottom flow, i.e. we assume that

$$U(z) \equiv 0 \quad \text{for } z \leqslant \max\{h(\boldsymbol{r})\} \tag{1}$$

and the velocity of the mean flow has no vertical component. To describe small oscillations of the fluid, we use the linearized hydrodynamic equations in the Boussinesq approximation (Miropol'sky 1981):

$$\mathrm{D}_t\boldsymbol{u} + w\frac{\mathrm{d}U}{\mathrm{d}z} + \frac{\boldsymbol{\nabla}P}{\rho_0} = 0, \quad (\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \frac{\partial w}{\partial z} = 0, \tag{2a}$$

$$\mathrm{D}_t w + \sigma + \frac{1}{\rho_0}\frac{\partial P}{\partial z} = 0, \quad \mathrm{D}_t\sigma - N^2 w = 0. \tag{2b}$$

Here $w(z, \boldsymbol{r}, t)$ and $\boldsymbol{u}(z, \boldsymbol{r}, t)$ are respectively the vertical and horizontal components of the velocity field caused by the internal-wave perturbations; $\sigma = g\rho'/\rho_0$ ($\rho'(z, \boldsymbol{r}, t)$ is the local density deviation from $\rho(z)$), ρ_0 is the value of the fluid density averaged over the depth of the layer, and $P(z, \boldsymbol{r}, t)$ is the pressure deviation from the hydrostatic value. Differential operators D_t, $\boldsymbol{\nabla}$ and ∇^2 are given by:

$$\mathrm{D}_t = \frac{\partial}{\partial t} + (\boldsymbol{U}\cdot\boldsymbol{\nabla}), \quad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \nabla^2 = (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}).$$

The set of equations (2) can be reduced to the single equaton for $w(z, \boldsymbol{r}, t)$:

$$\mathrm{D}_{tt}^2\left(\frac{\partial^2 w}{\partial z^2}+\nabla^2 w\right)-\left(\frac{\mathrm{d}^2 \boldsymbol{U}}{\mathrm{d}z^2}\cdot\boldsymbol{\nabla}\right)\mathrm{D}_t w+N^2\nabla^2 w=0. \tag{3}$$

At the surface we use the 'rigid lid' approximation:

$$w=0 \quad \text{for } z=H, \tag{4}$$

and at the lower boundary we use the generalized no-flow boundary condition, which together with (1) can be written in the form

$$w=(\boldsymbol{u}\cdot\boldsymbol{\nabla}h) \quad \text{when } z=h(\boldsymbol{r}). \tag{5}$$

To close (3)–(5), we have to express $\boldsymbol{u}$ in terms of w and substitute it in boundary condition (5). Starting from (2a), one can easily obtain the following equation:

$$\mathrm{D}_t\nabla^2\boldsymbol{u}=\boldsymbol{\nabla}\left[-\mathrm{D}_t\frac{\partial w}{\partial z}+\left(\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\cdot\boldsymbol{\nabla}w\right)\right]-\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\nabla^2 w. \tag{6}$$

We note that in view of geophysical applications we can assume that the value of the parameter $\max|h(\boldsymbol{r})|/H$ is considerably less than unity and it is possible to study (3)–(6) by asymptotic methods. The first step is to continue the boundary condition (3) to the level $z=0$ and expand it in a series in powers of h, truncating to the first (linear with respect to h) corrections to the zeroth approximation

$$w=-h\frac{\partial w}{\partial z}+(\boldsymbol{u}\cdot\boldsymbol{\nabla}h)=(\boldsymbol{\nabla}\cdot h\boldsymbol{u}) \quad \text{when } z=0. \tag{7}$$

With sufficient accuracy (6) yields

$$\boldsymbol{\nabla}^2\boldsymbol{u}=-\boldsymbol{\nabla}\frac{\partial w}{\partial z} \quad \text{when } z=0 \tag{8}$$

and we should rewrite (1) in the form

$$\boldsymbol{U}(0)=0. \tag{9}$$

Now we employ Fourier transforms of the fields w, $\boldsymbol{u}$ and h (transforms are in the horizontal plane only and are marked with index $\boldsymbol{k}$, for example:

$$w_{\boldsymbol{k}}(z,t)=(2\pi)^{-2}\int w(z,\boldsymbol{r},t)\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\,\mathrm{d}\boldsymbol{r} \qquad).$$

Let us introduce an auxiliary variable

$$W_{\boldsymbol{k}}=\begin{pmatrix}w_{\boldsymbol{k}}\\ v_{\boldsymbol{k}}\end{pmatrix},$$

where

$$v_{\boldsymbol{k}}=\mathrm{i}\frac{\partial w_{\boldsymbol{k}}}{\partial t}+(\boldsymbol{k}\cdot\boldsymbol{U})\,w_{\boldsymbol{k}} \tag{10}$$

(i.e. $v(z,\boldsymbol{r},t)=\mathrm{i}\,\mathrm{D}_t w(z,\boldsymbol{r},t)$). Using the new variables, (3), (4), (7) and (8) can be reduced after some calculations to the following boundary-value problem:

$$\mathrm{i}\hat{\mathbf{M}}_{\boldsymbol{k}}\frac{\partial W_{\boldsymbol{k}}}{\partial t}=\hat{\mathbf{L}}_{\boldsymbol{k}}W_{\boldsymbol{k}}; \tag{11a}$$

$$\left.\begin{aligned} w_{\boldsymbol{k}}|_{z=H}&=0,\\ w_{\boldsymbol{k}}|_{z=0}&=-\iint\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{k_1^2}\left.\frac{\partial w_{\boldsymbol{k}_1}}{\partial z}\right|_{z=0}h_{\boldsymbol{k}_2}\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2,\end{aligned}\right\} \tag{11b}$$

where $\delta(\boldsymbol{k})$ is the Dirac delta-function and matrix differential operators $\hat{\mathbf{M}}_k$ and $\hat{\mathbf{L}}_k$ are given by

$$\hat{\mathbf{M}}_k = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\partial^2}{\partial z^2} - k^2 \end{pmatrix},$$

$$\hat{\mathbf{L}}_k = \begin{pmatrix} -(\boldsymbol{k}\cdot\boldsymbol{U}) & 1 \\ -\left[k^2N^2 + 2\left(\boldsymbol{k}\cdot\dfrac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right)^2\right] & -(\boldsymbol{k}\cdot\boldsymbol{U})\left(\dfrac{\partial^2}{\partial z^2} - k^2\right) + 2\dfrac{\partial}{\partial z}\left(\boldsymbol{k}\cdot\dfrac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right) \end{pmatrix}.$$

In spite of the linearity, the boundary-value problem (11) is too complicated to be solved analytically. We shall explore it by using an asymptotic technique similar to the multiple-scale method (e.g. Nayfeh 1973), based on the fact that parameter h/H is small. We shall look for the solution of (11) in the form of a series in powers of h_k:

$$W_k = W_k^{(0)} + W_k^{(1)} + \dots .$$

As the zeroth approximation we take the solution describing the superposition of free internal waves of different modes with amplitudes A_k^ν slowly varying in time (ν is the mode number):

$$W_k^{(0)} = \sum_{\nu=-\infty}^{\infty} A_k^\nu(t)\exp(\mathrm{i}\omega_k^\nu t)\,\Phi_k^\nu(z), \quad \left|\frac{\partial A_k^\nu}{\partial t}\right| \ll |\omega_k^\nu A_k^\nu|. \tag{12}$$

Here ω_k^ν is the frequency,

$$\Phi_k^\nu = \begin{pmatrix} \phi_k^\nu \\ \psi_k^\nu \end{pmatrix}$$

describes the vertical structure of the waves of mode number ν, and for $\nu \neq 0$† satisfies the standard boundary-value problem for internal waves (although written in a somewhat unusual form):

$$\omega_k^\nu \hat{\mathbf{M}}_k \Phi_k^\nu + \hat{\mathbf{L}}_k \Phi_k^\nu = 0, \quad \phi_k^\nu|_{z=0} = \phi_k^\nu|_{z=H} = 0, \tag{13a}$$

and normalization condition

$$\langle \Phi_k^\nu | \Phi_k^\nu \rangle \equiv \int_0^H [(\phi_k^\nu)^2 + (\psi_k^\nu)^2]\,\mathrm{d}z = 1. \tag{13b}$$

The boundary-value problem (13) is invariant with respect to the replacement $\boldsymbol{k} \to -\boldsymbol{k}$, $\omega_k^\nu \to -\omega_k^\nu$, $\phi_k^\nu \to \phi_k^\nu$, $\psi_k^\nu \to -\psi_k^\nu$. This enables us to choose the enumeration of modes guaranteeing the validity of the following equalities:

$$\phi_{-k}^{-\nu} = \phi_k^\nu, \quad \psi_{-k}^{-\nu} = -\psi_k^\nu, \quad \omega_{-k}^{-\nu} = -\omega_k^\nu \tag{14a}$$

and to require that the amplitudes of the waves should satisfy analogous relations

$$A_{-k}^{-\nu} = \overset{*}{A}{}_k^\nu \tag{14b}$$

where the asterisk over A_k^ν means complex conjugation. It should be emphasized that relations (14) are chosen so that $w(z, \boldsymbol{r}, t)$ will be real (or so that the equivalent condition $w_{-k} = \overset{*}{w}_k$ holds). In order to fix finally the mode enumeration we shall use the well-known possibility of choosing the signs of ω_k^ν for $\nu > 0$ so that they will coincide with the signs of the energies of the appropriate waves (Voronovich 1979). Omitting technical details we note that this procedure leads to $\omega_k^\nu > 0$ for $(\boldsymbol{k}\cdot\boldsymbol{U}) > 0$ and $\omega_k^\nu < 0$ in a certain range of waves, for which the flow is reversed (cf. figure 2).

† We assume the value $\nu = 0$ to be omitted in (12) and in all subsequent summations.

Such waves diminish the energy of the mean flow, slowing it down, and in our terminology they have negative energy.†

Later we shall need the solution of the boundary-value problem adjoint to the problem (13 a, b)

$$\omega_k^\nu \hat{\mathbf{M}}_k^+ \tilde{\Phi}_k^\nu + \hat{\mathbf{L}}_k^+ \tilde{\Phi}_k^\nu = 0, \quad \tilde{\psi}_k^\nu|_{z=0} = \tilde{\psi}_k^\nu|_{z=H} = 0 \tag{15}$$

where $\hat{\mathbf{M}}_k^+ = \hat{\mathbf{M}}_k$ (since $\hat{\mathbf{M}}_k$ is a self-adjoint operator) and

$$\hat{\mathbf{L}}_k^+ = \begin{pmatrix} -(\boldsymbol{k}\cdot \boldsymbol{U}) & -\left[k^2N^2+2\left(\boldsymbol{k}\cdot\dfrac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right)^2\right] \\ 1 & -\left(\dfrac{\partial^2}{\partial z^2}-k^2\right)(\boldsymbol{k}\cdot\boldsymbol{U})-2\left(\boldsymbol{k}\cdot\dfrac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right)\dfrac{\partial}{\partial z} \end{pmatrix}.$$

One can ascertain a connection between the solutions of (13) and (15):

$$\phi_k^\nu = [\omega_k^\nu - (\boldsymbol{k}\cdot\boldsymbol{U})]^2 \tilde{\psi}_k^\nu. \tag{16}$$

(This relation is of great importance for the subsequent calculations.)

We require that the Miles (1961) stability criterion holds: $Ri = N^2/(\partial U/\partial z)^2 \geqslant \frac{1}{4}$, so that the boundary-value problems (13) and (15) describe the countable number of different modes with real dispersion laws $\omega = \omega_k^\nu$. We can also write the obvious orthogonality relation between the eigenfunctions of different modes:

$$\langle \tilde{\Phi}_k^\nu | \hat{\mathbf{M}}_k | \Phi_k^\mu \rangle \equiv \int_0^H \left[\tilde{\phi}_k^\nu \phi_k^\mu + \tilde{\psi}_k^\nu \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi_k^\mu\right] \mathrm{d}z = l_k^\nu \delta_{\nu\mu} \tag{17}$$

Here $\delta_{\nu\mu}$ is the Kronecker delta, and l_k^ν after some calculations attains the form

$$l_k^\nu = \int_0^H \left\{ \frac{2k^2N^2}{[\omega_k^\nu - (\boldsymbol{k}\cdot\boldsymbol{U})]^3} + \frac{\left(\dfrac{\boldsymbol{k}\cdot\mathrm{d}^2\boldsymbol{U}}{\mathrm{d}z^2}\right)}{[\omega_k^\nu - (\boldsymbol{k}\cdot\boldsymbol{U})]^2} \right\} (\phi_k^\nu)^2 \, \mathrm{d}z. \tag{18}$$

Now we derive equations for the first approximation to $W_k^{(1)}$:

$$\mathrm{i}\hat{\mathbf{M}}_k \frac{\partial W_k^{(1)}}{\partial t} + \mathrm{i} \sum_{\nu=-\infty}^{\infty} \frac{\partial A_k^\nu}{\partial t} \exp(\mathrm{i}\omega_k^\nu t)\, \hat{\mathbf{M}}_k \Phi_k^\nu = \hat{\mathbf{L}}_k W_k^{(1)}, \tag{19a}$$

$$w_k^{(1)}|_{z=H} = 0, \tag{19b}$$

$$w_k^{(1)}|_{z=0} = -\iint \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{k_1^2}\left[\sum_{\nu=-\infty}^{\infty} A_k^\nu \exp(\mathrm{i}\omega_k^\nu t) \left.\frac{\partial \phi_k^\nu}{\partial z}\right|_{z=0}\right] \times h_{k_2} \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2. \tag{19c}$$

Note that the solution of (19) is not unique. In fact, if $W_k^{(1)} = f_k(z,t)$ gives a solution to (19), then

$$W_k^{(1)} = f_k(z,t) + \sum_{\nu=-\infty}^{\infty} C_k^\nu \exp(\mathrm{i}\omega_k t)\, \Phi_k^\nu(z),$$

where C_k^ν are arbitrary functions of $\boldsymbol{k}$, also solves the equation. To make the solution unique, we need an additional condition. We require that $W_k^{(1)}$ is orthogonal to all the eigenfunctions $\tilde{\Phi}_k^\nu$ of the adjoint boundary-value problem (15) in the sense of the scalar product:

$$\langle W_k^{(1)} | \hat{\mathbf{M}}_k | \tilde{\Phi}_k^\nu \rangle = 0. \tag{20}$$

† In the simplest case of smooth dispersion curves ($|\partial\omega_k^\nu/\partial \boldsymbol{k}| < \infty$) we can only ascertain that the choice of frequency sign guarantees the validity of the unequality $\omega_k^\nu > 0$ for all $\boldsymbol{k}$ as $U \to 0$ (in the absence of the flow the energy of internal waves is known to be positive).

(The reasons for choosing this condition will be explained later.) We now multiply (19*a*) by $\tilde{\Phi}_k^\nu$ and integrate it with respect to z from 0 to H. Integrating by parts and making use of (15), (17) and (19*b, c*), we have

$$\left(\mathrm{i}\frac{\partial}{\partial t}+\omega_k^\nu\right)\langle W_k^{(1)}|\hat{\mathbf{M}}_k|\tilde{\Phi}_k^\nu\rangle-\mathrm{i}\left(\frac{\partial v_k^{(1)}}{\partial t}\frac{\partial\tilde{\psi}_k^\nu}{\partial z}\right)\bigg|_{z=0}+\mathrm{i}\frac{\partial A_k^\nu}{\partial t}\exp(\mathrm{i}\omega_k^\nu t)\,l_k^\nu=0. \quad (21)$$

Now, taking into account (9), (10) and (20), we obtain a closed set of equations for the amplitudes A_k^ν:

$$\mathrm{i}\frac{\partial A_k^\nu}{\partial t}=\sum_{\nu_1=-\infty}^{\infty}\iint\left\{\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{l_k^\nu}\left(\frac{\omega_{k_1}^{\nu_1}}{k_1}\right)^2\left(\frac{\partial\phi_{k_1}^{\nu_1}}{\partial z}\frac{\partial\tilde{\psi}_k^\nu}{\partial z}\right)\bigg|_{z=0}\right\}$$
$$\times\exp[\mathrm{i}(\omega_{k_1}^{\nu_1}-\omega_k^\nu)t]\,A_{k_1}^{\nu_1}\overset{*}{h}_{k_2}\,\delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2. \quad (22)$$

We note that in the wave spectrum described by the set of the functions A_k^ν there are pairs of harmonics $(\boldsymbol{k},\omega_k^\nu)$ and $(\boldsymbol{k}_1,\omega_{k_1}^{\nu_1})$ interacting resonantly with the 'help' of one of the harmonics h_{k_2}. The frequencies and wave vectors of these waves satisfy the resonant relations $\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2=0$, $\omega_k^\nu-\omega_{k_1}^{\nu_1}=0$. Clearly, in this case it is condition (20) (and, consequently, (22)) that must hold in order to eliminate the increasing secular terms $\sim t\exp(\mathrm{i}\omega_k^\nu t)$ in the solution of (19). This fact determines the choice of the condition (20). Let us now change the limits of summation with respect to ν in (22) from $(-\infty,\infty)$ to $(1,\infty)$ (with the help of (14)) and introduce a new field variable a_k^ν:

$$a_k^\nu=A_k^\nu\exp(\mathrm{i}\omega_k^\nu t)\,(l_k^\nu)^{\frac{1}{2}}\frac{(\omega_k^\nu)^2}{k} \quad (23)$$

Substituting (23) into (22) and making use of (16), we obtain after some straightforward calculations

$$\frac{\partial A_k^\nu}{\partial t}-\mathrm{i}\omega_k^\nu a_k^\nu=\mathrm{i}\sum_{\nu_1=1}^{\infty}\iint\{U_{kk_1}^{\nu\nu_1}a_{k_1}^{\nu_1}\overset{*}{h}_{k_2}\delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)$$
$$+V_{kk_1}^{\nu\nu_1}\overset{*}{a}_{k_1}^{\nu_1}\overset{*}{h}_{k_2}\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2, \quad (24a)$$

$$V_{kk_1}^{\nu\nu_1}=-U_{kk_1}^{\nu\nu_1}=\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{kk_1}(l_k^\nu l_{k_1}^{\nu_1})^{-\frac{1}{2}}\left(\frac{\partial\phi_k^\nu}{\partial z}\frac{\partial\phi_{k_1}^{\nu_1}}{\partial z}\right)\bigg|_{z=0}. \quad (24b)$$

Equation (24) keeps the main properties of the original dynamics and is the basis of all the results presented below. The quantity a_k^ν is the so-called normalized wave amplitude.† In therms of a_k^ν the structure functions $U_{kk_1}^{\nu\nu_1}$ and $V_{kk_1}^{\nu\nu_1}$ (matrix elements) have the symmetric form necessary for subsequent calculations.

3. The Hamiltonian statement of the problem

We first note that depending on the specific form of the coefficients ω_k^ν, $U_{kk_1}^{\nu\nu_1}$ and $V_{kk_1}^{\nu\nu_1}$, (24*a*) can describe any dynamic system with spatial inhomogeneity of parameters. For instance, the problem of the evolution of a barotropic Rossby-wave spectrum over the bottom topography can also be reduced to an equation of the type (24*a*) (Benilov 1985). But in the case of conservative systems there are a number of restrictions on the coefficients in (24*a*), namely

$$U_{k_1k}^{\nu_1\nu}=\overset{*}{U}_{kk_1}^{\nu\nu_1},\quad V_{k_1k}^{\nu_1\nu}=V_{kk_1}^{\nu\nu_1}. \quad (25)$$

† Henceforth we use the terminology of nonlinear wave dynamics.

(In our case (25) holds due to the key inequality

$$l_k^\nu \geqslant 0 \tag{26}$$

proved in Appendix A.)† In fact, for conservative systems the energy integral (the Hamiltonian) must be conserved. It has a form

$$\mathscr{H} = \sum_\nu \int \omega_k^\nu |a_k^\nu|^2 \, \mathrm{d}\boldsymbol{k} + \sum_\nu \sum_{\nu_1} \iiint \{U_{kk_1}^{\nu\nu_1} \overset{*}{a}{}_k^\nu a_{k_1}^{\nu_1} \overset{*}{h}_{k_2} \, \delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)$$
$$+ \tfrac{1}{2}(V_{kk_1}^{\nu\nu_1} \overset{*}{a}{}_k^\nu \overset{*}{a}{}_{k_1}^{\nu_1} \overset{*}{h}_{k_2} + \overset{*}{V}{}_{kk_1}^{\nu\nu_1} a_k^\nu a_{k_1}^{\nu_1} h_{k_2})\} \, \delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2) \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2, \tag{27}$$

while (25) guarantees that $\mathscr{H}$ is real and symmetric.‡) Now we can write (24 *a*) in Hamiltonian form:

$$\frac{\partial a_k^\nu}{\partial t} = \mathrm{i} \frac{\delta \mathscr{H}}{\delta \overset{*}{a}{}_k^\nu}; \tag{28}$$

here $\delta\mathscr{H}/\delta\overset{*}{a}{}_k^\nu$ denotes the variational derivative of the functional $\mathscr{H}$ with respect to $\overset{*}{a}{}_k^\nu$.

Representation (28) is canonical (with valency equal to i) for any dynamic equation written in terms of the normalized wave variable (Zakharov 1968, 1974). The relationship between a_k^ν and natural variables in specific physical systems often appears to be non-trivial, and in our case we only managed to ascertain it asymptotically (i.e. in (23) we truncate only to the first term in the expansion of $a_k^\nu[A_k^\nu, h_k]$ in powers of h_k, and the Hamiltonian $\mathscr{H}$ is a sum of the first two terms in the expansion of the energy of the original dynamic system). In principle, from the very beginning we could have used the Hamiltonian and canonical variables introduced by Voronovich (1979), but this results in much more complicated calculations.

Let us define the Poisson brackets between the functionals $F_1[a_k^\nu, \overset{*}{a}{}_k^\nu]$, $F_2[a_k^\nu, \overset{*}{a}{}_k^\nu]$ as

$$\{F_1, F_2\} = \mathrm{i} \sum_\nu \int \left(\frac{\delta F_1}{\delta a_k^\nu} \frac{\delta F_2}{\delta \overset{*}{a}{}_k^\nu} - \frac{\delta F_2}{\delta a_k^\nu} \frac{\delta F_1}{\delta \overset{*}{a}{}_k^\nu} \right) \mathrm{d}\boldsymbol{k}$$

and consider the following transforms of the field variable:

$$a_k^\nu = d_k^\nu + \sum_{\nu_1} \iint \{\tilde{U}_{kk_1}^{\nu\nu_1} d_{k_1}^{\nu_1} \overset{*}{h}_{k_2} \, \delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)$$
$$+ \tilde{V}_{kk_1}^{\nu\nu_1} \overset{*}{d}{}_{k_1}^{\nu_1} \overset{*}{h}_{k_2} \, \delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2. \tag{29}$$

We require that the dynamic equation (24 *a*) written in terms of the new variables d_k^ν should have a form (28) with the same Hamiltonian $\mathscr{H}$. One can easily see that in this case the following equalities must hold:

$$\{a_k^\nu, a_{k_1}^{\nu_1}\} = 0, \quad \{a_k^\nu, \overset{*}{a}{}_{k_1}^{\nu_1}\} = \mathrm{i}\delta_{\nu\nu_1} \delta(\boldsymbol{k}-\boldsymbol{k}_1). \tag{30}$$

† In the case of internal waves $U_{kk_1}^{\nu\nu_1}$ and $V_{kk_1}^{\nu\nu_1}$ appear to be real (cf. (24 *b*) and (26)) and the equalities (25) are consistent with (24 *b*) ($V_{kk_1}^{\nu\nu_1} = -U_{kk_1}^{\nu\nu_1}$).

‡ As we have expected the main part of the Hamiltonian ($\sim \omega_k^\nu |a_k^\nu|^2$) has the sign of the wave frequency and can be negative.

Relations (30) guarantee that the transform $a_k^\nu \to d_k^\nu$ is canonical. Applying (30) to the transform (29), we obtain restrictions on the structure functions: $\tilde{U}_{k_1 k}^{\nu_1 \nu} = -\tilde{U}_{kk_1}^{\nu\nu_1}$, $\tilde{V}_{k_1 k}^{\nu_1 \nu} = \tilde{V}_{kk_1}^{\nu\nu_1}$. One can see that by choosing

$$\tilde{U}_{kk_1}^{\nu\nu_1} = \frac{U_{kk_1}^{\nu\nu_1}}{\omega_k^\nu - \omega_{k_1}^{\nu_1}}, \quad \tilde{V}_{kk_1}^{\nu\nu_1} = 0 \tag{31a}$$

we can eliminate the term proportional to $U_{kk_1}^{\nu\nu_1}$ from the Hamiltonian (27) and equation (24*a*), while the transform (29) with

$$\tilde{U}_{kk_1}^{\nu\nu_1} = 0, \quad \tilde{V}_{kk_1}^{\nu\nu_1} = \frac{V_{kk_1}^{\nu\nu_1}}{\omega_k^\nu + \omega_{k_1}^{\nu_1}} \tag{31b}$$

terminates the term $\sim V_{kk_1}^{\nu\nu_1}$ (in both cases terms $\sim (h_k)^2$ appear which we have to drop within the bounds of the chosen accuracy). However, we note that since the denominators in (31) can vanish, non-integrable singularities can occur in the transforms (29), (31). In fact, if the sets of equations (resonant conditions)

$$\boldsymbol{k} - \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0, \quad \omega_k^\nu - \omega_{k_1}^{\nu_1} = 0; \tag{32a}$$

$$\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0, \quad \omega_k^\nu + \omega_{k_1}^{\nu_1} = 0 \tag{32b}$$

can be solved with respect to the vectors $\boldsymbol{k}$ and $\boldsymbol{k}_1$, then functions $\tilde{U}_{kk_1}^{\nu\nu_1}$ and $\tilde{V}_{kk_1}^{\nu\nu_1}$ respectively become singular. It is easy to see that (32*a*) can be solved for any $\boldsymbol{k}_2$ and the term $\sim U_{kk_1}^{\nu\nu_1}$ cannot be excluded from the Hamiltonian and dynamic equation (i.e. transferred into the term proportional to the next power of the small parameter). We note also that in the absence of flow, $\omega_k^\nu \geqslant 0$, conditions (32*b*) are not solvable and the term $\sim V_{kk_1}^{\nu\nu_1}$ *can* be excluded. This should be expected because as we shall see below this term in (24*a*) corresponds to instability, which is impossible in the absence of the flow.

To ascertain conditions of solvability of (32*b*) in the presence of the mean flow, we use the simplest two-layer model of stratification shown in figure 1:

$$\left.\begin{aligned} \rho(z) &= \rho_0 + \Delta\rho, \quad U(z) = U_0 \frac{z}{H_0} \quad \text{when } 0 \leqslant z \leqslant H_0; \\ \rho(z) &= \rho_0, \qquad\quad U(z) = U_0 \qquad \text{when } H_0 \leqslant z \leqslant H. \end{aligned}\right\} \tag{33}$$

In this case $$\frac{\partial^2 U}{\partial z^2} = -\frac{U_0}{H_0}\delta(z - H_0), \quad N^2 \approx g\frac{\Delta\rho}{\rho_0}\delta(z - H_0)$$

and the basic boundary-value problem describes only one mode of the internal waves with the dispersion law shown in figure 2. This simplest model describes satisfactorily the dynamics of the internal waves of the first mode in a shallow ocean with a quasi-homogeneous upper layer (the quantity $H - H_0$ means here the depth of pycnocline). One can see directly from figure 2(*a*) that the resonant conditions (32*b*) are solvable. We note one important fact, namely, that the existence of the waves with negative frequency is sufficient for (32*b*) to be solvable. This fact enables us to extend our study to the n-layer model with a piecewise-constant density approximation and broken-line velocity-profile approximation. Such a model describes $n-1$ internal-wave modes, each of them having a negative frequency in a certain range of the wave vector $\boldsymbol{k}$. We shall not discuss models with continuous fields $\mathrm{d}^2U(z)/\mathrm{d}z^2$ and $N^2(z)$, since in these cases, generally speaking, the so-called critical layers can occur (cf.

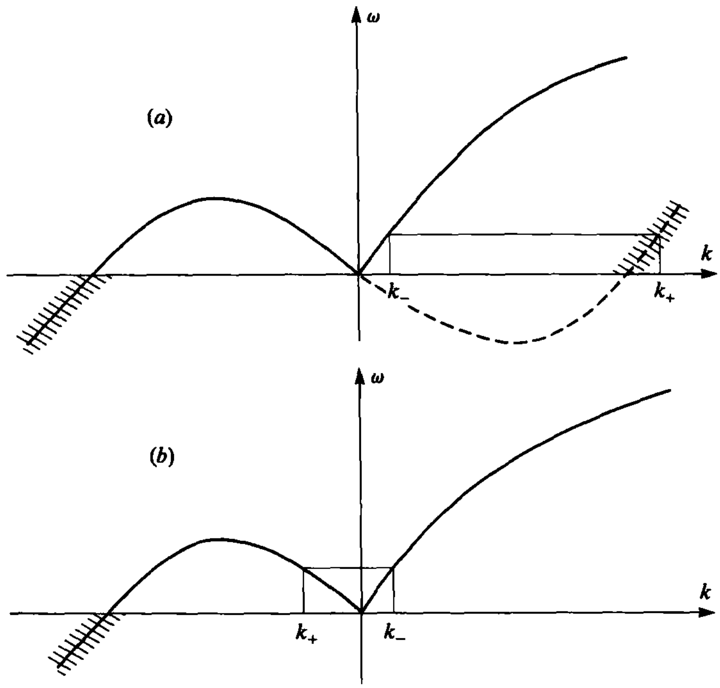

FIGURE 2. Dispersion relation of internal waves for the two-layer model of stratification (33). Two types of the wave resonance are shown: (*a*) interaction of the waves with energies (frequencies) of opposite signs; (*b*) interaction of the waves with positive energies (frequencies). ——, dispersion curve $\omega = \omega_k$; - - - -, inverse curve $\omega = -\omega_{-k}$; shaded portions of curves shows the region of the waves with negative energies.

Booker & Bretherton 1967), resulting in more cumbersome considerations. Note that the n-layer model is free from this shortcoming and at the same time describes the real ocean quite adequately when the number of layers is sufficiently large.

4. Instability of stratified flows over bottom topography

We first note that for the two-layer model of stratification (33) (when there is only one mode present), index ν and summation with respect to it must be dropped from all the previous calculations. Then (24*a*) can be rewritten in the following form:

$$\frac{\partial a_k}{\partial t} - \mathrm{i}\omega_k a_k = \mathrm{i}\iint \{U_{kk_1} a_{k_1} \overset{*}{h}_{k_2} \delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2) + V_{kk_1} \overset{*}{a}_{k_1} \overset{*}{h}_{k_2} \delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \quad (34)$$

and relations (28) and (27) are given by

$$\frac{\partial a_k}{\partial t} = \mathrm{i}\frac{\delta \mathscr{H}}{\delta \overset{*}{a}_k}, \quad \mathscr{H} = \mathscr{H}^{(0)} + \mathscr{H}^{(1)}; \quad (35a)$$

$$\mathscr{H}^{(0)} = \int \omega_k |a_k|^2\,\mathrm{d}\boldsymbol{k}, \quad (35b)$$

$$\mathscr{H}^{(1)} = \iiint \{U_{kk_1} \overset{*}{a}_k a_{k_1} \overset{*}{h}_{k_2} \delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2) + \tfrac{1}{2}(V_{kk_1} \overset{*}{a}_k \overset{*}{a}_{k_1} \overset{*}{h}_{k_2} + \overset{*}{V}_{kk_1} a_k a_{k_1} h_{k_2})\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2. \quad (35c)$$

In what follows, to make the calculations easier and to clarify the physical sense of the problem, we shall consider only the simplest model (the results obtained can be trivially generalized to the multiple mode case without any changes in the techniques used). We also note that the Hamiltonian (35*b*, *c*) determining the dynamic system (35a) is of the most general form for the conservative systems with one mode. That is why all the results below will be formulated so that one can easily extend them to any Hamiltonian system with spatial inhomogeneity of parameters. First we formulate the results obtained earlier for the problems of the stability of flows over sinusoidal topography (cf. Yih 1976; Charney & Flierl 1981; Benilov & Chernyak 1985) in the terminology of (35).

(i) Let $h(\boldsymbol{r}) = h_0 \cos(\boldsymbol{\kappa}\cdot\boldsymbol{r})$. Then $h_k = \frac{1}{2}h_0[\delta(\boldsymbol{k}-\boldsymbol{\kappa})+\delta(\boldsymbol{k}+\boldsymbol{\kappa})]$ and (34) yields

$$\frac{\partial a_k}{\partial t} - \mathrm{i}\omega_k a_k = \tfrac{1}{2}\mathrm{i}h_0\{U_{k,\,k+\kappa}\,a_{k+\kappa} + U_{k,\,k-\kappa}\,a_{k-\kappa} + V_{k,\,-(k+\kappa)}\,\overset{*}{a}_{-(k+\kappa)} + V_{k,\,-(k-\kappa)}\,\overset{*}{a}_{-(k-\kappa)}\}. \quad (36)$$

We shall look for a solution describing the superposition of two waves with wave vectors and frequencies connected by resonant relations (32*b*) (cf. figure 2*a*):

$$a_k(t) = A_+(t)\exp(\mathrm{i}\omega_{k_+}t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_+) + A_-(t)\exp(\mathrm{i}\omega_{k_-}t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_-) + O(h_0); \quad (37a)$$

$$\boldsymbol{k}_+ + \boldsymbol{k}_- + \boldsymbol{\kappa} = 0, \quad \omega_{k_+} + \omega_{k_-} = 0. \quad (37b)$$

Substituting (37) in (36) and keeping the terms $\sim h_0$, we have

$$\frac{\mathrm{d}A_+}{\mathrm{d}t} = \tfrac{1}{2}\mathrm{i}h_0\,V_{k_+k_-}\overset{*}{A}_-, \quad \frac{\mathrm{d}A_-}{\mathrm{d}t} = \tfrac{1}{2}\mathrm{i}h_0\,V_{k_-k_+}\overset{*}{A}_+. \quad (38)$$

The system (38) can be trivially integrated and has a particular solution

$$A_\pm(t) = A_\pm(0)\,\mathrm{e}^{\lambda t}; \quad |A_+(0)| = |A_-(0)|, \quad \arg A_+(0) + \arg A_-(0) = \tfrac{1}{2}\pi + \arg V,$$

where
$$\lambda = \tfrac{1}{2}h_0|V| > 0 \quad (39)$$

and $V = V_{k_+k_-} = V_{k_-k_+}$. When $\boldsymbol{k}_+$ is parallel to $\boldsymbol{k}_-$, expression (39) coincides with a similar formula for the growth rate of the instability obtained in another way by Benilov & Chernyak (1985). Thus, we have shown that (36) has a solution exponentially growing from zero, and this proves the instability of the 'vacuum' state (i.e. the instability of the basic unperturbed flow described by the solution $a_k \equiv 0$). The cause for the instability is the birth of a pair of wave-field 'quanta'† with frequencies of different signs, and this is essential for understanding the physical sense of the discovered instability. Indeed, as we have already seen the signs of the wave frequencies coincide with the signs of the wave energies, and the amplitudes of two waves with energies of opposites signs can grow, while *the energy of the whole system is equal to zero*. In this sense the discovered instability is a linear analogue of the explosive instability in nonlinear media with negative dispersion (e.g. Craik 1985; Zakharov 1974). The explosive instability is based on the nonlinear interaction of triplets of waves with wave vectors and frequencies satisfying the resonant relations $\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2 = 0$, $\omega_k + \omega_{k_1} + \omega_{k_2} = 0$ coinciding with the relations (37*b*) when $\boldsymbol{k} = \boldsymbol{k}_+$, $\boldsymbol{k}_1 = \boldsymbol{k}_-$, $\boldsymbol{k}_2 = \boldsymbol{\kappa}$ and $\omega_{k_2} = 0$ (the bottom topography corresponds to the wave with zero frequency). The only distinction between the two types of instabilities is the

† The terminology is borrowed from the paper by Zakharov (1974), who in turn had borrowed it from the formalism of secondary quantization in quantum mechanics.

character of the wave growth: in our (linear) case it is exponential; in the case of the nonlinear explosive instability it is a power law (proportional to $1/(t_0-t)$, where t_0 is the moment of the 'explosion').

Considering the instability of the flow over bottom topography as one of the possible mechanisms of generation of internal waves in the ocean, we give a numerical estimate of the characteristic time $\tau = 1/\lambda$ of instability development for typical parameters of the ocean shelf. For $H \approx 200$ m, $H_0 \approx 150$ m, $\Delta\rho/\rho_0 \approx 0.5 \times 10^{-3}$, $h_0 \approx 20$ m, $\kappa \approx 0.02\ \mathrm{m}^{-1}$ (the horizontal scale of the topography $2\pi/\kappa \approx 300$ m) and $U_0 \approx 0.4$ m/s we have $\tau \approx 65$ h. We note that within the bounds of the model (33) the explosive instability proper cannot occur for any values of parameters H, H_0, U_0, $\Delta\rho/\rho_0$. We shall not dwell here on the case of sinusoidal topography, but note only that if the type of the resonance is changed (cf. figure 2*b*), i.e. condition (32*b*) is replaced by condition (32*a*) in formula (37*b*), then we have stable periodic modulations of the basic wave amplitudes. In this case interaction of waves with frequencies (energies) of the *same* sign takes place and *the energy conservation law forbids the growth of their amplitude.*

(ii) Let $h(\boldsymbol{r})$ now be a random function with Gaussian statistics. We require also that the two-point correlation function should depend only on the distance between the points:†

$$\overline{h(\boldsymbol{r})\,h(\boldsymbol{r}+\boldsymbol{R})} = B(\boldsymbol{R}), \tag{40}$$

where the overbar denotes the average over the ensemble of realizations. Condition (40) implies

$$\overline{h_k \overset{*}{h}_{k_1}} = H_k\,\delta(\boldsymbol{k}-\boldsymbol{k}_1), \tag{41a}$$

where
$$H_k = (2\pi)^{-2} \int B(R)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R})}\,\mathrm{d}\boldsymbol{R}.$$

As before, we consider small spatial inhomogeneities and describe the wave field as a wide spectrum of free weakly interacting waves with random phases. According to this a_k satisfies the condition, analogous to (41*a*):

$$\overline{a_k \overset{*}{a}_{k_1}} = n_k\,\delta(\boldsymbol{k}-\boldsymbol{k}_1) \tag{41b}$$

(the equality (41*b*) can be taken as a formal definition of the second-order cumulant spectral density n_k). Such a description is called weakly turbulent and was introduced in the papers by Hasselman (1962, 1963), Zakharov (1965) and Zakharov & Filonenko (1966, 1967) to describe nonlinear surface waves. The most effective tool in studying weak turbulence is the kinetic equation governing the evolution of n_k, and in this paper we shall use its analogue for our problem:

$$\frac{\partial n_k}{\partial t} = 2\pi \iint \{|U_{kk_1}|^2 H_{k_2}(n_{k_1}-n_k)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)\,\delta(\omega_k-\omega_{k_2}) + |V_{kk_1}|^2 H_{k_2}(n_{k_1}+n_k)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\,\delta(\omega_k+\omega_{k_1})\}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \tag{42}$$

(which is derived in Appendix B).

To prove the instability of the vacuum state ($n_k \equiv 0$), we shall look for the solution in the form $n_k(t) = N_k\,\mathrm{e}^{\lambda t}$, where $N_k > 0$ and λ are the eigenfunction and eigenvalue

† In other words, we require that the bottom irregularities should be spatially homogeneous (in the statistical sense).

of the Fredholm integral equation of the second type which can be obtained from (42) via the replacement $n_k \to N_k$, $\partial/\partial t \to \lambda$. To prove the instability, we need only to show that $\lambda > 0$. Integrating the Fredholm equation with respect to all $\boldsymbol{k}$ we find the required estimate:

$$\lambda = \frac{\iiint |V_{kk_1}|^2 H_{k_2}(N_{k_1}+N_k)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\,\delta(\omega_k+\omega_{k_1})\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2}{\int N_k\,\mathrm{d}\boldsymbol{k}} > 0 \tag{43}$$

(the term $\sim |U_{kk_1}|^2$ vanished owing to the antisymmetry of the integral function when replacing $\boldsymbol{k} \leftrightarrow \boldsymbol{k}_1$). More precisely, to prove the instability, we have to prove that a solution of the Fredholm equation exists. We shall not dwell on this question, but note only that apparently a solution exists if the function H_k decreases fast enough as $k \to \infty$. Summarizing the discussion of the random bottom irregularities we note that in this case the growth rate of instability λ is proportional to the *second* power of the small parameter h/H. Thus this instability is weaker than that in the case of sinusoidal topography (compare (39) to (43)). On the other hand the spectrum of unstable waves due to the random inhomogeneities is much wider and spreads over the whole $\boldsymbol{k}$-plane.

5. Influence of the near-bottom flow

In this section we study the influence of non-zero near-bottom flow ($\boldsymbol{U}(0) \neq 0$) using the example of homogeneous fluid with a free boundary over the bottom topography. In this case, in contrast to that considered above, the 'vacuum state' does not exist, since the interaction of the near-bottom flow and bottom topography results in the appearance of topographic surface waves (lee waves). Note that we do not consider stratified flows, only in order to simplify the calculations, since the case of inhomogeneous fluid is quite similar in concept to that studied below but is impeded by the extremely cumbersome calculations. The case of homogeneous fluid is also very attractive because it enables us to remain from the very beginning in the framework of the Hamiltonian formalism and to use canonical variables without applying 'canonizing' transforms of the type (23).

Consider the potential oscillations of an ideal fluid in a basin with an uneven bottom. In this case the hydrodynamic potential $\Phi(z,\boldsymbol{r},t)$ satisfies Laplace's equation

$$\nabla^2\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{44a}$$

in the region $H+\eta(\boldsymbol{r},t) > z > h(\boldsymbol{r})$ (η is the vertical displacement of the free surface). We must solve (44a) with the no-flow boundary condition on the bottom:

$$\frac{\partial\Phi}{\partial z} = (\nabla h\cdot\nabla\Phi) \quad \text{when } z = h(\boldsymbol{r}) \tag{44b}$$

and kinematic and dynamic conditions on the surface of the fluid (when $z = H+\eta(\boldsymbol{r},t)$):

$$\frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} + (\nabla\eta\cdot\nabla\Phi) = 0, \qquad \frac{\partial\Phi}{\partial t} + \frac{1}{2}\left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] + g\eta = 0. \tag{44c}$$

We introduce the potential $\chi(z, \boldsymbol{r}, t)$ describing the surface waves against the background of the mean flow $\boldsymbol{U}_0$, which is constant within the depth of the fluid

$$\Phi = (\boldsymbol{U}_0 \cdot \boldsymbol{r}) - \tfrac{1}{2} U_0^2 t + \chi.$$

Equations (44) have an integral of the motion and its physical interpretation is the energy of fluid oscillations:

$$\mathscr{H} = \frac{1}{2} \int \left\{ g\eta^2 + \int_h^{H+\eta} \left[2(\boldsymbol{U}_0 \cdot \boldsymbol{\nabla}\chi) + (\boldsymbol{\nabla}\chi)^2 + \left(\frac{\partial \chi}{\partial z}\right)^2 \right] \mathrm{d}z \right\} \mathrm{d}\boldsymbol{r}.$$

It is easy now to identify the canonical variables representing the dynamical system (44) as Hamiltonian. Introducing the quantity

$$\Psi(\boldsymbol{r}, t) = \chi(z, \boldsymbol{r}, t)|_{z=H+\eta}$$

determining the boundary value of potential χ, we represent (44*c*) in the form

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$

(one can check the relations (45) by calculating directly the variational derivatives $\delta\mathscr{H}/\delta\Psi$ and $\delta\mathscr{H}/\delta\eta$). We can see from (45) that η and Ψ are canonically adjoint variables and we have to understand (44*a, b*) as conditions defining the function $\chi[\eta, \Psi]$ included in the Hamiltonian. (The canonical formalism for this problem is similar to the case of potential waves in a motionless fluid of infinite depth (Zakharov 1968).) Henceforth we can perform all calculations directly with the Hamiltonian and not with the dynamic equations (it is only necessary to make sure that all the transformations of variables are canonical). However, an attempt to express the Hamiltonian only in terms of the dynamic variables η and Ψ (i.e. to exclude χ from $\mathscr{H}$) turns out to be rather complicated. We shall not dwell upon this question, but refer to the similar procedure for the case of waves in a motionless fluid of infinite or finite (uniform) depth (Zakharov 1968; Yuen & Lake 1982; Lavrova 1983). Here we shall only remark that we can expand the Hamiltonian in a series of powers of small parameters $\epsilon_1 \sim kh$ and $\epsilon_2 \sim k\eta$:

$$\mathscr{H} = \mathscr{H}^{(0)} + \mathscr{H}_1^{(1)} + \mathscr{H}_2^{(1)} + \dots,$$

where $\mathscr{H}_1^{(1)} \sim \epsilon_1$, $\mathscr{H}_2^{(1)} \sim \epsilon_2$ (ϵ_2 has the physical meaning of a characteristic wave slope and gives a measure of the wave nonlinearity). The normalized wave amplitude a_k is introduced proceeding from the natural condition of diagonalization of $\mathscr{H}^{(0)}$:

$$\eta_k = \frac{1}{2}\left(\frac{k \tanh kH}{g}\right)^{\frac{1}{4}} (a_k + \overset{*}{a}_{-k}), \quad \Psi_k = -\tfrac{1}{2}\mathrm{i}\left(\frac{g}{k \tanh kH}\right)^{\frac{1}{4}} (a_k - \overset{*}{a}_{-k}).$$

In new variables

$$\mathscr{H}^{(0)} = \int \omega_k |a_k|^2 \,\mathrm{d}\boldsymbol{k} + \int (F_k \overset{*}{h}_k \overset{*}{a}_k + \overset{*}{F}_k h_k a_k) \,\mathrm{d}\boldsymbol{k},$$

where

$$\omega_k = (gk \tanh kH)^{\frac{1}{2}} + (\boldsymbol{k} \cdot \boldsymbol{U}_0), \quad F_k = -(\boldsymbol{k} \cdot \boldsymbol{U}_0) \operatorname{sech}(kH) \left[\frac{g}{k \tanh kH}\right]^{\frac{1}{4}}.$$

$\mathscr{H}_2^{(1)}$ attains the form

$$\mathscr{H}_2^{(1)} = \iiint \{(U_{kk_1k_2} \overset{*}{a}_k a_{k_1} \overset{*}{a}_{k_2} + \overset{*}{U}_{kk_1k_2} a_k \overset{*}{a}_{k_1} a_{k_2})\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 + \boldsymbol{k}_2)$$
$$+ (V_{kk_1k_2} \overset{*}{a}_{k_1} \overset{*}{a}_{k_2} + \overset{*}{V}_{kk_1k_2} a_k a_{k_1} a_{k_2})\, \delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2)\} \,\mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2$$

and $\mathscr{H}_1^{(1)}$ has the standard form (35*c*) with

$$U_{kk_1} = \frac{1}{8}\left(\frac{k\tanh kH}{g}\right)^{\frac{1}{4}}\left(\frac{k_1\tanh k_1 H}{g}\right)^{\frac{1}{4}}((\boldsymbol{k}+\boldsymbol{k}_1)\cdot \boldsymbol{U}_0)\,\mathrm{sech}\,(|\boldsymbol{k}-\boldsymbol{k}_1|H)$$

$$V_{kk1} = \frac{1}{8}\left(\frac{k\tanh kH}{g}\right)^{\frac{1}{4}}\left(\frac{k_1\tanh k_1 H}{g}\right)^{\frac{1}{4}}(\boldsymbol{k}\cdot\boldsymbol{k}_1)\,(\mathrm{sech}\,kH+\mathrm{sech}\,k_1 H)+U_{kk_1}.$$

The three-indexed matrix elements $U_{kk_1k_2}$ and $V_{kk_1k_2}$ corresponding to the nonlinear wave interactions coincide with the matrix elements calculated by Lavrova (1983) for the case of waves in a motionless fluid of uniform depth. The dynamic equation (35*a*) yields

$$\frac{\partial a_k}{\partial t} = \mathrm{i}\omega_k a_k + \mathrm{i}F_k \overset{*}{h}_k + \mathrm{i}\frac{\delta}{\delta \overset{*}{a}_k}(\mathscr{H}_1^{(1)}+\mathscr{H}_2^{(1)}+\dots). \tag{46}$$

Equation (46) has a stationary solution a_k^{st} describing topographic waves on the flow, and below we shall study the stability of this flow. Keeping the first non-trivial terms in (46), we have:

$$a_k^{\mathrm{st}} = -\frac{\overset{*}{h}_k F_k}{\omega_k}. \tag{47}$$

Let us now linearize (46) against the background of the stationary solution obtained:

$$a_k = a_k^{\mathrm{st}} + b_k, \quad |b_k| \ll |a_k^{\mathrm{st}}|.$$

It is easy to see that the linearized equation (46) coincides with (34), describing the internal waves over bottom topography in the absence of near-bottom flow, up to the specific form of the coefficients and the replacement $a_k \to b_k$. Thus, our case is reduced to that already studied.

However, it is necessary to remark on one fact distinguishing the case of non-zero near-bottom flow. In fact, we see from (47) that the spectrum of topographic waves has a singularity ($a_k^{\mathrm{st}} \to \infty$) for the wave vectors satisfying the equality $\omega_k = 0$ and corresponding to surface waves with zero phase velocity. The singularity occurs because we have neglected the small nonlinear terms when transferring from (46) to (47), while the exact stationary solution of (46) is regular on the whole $\boldsymbol{k}$-plane, although it has a spectral peak.† Nevertheless, in a number of cases we can use (47) directly, adding the infinitely small imaginary correction corresponding to the wave damping (this procedure is called regularization):

$$\frac{1}{\omega_k} \to \frac{1}{\omega_k + \mathrm{i}0} = \mathscr{P}\left(\frac{1}{\omega_k}\right) - \mathrm{i}\pi\delta(\omega_k),$$

where the distribution $\mathscr{P}(1/\omega)$ is defined by the equality

$$\mathscr{P}\left(\frac{1}{\omega}\right)f(\omega) = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{f(\omega)}{\omega}\,\mathrm{d}\omega.$$

To study the stability of the flow over random topography, we shall need the relation between the second-order cumulant of the topographic wave field $\overline{a_k^{\mathrm{st}} \overset{*}{a}{}_{k_1}^{\mathrm{st}}}$ and the correlation function $\overline{h_k \overset{*}{h}_{k_1}}$. Unfortunately, we cannot regularize the quantity $\overline{a_k^{\mathrm{st}} \overset{*}{a}{}_{k_1}^{\mathrm{st}}}$, where a_k^{st} is given by (47), for want of a satisfactory definition of the product

† This fact is similar to the limitation of the amplitude of nonlinear pendulum oscillations under resonant external forcing (while the amplitude of linear pendulum oscillations increases infinitely in time).

of distributions. Under no circumstances does this invalidate the proof of the instability, since, as we have seen above, the exact stationary solution of (46) is regular and all the intermediate calculations do not depend on its particular form.

6. Conclusions

Here, we have considered two problems of the stability of flows over bottom topography: the case of shear stratified flow (vanishing at the bottom) with a 'rigid-lid' at the surface, and homogeneous potential flow with a free boundary. Both flows turned out to be unstable for:

(*a*) sinusoidal (and any periodic) topography;

(*b*) random spatially homogeneous bottom topography.

The proof of the instability is based on the Hamiltonian approach and if the canonical variables are not known *a priori* (or if they are inconvenient), special symmetries of the matrix elements of the basic equations are required. Universality of the technique used in studying both these particular cases enables us to make a rather general statement: Any spatially inhomogeneous medium with sign-changing dispersion is unstable. The mechanism of the instability is based on the interaction (mutual generation) of waves with energies (frequencies) of opposite signs and in this sense is similar to the explosive instability.

Note that even the trivial Doppler shift of the wave frequencies $\omega_k \to \omega_k + (\boldsymbol{k}\cdot\boldsymbol{U}_0)$ in the isotropic medium moving as a solid results in the medium's instability (only if the inhomogeneities are not moving together with the medium). We note also that the explosive instability proper does not occur in this case.

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Appendix A. Proof of the inequality (26)

Let us rewrite (13*a*) in a more traditional form:

$$\frac{\partial^2\phi_k^\nu}{\partial z^2}+\left[\frac{k(\mathrm{d}^2V/\mathrm{d}z^2)}{\omega_k^\nu-kV}+\frac{k^2N^2}{(\omega_k^\nu-kV)^2}-k^2\right]\phi_k^\nu=0. \qquad \text{(A 1)}$$

Here $k=|\boldsymbol{k}|$, $V(z)=|\boldsymbol{U}(z)|\cos\beta$, where β is the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{U}$. We now differentiate (A 1) with respect to k, multiply it by ϕ_k^ν and integrate with respect to z within the limits 0 and H. After some simple calculations $\partial\phi_k^\nu/\partial k$ will be excluded from the obtained equality, which now attains the form

$$c_\mathrm{p}-c_\mathrm{g}=\frac{2k}{l_k^\nu}, \qquad \text{(A 2)}$$

where l_k^ν is determined by (18) and the quantities $c_\mathrm{p}=\omega_k^\nu/k$ and $c_\mathrm{g}=\partial\omega_k^\nu/\partial k$ coincide for $\boldsymbol{k}$ parallel to $\boldsymbol{U}$ with the phase and group velocities respectively. One can see from equality (A 2) that in the case of smooth dispersion curves $\omega=\omega_k^\nu$ ($c_\mathrm{p}\neq\infty$, $c_\mathrm{g}\neq\infty$) the quantity l_k^ν does not vanish on the $\boldsymbol{k}$-plane, except perhaps for $\boldsymbol{k}=0$. Now, to prove the inequality $l_k^\nu \geqslant 0$, we need only to make sure that l_k^ν is positive at least at one point, for instance, when $\boldsymbol{k}$ is orthogonal to $\boldsymbol{U}$. In this case internal waves do not 'feel' the flow at all (cf. (A 1) with $\beta=\frac{1}{2}\pi$), and l_k^ν is known to be positive (this follows from the fact that the internal-wave frequency ω_k^ν is positive in the absence of the flow). Thus, we have proved the desired inequality.

Appendix B. Derivation of the kinetic equation (42)

We first note that since the wave interaction is weak, we can assume the field to be close to Gaussian, while the condition of random phases gives us equalities

$$\overline{a_k a_{k_1}} = 0, \quad \overline{h_k a_{k_1}} = 0. \tag{B 1}$$

Following the usual scheme we can obtain from (34) an infinite series of equations for cumulants of increasing orders. The first three equations of the series have the following form:

$$\frac{\partial}{\partial t}\overline{a_k \overset{*}{a}_{k_3}} = 2\,\mathrm{Im}\iint \{\overset{*}{U}_{kk_1}\overline{\overset{*}{a}_{k_1} h_{k_2} a_{k_3}}\,\delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2) + \overset{*}{V}_{kk_1}\overline{a_{k_1} h_{k_2} a_{k_3}}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2; \tag{B 2a}$$

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t}-\mathrm{i}\Omega_1\right)\overline{\overset{*}{a}_{k_1} h_{k_2} a_{k_3}} &= \mathrm{i}\Sigma_1, \quad \Omega_1 = \omega_{k_3}-\omega_{k_1}; \\ \left(\frac{\partial}{\partial t}-\mathrm{i}\Omega_2\right)\overline{a_{k_1} h_{k_2} a_{k_3}} &= \mathrm{i}\Sigma_2, \quad \Omega_2 = \omega_{k_3}+\omega_{k_1} \end{aligned}\right\} \tag{B 2b}$$

where Σ_1 and Σ_2 are the corresponding sums of the fourth-order cumulants. To close (B 2), we must split the cumulants of the fourth order through the cumulants of the second order (this procedure is allowable because the distributions of a_k and h_k are close to Gaussian distributions). It is easy to see that owing to (41) and (B 1) only the following terms come through:

$$\overline{a_k h_{k_1} \overset{*}{a}_{k_2} \overset{*}{h}_{k_3}} = n_k H_{k_1}\,\delta(\boldsymbol{k}-\boldsymbol{k}_2)\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_3),$$

$$\overline{a_k h_{k_1} \overset{*}{a}_{k_2} h_{k_3}} = n_k H_{k_1}\,\delta(\boldsymbol{k}-\boldsymbol{k}_2)\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_3),$$

Finally, we have

$$\left.\begin{aligned} \Sigma_1 &= U_{k_3k_1} H_{k_2}(n_{k_3}-n_{k_1})\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3), \\ \Sigma_2 &= V_{k_3k_1} H_{k_2}(n_{k_3}+n_{k_1})\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3). \end{aligned}\right\} \tag{B 3}$$

Notice that we have two timescales in the problem: characteristic wave period $T_1 \sim 1/\omega_k$ and characteristic time of the wave interaction $T_2 \gg T_1$. Thus, (B 2*b*) are the equations of 'fast' pendulums with frequencies $\Omega_{1,2}$ under the 'slow' external forces $\Sigma_{1,2}$. We are interested in the established regime of movement when fast eigenoscillations of the third-order cumulants die out owing to the viscosity (so far not taken into account), i.e. the pendulums perform slow (forced) movements in the neighbourhood of the equilibrium state. Accordingly we introduce an infinitely small damping in (B 2*b*) and neglect the derivatives with respect to time:

$$\left.\begin{aligned} \overline{\overset{*}{a}_{k_1} h_{k_2} a_{k_3}} &= -\frac{\Sigma_1}{\Omega_1+\mathrm{i}0}, \\ \overline{a_{k_1} h_{k_2} a_{k_3}} &= -\frac{\Sigma_2}{\Omega_2+\mathrm{i}0}. \end{aligned}\right\} \tag{B 4}$$

Substituting (41*b*), (B 3) and (B 4) into (B 2*a*) and making use of the equality

$$\mathrm{Im}\frac{1}{\Omega+\mathrm{i}0} = -\pi\delta(\Omega),$$

we get the desired kinetic equation (42).

We should like to emphasize that if we had used the dynamic equation with non-symmetrized (by the transform (23)) matrix elements, the kinetic equation would have had a considerably more cumbersome form.

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