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Oblique liquid curtains with a large Froude number

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This paper examines two-dimensional liquid curtains ejected at an angle to the horizontal and affected by gravity and surface tension. The flow in the curtain is, generally, sheared. The Froude number based on the injection velocity and the outlet's width is assumed large; as a result, the streamwise scale of the curtain exceeds its thickness. A set of asymptotic equations for such (slender) curtains is derived and its steady solutions are examined. It is shown that, if the surface tension exceeds a certain threshold, the curtain – quite paradoxically – bends upwards, i.e. against gravity. Once the flow reaches the height where its initial supply of kinetic energy can take it, the curtain presumably breaks up and splashes down.

Key words: interfacial flows (free surface), jets, thin films

1. Introduction

Liquid curtains are used in the manufacturing of paper and similar industrial processes. To produce paper, the pulp is ejected through a narrow horizontal slot (outlet) and falls onto a conveyor belt, which carries it away. The productivity of this process depends on the belt speed – which, clearly, cannot be increased without a matching increase of the pulp ejection velocity and a change of the ejection angle. This suggests a need for studying high-velocity, oblique curtains.

Industrial applications, however, are not the only motivation for the present work, as liquid curtains have been part of classical fluid mechanics for almost 60 years – since the experimental work by Brown (1961) and theoretical result by G. I. Taylor (described in Brown's paper as a private communication). Both examined vertical curtains, as did all follow-up researchers (e.g. Finnicum, Weinstein & Ruschak 1993; Li 1993, 1994; Ramos 1996; Roche *et al.* 2006; Dyson *et al.* 2009; Benilov, Barros & O'Brien 2016; Lhuissier, Brunet & Dorbolo 2016). No attempt was made to examine oblique curtains, whose curved trajectories make their dynamics different from that of their vertical counterparts.

In this respect, oblique curtains have much in common with jets, so one could try to borrow methodology developed for those, e.g. the slender-jet equations derived by Wallwork (2001), Decent, King & Wallwork (2002) and Wallwork *et al.* (2002). Those papers employed curvilinear coordinates linked to the jet's centreline and assumed them to be orthogonal – which they were not, as it turned out (Entov

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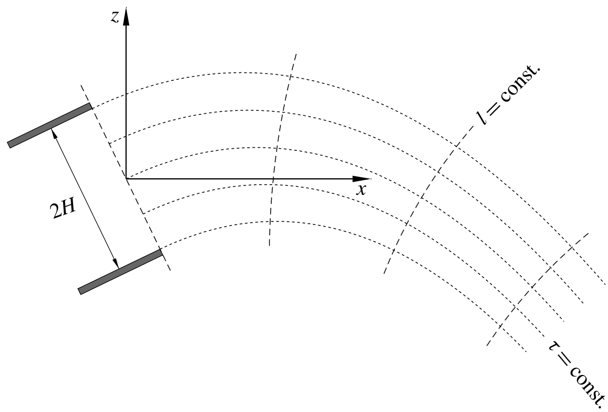

FIGURE 1. The setting: a liquid curtain is ejected from an outlet of width $2H$. Coordinates (x, z) are the Cartesian coordinates and (l, τ) are the curvilinear orthogonal coordinates associated with the curtain's centreline.

& Yarin 1984; Shikhmurzaev & Sisoev 2017) – although the asymptotic equations derived by Wallwork, Decent and collaborators still happened to be correct, as the non-orthogonality of the coordinates for slender jets is weak (Decent *et al.* 2018).

For two-dimensional flows, however, the coordinates of Decent *et al.* (2002) and Wallwork *et al.* (2002) are orthogonal – so, in principle, they could be used in the present work. Still, a different approach will be employed: the (orthogonal) coordinates will be sought together with the asymptotic solution of the Navier–Stokes equations, i.e. the orthogonality condition will be treated as one of the governing equations. The main advantage of this approach is that it can be readily extended to three dimensions, providing for these a much simpler framework than that based on non-orthogonal coordinates. In two dimensions, in turn, it has the added benefit of making the coordinates preserve the elemental area and angles.

This paper has the following structure. In § 2, we formulate the problem and, in § 3, derive asymptotic equations for curtains with a large Froude number. Steady solutions of these equations are examined for the limits of high and low viscosity in §§ 4 and 5, respectively. In § 6, the results obtained are summarised, discussed and expressed in terms of measurable parameters.

2. Formulation of the problem

Consider an incompressible liquid (of density ρ, kinematic viscosity ν and surface tension σ) ejected from an infinitely long horizontal slot (outlet). Let the flow be homogeneous in the along-the-outlet direction, i.e. depend on a single horizontal variable x and the vertical coordinate z (see figure 1).

Consider a set of curvilinear coordinates (l, τ) such that the Cartesian coordinates are related to them by

$$x = x(l, \tau, t), \qquad z = z(l, \tau, t), \tag{2.1a,b}$$

where t is the time. The curve $\tau = 0$ is assumed to coincide with the curtain's centreline, and l on this curve is the centreline's arc length – but other than that,

(l, τ) are not related to the coordinates used in Decent *et al.* (2002), Wallwork *et al.* (2002) and Shikhmurzaev & Sisoev (2017).

Let (l, τ) be orthogonal and with a unit Jacobian, i.e.

$$\frac{\partial x}{\partial l}\frac{\partial x}{\partial \tau}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial \tau}=0, \quad \frac{\partial x}{\partial l}\frac{\partial z}{\partial \tau}-\frac{\partial x}{\partial \tau}\frac{\partial z}{\partial l}=1. \tag{2.2a,b}$$

Since (2.2) are not complemented with boundary conditions, they do not fix uniquely the relationship of (x, z) to (l, τ). One may choose the solution of (2.2) that makes the forthcoming calculations simpler.

Let the flow be characterised by its pressure p and vector velocity $\boldsymbol{u}$, and let the components of the latter with respect to the curvilinear coordinates be u_s and u_τ. Introduce also so-called Lamé coefficients

$$h_l=\sqrt{\left(\frac{\partial x}{\partial l}\right)^2+\left(\frac{\partial z}{\partial l}\right)^2}, \quad h_\tau=\sqrt{\left(\frac{\partial x}{\partial \tau}\right)^2+\left(\frac{\partial z}{\partial \tau}\right)^2}. \tag{2.3a,b}$$

Representing the gravitational force by $-\rho g\boldsymbol{\nabla} z$ (g is the acceleration due to gravity), one can write the Navier–Stokes equations in the form

$$\left.\begin{aligned}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_l+\frac{1}{\rho h_l}\frac{\partial p}{\partial l}=\nu(\nabla^2\boldsymbol{u})_l-\frac{g}{h_l}\frac{\partial z}{\partial l},\\ \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_\tau+\frac{1}{\rho h_\tau}\frac{\partial p}{\partial \tau}=\nu(\nabla^2\boldsymbol{u})_\tau-\frac{g}{h_\tau}\frac{\partial z}{\partial \tau},\end{aligned}\right\} \tag{2.4}$$

$$\frac{\partial(u_l h_\tau)}{\partial l}+\frac{\partial(u_\tau h_l)}{\partial \tau}=0, \tag{2.5}$$

where the l- and τ-components of the material derivative and viscous term are

$$\begin{aligned}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_l&=\frac{\partial u_l}{\partial t}+\frac{1}{h_l}\left[u_l-\frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\left(\frac{\partial u_l}{\partial l}+\frac{u_\tau}{h_\tau}\frac{\partial h_l}{\partial \tau}\right)\\ &\quad+\frac{1}{h_\tau}\left[u_\tau-\frac{1}{h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\frac{\partial u_l}{\partial \tau}-\frac{u_\tau^2}{h_l h_\tau}\frac{\partial h_\tau}{\partial l}\\ &\quad+\frac{u_\tau}{h_l h_\tau}\left[\frac{\partial x}{\partial l}\frac{\partial^2 x}{\partial t\,\partial \tau}+\frac{\partial z}{\partial l}\frac{\partial^2 z}{\partial t\,\partial \tau}-\frac{1}{h_\tau^2}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\left(\frac{\partial x}{\partial l}\frac{\partial^2 x}{\partial \tau^2}+\frac{\partial z}{\partial l}\frac{\partial^2 z}{\partial \tau^2}\right)\right],\end{aligned} \tag{2.6}$$

$$\begin{aligned}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_\tau&=\frac{\partial u_\tau}{\partial t}+\frac{1}{h_l}\left[u_l-\frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\frac{\partial u_\tau}{\partial l}\\ &\quad+\frac{1}{h_\tau}\left[u_\tau-\frac{1}{h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\left(\frac{\partial u_\tau}{\partial \tau}+\frac{u_l}{h_l}\frac{\partial h_\tau}{\partial l}\right)-\frac{u_l^2}{h_l h_\tau}\frac{\partial h_l}{\partial \tau}\\ &\quad+\frac{u_l}{h_l h_\tau}\left[\frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial t\,\partial l}+\frac{\partial z}{\partial \tau}\frac{\partial^2 z}{\partial t\,\partial l}-\frac{1}{h_l^2}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\left(\frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial l^2}+\frac{\partial z}{\partial \tau}\frac{\partial^2 z}{\partial l^2}\right)\right],\end{aligned} \tag{2.7}$$

$$(\nabla^2\boldsymbol{u})_l=\frac{1}{h_l}\frac{\partial}{\partial l}\left\{\frac{1}{h_l h_\tau}\left[\frac{\partial(u_l h_\tau)}{\partial l}+\frac{\partial(u_\tau h_l)}{\partial \tau}\right]\right\}+\frac{1}{h_\tau}\frac{\partial}{\partial \tau}\left\{\frac{1}{h_l h_\tau}\left[\frac{\partial(u_l h_l)}{\partial \tau}-\frac{\partial(u_\tau h_\tau)}{\partial l}\right]\right\}, \tag{2.8}$$

$$(\nabla^2 \boldsymbol{u})_\tau = \frac{1}{h_\tau}\frac{\partial}{\partial \tau}\left\{\frac{1}{h_\tau h_l}\left[\frac{\partial(u_l h_\tau)}{\partial l} + \frac{\partial(u_\tau h_l)}{\partial \tau}\right]\right\} + \frac{1}{h_l}\frac{\partial}{\partial l}\left\{\frac{1}{h_\tau h_l}\left[\frac{\partial(u_\tau h_\tau)}{\partial l} - \frac{\partial(u_l h_l)}{\partial \tau}\right]\right\}. \tag{2.9}$$

Expressions (2.8) and (2.9) can be found in most fluid mechanics texts (e.g. Kochin, Kibel & Roze 1964), whereas (2.6) and (2.7) comprise the standard expressions derived for curvilinear coordinates plus extra terms due to the dependence of the coordinates on t (see appendix A).

Let the liquid be bounded by two free surfaces described by equations $f_\pm(l, \tau, t) = 0$. Then, the boundary conditions for the Navier–Stokes equations are

$$\begin{aligned}&\frac{\partial f_\pm}{\partial t} + \frac{1}{h_l}\left[u_l - \frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\frac{\partial f_\pm}{\partial l}\\ &\quad + \frac{1}{h_\tau}\left[u_\tau - \frac{1}{h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\left[\rho\nu\left(\frac{1}{h_l}\frac{\partial u_l}{\partial l} + \frac{u_\tau}{h_l h_\tau}\frac{\partial h_l}{\partial \tau}\right) - p + \sigma c_\pm\right]\frac{1}{h_l}\frac{\partial f_\pm}{\partial l}\\ &\quad + \rho\nu\left[\frac{h_l}{h_\tau}\frac{\partial}{\partial \tau}\left(\frac{u_l}{h_l}\right) + \frac{h_\tau}{h_l}\frac{\partial}{\partial l}\left(\frac{u_\tau}{h_\tau}\right)\right]\frac{1}{h_\tau}\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\rho\nu\left[\frac{h_l}{h_\tau}\frac{\partial}{\partial \tau}\left(\frac{u_l}{h_l}\right) + \frac{h_\tau}{h_l}\frac{\partial}{\partial l}\left(\frac{u_\tau}{h_\tau}\right)\right]\frac{1}{h_l}\frac{\partial f_\pm}{\partial l}\\ &\quad + \left[\rho\nu\left(\frac{1}{h_\tau}\frac{\partial u_\tau}{\partial \tau} + \frac{u_l}{h_l h_\tau}\frac{\partial h_\tau}{\partial l}\right) - p + \sigma c_\pm\right]\frac{1}{h_\tau}\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0,\end{aligned} \tag{2.12}$$

where the curvature of the free surfaces is

$$c_\pm = \frac{1}{h_l h_\tau}\frac{\partial}{\partial l}\left[\frac{\dfrac{h_\tau}{h_l}\dfrac{\partial f_\pm}{\partial l}}{\sqrt{\left(\dfrac{1}{h_l}\dfrac{\partial f_\pm}{\partial l}\right)^2 + \left(\dfrac{1}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}\right)^2}}\right] + \frac{1}{h_l h_\tau}\frac{\partial}{\partial \tau}\left[\frac{\dfrac{h_l}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}}{\sqrt{\left(\dfrac{1}{h_l}\dfrac{\partial f_\pm}{\partial l}\right)^2 + \left(\dfrac{1}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}\right)^2}}\right]. \tag{2.13}$$

Equation (2.10) is the usual free-boundary kinematic condition, and (2.11) and (2.12) are equivalent to the standard dynamic conditions (e.g. Benilov, Lapin & O'Brien 2011). Expression (2.13) has been obtained through the usual formula $c = \boldsymbol{\nabla}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal. Note that (2.13) implies that

$$f_\pm < 0 \quad \text{in the region occupied by the liquid.} \tag{2.14}$$

Subject to specific expressions for $x(l, \tau, t)$ and $z(l, \tau, t)$ (satisfying (2.2)) and a suitable boundary condition at the outlet, the boundary-value problem (2.4)–(2.13) fully determines the evolution of p, u_l, u_τ and $f_\pm$.

Note that, despite the intimidating size of the exact equations, the asymptotic ones turn out to be compact.

3. Asymptotic analysis

3.1. *The scaling*

Consider a slender liquid curtain, i.e. such that the streamwise spatial scale L exceeds the cross-stream scale H (the latter can be identified with, say, half of the outlet's width). Since the curvature of an oblique curtain is determined by gravity, let

$$L=\frac{U^2}{g}, \tag{3.1}$$

where U is the velocity scale. Then, the slenderness requirement is $\varepsilon \ll 1$, where

$$\varepsilon=\frac{gH}{U^2} \tag{3.2}$$

is the reciprocal of the Froude number.

One would also be tempted to assume that, due to the slenderness, the streamwise velocity u_l exceeds its cross-stream counterpart u_τ – but this is true only for stationary curtains. For evolving ones, the motion of the curtain as a whole can make u_τ be comparable to u_l.

It is still possible to take advantage of the asymmetry between the streamwise and cross-stream directions by representing (u_l, u_τ) as the sum of the curtain's velocity and the relative velocity $(\hat{u}_l, \hat{u}_\tau)$. To find the former, recall that, since the coordinates (x, z) are associated with the curtain, its velocity can be approximated by $\boldsymbol{v}=(\partial x/\partial t, \partial z/\partial t)$ – as seen later, $\boldsymbol{v}$ changes weakly in the cross-stream direction and, thus, can be identified with the velocity of the centreline. Using the standard expressions for the unit vectors corresponding to l and τ, one can separate in $\boldsymbol{v}$ the streamwise and cross-stream components, and then set

$$u_l=\frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)+\hat{u}_l, \tag{3.3}$$

$$u_\tau=\frac{1}{h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)+\hat{u}_\tau. \tag{3.4}$$

As shown later, the two terms on the right-hand side of (3.3) are comparable, making this substitution optional – but it makes the forthcoming asymptotic equations simpler. Substitution (3.4), in turn, is crucial, as the first term on its right-hand side will be shown to exceed the second term by an order of magnitude.

The following non-dimensional variables (marked with the subscript *nd*) will be used:

$$l=L\,l_{nd}, \quad \tau=H\,\tau_{nd}, \quad t=T\,t_{nd}, \tag{3.5a–c}$$

$$u_l=U(u_l)_{nd}, \quad \hat{u}_l=U(\hat{u}_l)_{nd}, \quad u_\tau=U(u_\tau)_{nd}, \quad \hat{u}_\tau=\frac{UH}{L}(\hat{u}_\tau)_{nd}, \tag{3.6a–d}$$

$$p=P\,p_{nd}, \quad f_\pm=H(f_\pm)_{nd}, \quad c_\pm=\frac{1}{L}(c_\pm)_{nd}, \tag{3.7a–c}$$

$$x=L\,x_{nd}, \quad z=L\,z_{nd}, \quad h_l=(h_l)_{nd}, \quad h_\tau=(h_\tau)_{nd}, \tag{3.8a–d}$$

$$\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_l=\frac{U}{T}\left[\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_l\right]_{nd}, \quad \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_\tau=\frac{U}{T}\left[\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_\tau\right]_{nd}, \tag{3.9a,b}$$

$$(\nabla^2 \boldsymbol{u})_l = \frac{U}{H^2}\left[(\nabla^2 \boldsymbol{u})_l\right]_{nd}, \quad (\nabla^2 \boldsymbol{u})_\tau = \frac{U}{HL}[(\nabla^2 \boldsymbol{u})_\tau]_{nd}, \tag{3.10a,b}$$

where L is defined by (3.1) and the time and pressure scales are, respectively, kinematic and hydrostatic, i.e.

$$T = \frac{L}{U}, \quad P = \rho g H. \tag{3.11a,b}$$

Substituting (3.5)–(3.11), (3.1) into (2.2)–(2.13), (3.3)–(3.4) and omitting the subscript nd, one obtains

$$\frac{\partial x}{\partial l}\frac{\partial x}{\partial \tau} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial \tau} = 0, \quad \frac{\partial x}{\partial l}\frac{\partial z}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial z}{\partial l} = \varepsilon, \tag{3.12a,b}$$

$$h_l = \sqrt{\left(\frac{\partial x}{\partial l}\right)^2 + \left(\frac{\partial z}{\partial l}\right)^2}, \quad h_\tau = \frac{1}{\varepsilon}\sqrt{\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2}, \tag{3.13a,b}$$

$$\left.\begin{aligned} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_l + \frac{\varepsilon}{h_l}\frac{\partial p}{\partial l} &= \mu(\nabla^2 \boldsymbol{u})_l - \frac{1}{h_l}\frac{\partial z}{\partial l}, \\ \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_\tau + \frac{1}{h_\tau}\frac{\partial p}{\partial \tau} &= \varepsilon\mu(\nabla^2 \boldsymbol{u})_\tau - \frac{1}{\varepsilon h_\tau}\frac{\partial z}{\partial \tau}, \end{aligned}\right\} \tag{3.14}$$

$$\frac{\partial (u_l h_\tau)}{\partial l} + \frac{1}{\varepsilon}\frac{\partial (u_\tau h_l)}{\partial \tau} = 0, \tag{3.15}$$

$$\begin{aligned} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_l = {} & \frac{\partial u_l}{\partial t} + \frac{1}{h_l}\left[u_l - \frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\left(\frac{\partial u_l}{\partial l} + \frac{u_\tau}{\varepsilon h_\tau}\frac{\partial h_l}{\partial \tau}\right) \\ & + \frac{1}{\varepsilon h_\tau}\left[u_\tau - \frac{1}{\varepsilon h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\frac{\partial u_l}{\partial \tau} - \frac{u_\tau^2}{h_l h_\tau}\frac{\partial h_\tau}{\partial l} \\ & + \frac{u_\tau}{h_l h_\tau}\left[\frac{1}{\varepsilon}\left(\frac{\partial x}{\partial l}\frac{\partial^2 x}{\partial t\,\partial \tau} + \frac{\partial z}{\partial l}\frac{\partial^2 z}{\partial t\,\partial \tau}\right) - \frac{1}{\varepsilon^3 h_\tau^2}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right. \\ & \left. \times \left(\frac{\partial x}{\partial l}\frac{\partial^2 x}{\partial \tau^2} + \frac{\partial z}{\partial l}\frac{\partial^2 z}{\partial \tau^2}\right)\right], \end{aligned} \tag{3.16}$$

$$\begin{aligned} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_\tau = {} & \frac{\partial u_\tau}{\partial t} + \frac{1}{h_l}\left[u_l - \frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\frac{\partial u_\tau}{\partial l} \\ & + \frac{1}{h_\tau}\left[u_\tau - \frac{1}{\varepsilon h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\left(\frac{1}{\varepsilon}\frac{\partial u_\tau}{\partial \tau} + \frac{u_l}{h_l}\frac{\partial h_\tau}{\partial l}\right) - \frac{u_l^2}{\varepsilon h_l h_\tau}\frac{\partial h_l}{\partial \tau} \\ & + \left[\frac{1}{\varepsilon}\left(\frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial t\,\partial l} + \frac{\partial z}{\partial \tau}\frac{\partial^2 z}{\partial t\,\partial l}\right) - \frac{1}{\varepsilon h_l^2}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right. \\ & \left. \times \left(\frac{\partial x}{\partial \tau}\frac{\partial^2 x}{\partial l^2} + \frac{\partial z}{\partial \tau}\frac{\partial^2 z}{\partial l^2}\right)\right], \end{aligned} \tag{3.17}$$

$$(\nabla^2 \boldsymbol{u})_l = \frac{1}{h_l}\frac{\partial}{\partial l}\left\{\frac{1}{h_l h_\tau}\left[\varepsilon^2 \frac{\partial (u_l h_\tau)}{\partial l} + \varepsilon\frac{\partial (u_\tau h_l)}{\partial \tau}\right]\right\} + \frac{1}{h_\tau}\frac{\partial}{\partial \tau}\left\{\frac{1}{h_l h_\tau}\left[\frac{\partial (u_l h_l)}{\partial \tau} - \varepsilon\frac{\partial (u_\tau h_\tau)}{\partial l}\right]\right\}, \tag{3.18}$$

$$(\nabla^2 \boldsymbol{u})_\tau = \frac{1}{h_\tau}\frac{\partial}{\partial \tau}\left\{\frac{1}{h_\tau h_l}\left[\frac{\partial(u_l h_\tau)}{\partial l} + \frac{1}{\varepsilon}\frac{\partial(u_\tau h_l)}{\partial \tau}\right]\right\} + \frac{1}{h_l}\frac{\partial}{\partial l}\left\{\frac{1}{h_\tau h_l}\left[\varepsilon\frac{\partial(u_\tau h_\tau)}{\partial l} - \frac{\partial(u_l h_l)}{\partial \tau}\right]\right\}, \tag{3.19}$$

$$\frac{\partial f_\pm}{\partial t} + \frac{1}{h_l}\left[u_l - \frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right)\right]\frac{\partial f_\pm}{\partial l} + \frac{1}{\varepsilon h_\tau}\left[u_\tau - \frac{1}{\varepsilon h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right)\right]\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0, \tag{3.20}$$

$$\varepsilon\left[\varepsilon\left(\frac{1}{h_l}\frac{\partial u_l}{\partial l} + \frac{u_\tau}{h_l h_\tau}\frac{\partial h_l}{\partial \tau}\right) - \frac{p - \gamma c_\pm}{\mu}\right]\frac{1}{h_l}\frac{\partial f_\pm}{\partial l} + \left[\frac{h_l}{h_\tau}\frac{\partial}{\partial \tau}\left(\frac{u_l}{h_l}\right) + \frac{\varepsilon^2 h_\tau}{h_l}\frac{\partial}{\partial l}\left(\frac{u_\tau}{h_\tau}\right)\right]\frac{1}{h_\tau}\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0, \tag{3.21}$$

$$\varepsilon\left[\frac{h_l}{h_\tau}\frac{\partial}{\partial \tau}\left(\frac{u_l}{h_l}\right) + \frac{\varepsilon^2 h_\tau}{h_l}\frac{\partial}{\partial l}\left(\frac{u_\tau}{h_\tau}\right)\right]\frac{1}{h_l}\frac{\partial f_\pm}{\partial l} + \left[\varepsilon\left(\frac{1}{h_\tau}\frac{\partial u_\tau}{\partial \tau} + \frac{u_l}{h_l h_\tau}\frac{\partial h_\tau}{\partial l}\right) - \frac{p - \gamma c_\pm}{\mu}\right]\frac{1}{h_\tau}\frac{\partial f_\pm}{\partial \tau} = 0 \quad \text{if } f_\pm = 0, \tag{3.22}$$

$$\varepsilon c_\pm = \frac{\varepsilon^2}{h_l h_\tau}\frac{\partial}{\partial l}\left[\frac{\dfrac{h_\tau}{h_l}\dfrac{\partial f_\pm}{\partial l}}{\sqrt{\varepsilon^2\left(\dfrac{1}{h_l}\dfrac{\partial f_\pm}{\partial l}\right)^2 + \left(\dfrac{1}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}\right)^2}}\right] + \frac{1}{h_l h_\tau}\frac{\partial}{\partial \tau}\left[\frac{\dfrac{h_l}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}}{\sqrt{\varepsilon^2\left(\dfrac{1}{h_l}\dfrac{\partial f_\pm}{\partial l}\right)^2 + \left(\dfrac{1}{h_\tau}\dfrac{\partial f_\pm}{\partial \tau}\right)^2}}\right], \tag{3.23}$$

$$u_l = \frac{1}{h_l}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial l}\frac{\partial z}{\partial t}\right) + \hat{u}_l, \quad u_\tau = \frac{1}{\varepsilon h_\tau}\left(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial \tau}\frac{\partial z}{\partial t}\right) + \varepsilon\hat{u}_\tau, \tag{3.24a,b}$$

where ε is defined by (3.2) and

$$\mu = \frac{\nu U}{g H^2}, \quad \gamma = \frac{\sigma}{\rho H U^2}. \tag{3.25a,b}$$

Note that γ is the reciprocal of the Weber number and μ is the reciprocal of the reduced Reynolds number (the latter is discussed in detail in § 3.5).

It turns out that the most general characteristic limit is

$$\gamma = O(1), \quad \mu = O(1) \quad \text{as } \varepsilon \to 0. \tag{3.26a,b}$$

The cases where γ and/or μ are large and/or small result in asymptotic equations that can be obtained by adapting the general case to the corresponding limit.

3.2. *How should the curvilinear coordinates be chosen?*

The approximation of a slender curtain corresponds to expanding the solution of (3.12) in powers of the cross-stream variable, i.e.

$$x = \sum_{n=0}^{\infty} (\varepsilon\tau)^n x^{(n)}(l, t), \quad z = \sum_{n=0}^{\infty} (\varepsilon\tau)^n z^{(n)}(l, t). \tag{3.27a,b}$$

Observe that the leading terms of these expansions satisfy (3.12) identically – hence, $x^{(0)}(l, t)$ and $x^{(0)}(l, t)$ can be chosen at will. In the next two orders, equations (3.12) yield equations for $x^{(1)}$, $z^{(1)}$, $x^{(2)}$ and $z^{(2)}$:

$$\left.\begin{aligned} \frac{\partial x^{(0)}}{\partial l} x^{(1)} + \frac{\partial z^{(0)}}{\partial l} z^{(1)} = 0, \\ \frac{\partial x^{(0)}}{\partial l} z^{(1)} - \frac{\partial z^{(0)}}{\partial l} x^{(1)} = 1, \end{aligned}\right\} \tag{3.28}$$

$$\left.\begin{aligned} \frac{\partial x^{(0)}}{\partial l} x^{(2)} + \frac{\partial z^{(0)}}{\partial l} z^{(2)} = -\frac{\partial x^{(1)}}{\partial l} \frac{x^{(1)}}{2} - \frac{\partial z^{(1)}}{\partial l} \frac{z^{(1)}}{2}, \\ \frac{\partial x^{(0)}}{\partial l} z^{(2)} - \frac{\partial z^{(0)}}{\partial l} x^{(2)} = -\frac{\partial x^{(1)}}{\partial l} \frac{z^{(1)}}{2} + \frac{\partial z^{(1)}}{\partial l} \frac{x^{(1)}}{2}. \end{aligned}\right\} \tag{3.29}$$

It turns out that the most convenient choice for $x^{(0)}$ and $z^{(0)}$ is such that

$$\frac{\partial x^{(0)}}{\partial l} = \cos\alpha, \quad \frac{\partial z^{(0)}}{\partial l} = \sin\alpha, \tag{3.30a,b}$$

where, physically, $\alpha(l, t)$ is the local angle between the curtain's centreline and the horizontal.

Given (3.31), equations (3.28)–(3.29) yield

$$x^{(1)} = -\sin\alpha, \quad z^{(1)} = \cos\alpha, \tag{3.31a,b}$$

$$x^{(2)} = -\frac{1}{2}\frac{\partial\alpha}{\partial l}\sin\alpha, \quad z^{(2)} = \frac{1}{2}\frac{\partial\alpha}{\partial l}\cos\alpha, \tag{3.32a,b}$$

and expressions (3.2) yield

$$h_l = 1 - \varepsilon\tau\frac{\partial\alpha}{\partial l} + O(\varepsilon^2), \quad h_\tau = 1 + \varepsilon\tau\frac{\partial\alpha}{\partial l} + O(\varepsilon^2). \tag{3.33a,b}$$

3.3. *The asymptotic equations*

The simplest choice for $f_\pm$ – such that it satisfies requirement (2.14) – is

$$f_\pm(l, \tau, t) = \pm[\tau - \tau_\pm(l, t)], \tag{3.34}$$

where $\tau_\pm(l, t)$ are the new unknowns.

Now, substitute expansions (3.27) into (3.14)–(3.24), take into account (3.31)–(3.34) and take the limit $\varepsilon \to 0$. After straightforward algebra, one obtains (small terms, hats and the superscript (0) omitted)

$$\frac{\partial u_l}{\partial t} + u_l\frac{\partial u_l}{\partial l} + u_\tau\frac{\partial u_l}{\partial \tau} = \mu\frac{\partial^2 u_l}{\partial \tau^2} - \sin\alpha - \frac{\partial^2 x}{\partial t^2}\cos\alpha - \frac{\partial^2 z}{\partial t^2}\sin\alpha, \tag{3.35}$$

$$2\frac{\partial\alpha}{\partial t}u_l+\frac{\partial\alpha}{\partial l}u_l^2+\frac{\partial p}{\partial\tau}=-\cos\alpha+\frac{\partial^2 x}{\partial t^2}\sin\alpha-\frac{\partial^2 z}{\partial t^2}\cos\alpha, \tag{3.36}$$

$$\frac{\partial u_l}{\partial l}+\frac{\partial u_\tau}{\partial\tau}=0, \tag{3.37}$$

$$\frac{\partial\tau_\pm}{\partial t}+u_l\frac{\partial\tau_\pm}{\partial l}-u_\tau=0\quad\text{if }\tau=\tau_\pm, \tag{3.38}$$

$$\frac{\partial u_l}{\partial\tau}=0\quad\text{if }\tau=\tau_\pm, \tag{3.39}$$

$$p=\mp\gamma\frac{\partial\alpha}{\partial l}\quad\text{if }\tau=\tau_\pm. \tag{3.40}$$

Finally, omit the superscript (0) from (3.31):

$$\frac{\partial x}{\partial l}=\cos\alpha,\qquad\frac{\partial z}{\partial l}=\sin\alpha. \tag{3.41a,b}$$

Equations (3.35)–(3.41) form the desired asymptotic set.

3.4. *Discussion*

(1) To better understand the effects governing slender curtains, one needs to identify the physical meaning of the terms in the equations derived. While doing so, keep in mind that the coordinate system (l,τ) depends on t and, thus, is non-inertial.

   (i) The terms involving ∂^2x/∂^2t and ∂^2z/∂^2t in (3.35)–(3.36) describe the force of inertia due to the curtain's linear acceleration as a whole.

   (ii) The first term in (3.36) describes the Coriolis force, with $\partial\alpha/\partial t$ being the local angular velocity of the curtain's rotation as a whole. Note that a similar Coriolis term does not appear in (3.35) because the cross-stream velocity is much smaller than the streamwise one.

   (iii) The second term in (3.36) describes the centripetal acceleration of liquid particles, with $\partial\alpha/\partial l$ being the curtain's local curvature.

   (iv) The terms involving $\sin\alpha$ in (3.35) and $\cos\alpha$ in (3.36) describe gravity.

   (v) As follows from the boundary condition (3.40), the capillary force depends only on the centreline's curvature – as the approximation of a slender curtain makes the curvature of its boundaries be negligible.

(2) Observe that the pressure has dropped out from the streamwise equation (3.36). This allows one to eliminate it altogether – by integrating (3.36) with respect to τ over (τ_-,τ_+) and taking into account the boundary condition (3.40), which yields

$$2\frac{\partial\alpha}{\partial t}\int_{\tau_-}^{\tau_+}u_l\,\mathrm{d}\tau+\frac{\partial\alpha}{\partial l}\left(\int_{\tau_-}^{\tau_+}u_l^2\,\mathrm{d}\tau-2\gamma\right)$$
$$=(\tau_+-\tau_-)\left(-\cos\alpha+\frac{\partial^2 x}{\partial t^2}\sin\alpha-\frac{\partial^2 z}{\partial t^2}\cos\alpha\right). \tag{3.42}$$

This equation governs $\alpha(l,t)$, and it will be used instead of (3.36) and (3.40).

The physical meaning of the terms in (3.42) indicates that the curtain's trajectory is determined by the Coriolis force, centripetal acceleration, surface tension, gravity and force of inertia.

3.5. *The boundary conditions at the outlet*

Note that the asymptotic set (3.35)–(3.41) is inapplicable to the small region near the outlet, where the streamwise and cross-stream scales are both comparable to H. The solution in this region is, generally, difficult to find due to the absence of small parameters – which is unfortunate, as the flow 'exiting' from it supplies the 'entrance' boundary condition for the asymptotic set.

One can safely assume, however, that, once the flow has emerged from the outlet, it begins to homogenise in the cross-stream direction due to viscosity – and, after a certain distance L_h, turns into a plug flow. This implies the following boundary condition for the asymptotic set:

$$u_l = 1, \quad u_\tau = 0 \quad \text{at } \tau \in [-1, 1]. \tag{3.43a,b}$$

This condition is applicable only if the homogenisation occurs faster than the dynamics described by the asymptotic model derived, i.e. if L_h is much smaller than the asymptotic scale L. Estimating that $L_h \sim H\,Re$ (Tillett 1968), with the Reynolds number given by

$$Re = \frac{UH}{\nu}, \tag{3.44}$$

and recalling definition (3.1) of L, one can see that (3.43) is applicable if

$$\mu \gg 1, \tag{3.45}$$

where μ is defined by (3.25).

Next, consider the limit of high Reynolds number, $Re \gg 1$, in which case the emerging flow is very close to the Poiseuille profile and remains so within a long distance from the outlet (Tillett 1968; Khayat 2014). This implies the following boundary condition for the asymptotic set:

$$u_l = \tfrac{3}{2}(1 - \tau^2), \quad u_\tau = 0 \quad \text{at } \tau \in [-1, 1], \tag{3.46a,b}$$

where the coefficient 3/2 was included to make the non-dimensional flow rate equal that for the plug flow (3.43).

To clarify when (3.46) applies, note that $\mu = 1/(\varepsilon\,Re)$, so that the condition $Re \gg 1$ corresponds to

$$\mu \ll \varepsilon^{-1}. \tag{3.47}$$

Observe that the applicability conditions (3.45) and (3.47) overlap in the region

$$1 \ll \mu \ll \varepsilon^{-1}. \tag{3.48}$$

This seems to suggest that, for range (3.48), two different boundary conditions can be used – and, in a sense, they can. Since μ is large in this range, the viscosity term dominates the asymptotic equation (3.35) – hence, for any boundary condition whatsoever, the flow homogenises within a short distance from the outlet. One can just as well assume that the flow was homogeneous from the start, i.e. use condition (3.43).

4. Steady curtains: the high-viscosity limit

Consider steady curtains with $\mu \gg 1$, which will be referred to as the high-viscosity limit. Due to the homogeneity of the boundary condition (3.43), the whole flow is homogeneous in the cross-stream direction – which suggests the following substitution:

$$\left.\begin{aligned} u_l(l,\tau,t) = u(l), \quad u_\tau(l,\tau,t) = w(l) - \tau\frac{\partial u(l)}{\partial l}, \\ x(l,t) = x(l), \quad z(l,t) = z(l). \end{aligned}\right\} \tag{4.1}$$

Observe that this ansatz eliminates the viscous term in (3.35) – i.e. once the flow was homogenised in the near-outlet region, viscosity has no effect on the solution elsewhere.

Substituting (4.1) into (3.35), (3.37)–(3.39), (3.41)–(3.42), eliminating $w(l)$ by subtracting $(3.38)_-$ from $(3.38)_+$ and omitting the subscript l, one obtains

$$u\frac{\mathrm{d}u}{\mathrm{d}l} = -\sin\alpha, \tag{4.2}$$

$$\frac{\mathrm{d}(uW)}{\mathrm{d}l} = 0, \tag{4.3}$$

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \cos\alpha, \tag{4.4}$$

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \sin\alpha, \tag{4.5}$$

$$(2\gamma - Wu^2)\frac{\mathrm{d}\alpha}{\mathrm{d}l} = W\cos\alpha, \tag{4.6}$$

where $W = \tau_+ - \tau_-$ is the curtain's thickness.

Let the origin of the Cartesian coordinates be the middle point of the outlet, and the outlet's boundaries be at $\tau_\pm = \pm 1$, i.e.

$$x = 0, \quad z = 0 \quad \text{at } l = 0, \tag{4.7a,b}$$

$$W = 2 \quad \text{at } l = 0, \tag{4.8}$$

and assume that the curtain is ejected from the outlet at an angle α_0,

$$\alpha = \alpha_0 \quad \text{at } l = 0, \tag{4.9}$$

where $\alpha_0 > 0$ corresponds to an initially upward flow.

Equations (4.2)–(4.6) and boundary conditions (3.43), (4.7)–(4.9) fully determine u, W, x, z and α.

4.1. *Upwards-bending curtains*

The above boundary-value problem has a highly counterintuitive property: if

$$\gamma > \tfrac{1}{2}Wu^2, \tag{4.10}$$

it follows from (4.6) that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}l} > 0, \tag{4.11}$$

i.e. the curtain bends upwards (against the effect of gravity)!

Furthermore, disregarding the trivial case of vertical curtains, one can show that the dynamics of an oblique curtain depends on whether (4.10) holds at the outlet. Substituting thus the boundary conditions (3.43), (4.8) into (4.10), one obtains

$$\gamma > 1. \tag{4.12}$$

If (4.12) holds, the right-hand side of (4.6) always remains positive (see appendix B), and the curtain bends upwards. (Here and hereinafter, the word 'bend' is used when referring to the curtain's curvature, with the word 'turn' implying a change of direction (up to down and *vice versa*).) In fact, even if it is initially ejected downwards ($\alpha_0 \in (-1/2\pi, 0)$), it will eventually turn upwards.

To find out what happens after that, one should use (4.5) to replace $\sin\alpha$ with $\mathrm{d}z/\mathrm{d}l$ in (4.2), integrate this equation, as well as (4.3), with respect to l, and use the boundary conditions (4.7)–(4.8) to obtain

$$u = \sqrt{1-2z}, \quad W = \frac{2}{\sqrt{1-2z}}. \tag{4.13a,b}$$

Thus,

(a) at $z_{max} = 1/2$, the velocity of the curtain vanishes, so it 'stops';

(b) as $z \to z_{max}$, the curtain's thickness tends to infinity.

To interpret result (a) physically, recall that, in the case under consideration, the flow is not sheared – hence, viscosity is not a factor, and the energy is conserved. This makes z_{max} the height to which the liquid particles can climb given their initial supply of (non-dimensional) kinetic energy.

Result (b), in turn, violates the approximation of slender curtains – hence, the solution obtained cannot be trusted near the stoppage point. Still, the general tendency of a rising curtain to become thicker can be trusted; one can also assume that, having reached z_{max}, the curtain breaks up and splashes down (similar to a jet directed vertically upwards).

To illustrate the above conclusions, use (4.12) to reduce (4.6) to

$$\frac{\mathrm{d}\alpha}{\mathrm{d}l} = \frac{\cos\alpha}{(\gamma - \sqrt{1-2z})\sqrt{1-2z}}. \tag{4.14}$$

Equations (4.4)–(4.5), (4.14) were solved numerically with the boundary conditions (4.7), (4.9), for various values of γ and α_0. Some of these solutions are shown in figure 2: one can see that, as $\gamma \to 1^-$, the curtain bends increasingly steeply downwards and, in the limit $\gamma \to 1^+$, upwards.

4.2. *Discussion*

Given the highly counterintuitive nature of upwards-bending curtains, the reader may not help but think that they arise due to a sign error in the equations derived. To this end, it should be reassuring to show that these equations also admit solutions with a clear physical interpretation – and even more so, since they happen to clarify the physics of the upwards-bending curtains.

(1) Without surface tension, the trajectory of a curtain should coincide with that of a free-falling object launched with a unit velocity, at an angle α_0.

To show this, set $\gamma = 0$ in (4.6) and use (4.3) to rewrite it in terms of x instead of l. Doing the same with (4.5) and taking into account (4.13a,b), one obtains

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \tan\alpha, \quad \frac{\mathrm{d}\alpha}{\mathrm{d}x} = \frac{1}{2z-1}. \tag{4.15a,b}$$

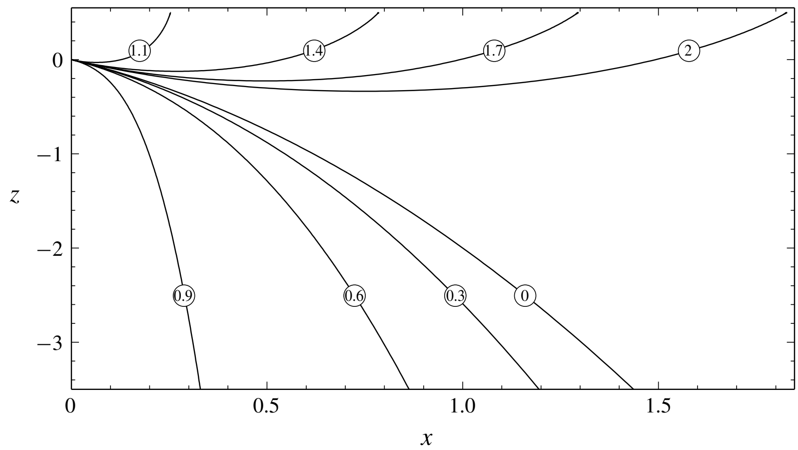

FIGURE 2. Steady curtains under high viscosity. The curves are marked with the corresponding value of the capillary parameter γ (the threshold value of γ separating upwards- and downwards-bending curtains is $\gamma = 1$). In all cases, $\alpha_0 = -(1/4)\pi$.

It can be verified by substitution that these equations and the boundary conditions (4.7), (4.9) are satisfied if

$$z = x \tan\alpha_0 - \frac{x^2}{2\cos^2\alpha_0}, \quad \cos\alpha = \frac{\cos\alpha_0}{\sqrt{1-2z}}. \tag{4.16a,b}$$

These equalities do describe the trajectory of a free-falling object.

(2) For curtains falling vertically ($\alpha = -1/2\pi$), equation (4.2) becomes

$$u\frac{\mathrm{d}u}{\mathrm{d}l} = 1. \tag{4.17}$$

To compare this equation with that of G. I. Taylor (see the appendix in Brown (1961)), rewrite the latter in terms of the present paper's variables, which yields

$$u\frac{\mathrm{d}u}{\mathrm{d}l} - 1 = 4\mu\varepsilon^2 u \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}l}\right). \tag{4.18}$$

This equation and (4.17) agree if $\varepsilon \ll 1$ (which is indeed the applicability condition of the latter).

(3) Our asymptotic equations admit a solution describing a free-falling liquid sheet. It is described by a modification of ansatz (4.1), where the expression for z is replaced by

$$z(l, t) = z(l) - \tfrac{1}{2}t^2. \tag{4.19}$$

Substituting the modified ansatz into (3.35), (3.37)–(3.39), (3.41)–(3.42) and disregarding the boundary conditions at the outlet, one obtains

$$u\frac{\mathrm{d}u}{\mathrm{d}l} = 0, \quad \frac{\mathrm{d}(Wu)}{\mathrm{d}l} = 0, \quad (2\gamma - Wu^2)\frac{\mathrm{d}\alpha}{\mathrm{d}l} = 0. \tag{4.20a–c}$$

Thus, u and W are constants and, if $\mathrm{d}\alpha/\mathrm{d}l \neq 0$ (non-zero curvature), then

$$Wu^2 = 2\gamma. \tag{4.21}$$

This equation shows that the centripetal acceleration associated with the flow's curvature matches the capillary pressure difference between the sheet's convex and concave boundaries (the latter pressure is sub-atmospheric, the former is super-atmospheric). Since the centripetal acceleration and pressure difference are both proportional to the curvature, $d\alpha/dl$ cancels from condition (4.21) of their balance.

As α is not restricted by the leading-order equations, one should assume that it is determined by higher-order effects. There is one particular case, however, where a two-dimensional non-sheared flow bounded by capillary surfaces can be safely conjectured to be steady in all orders: when these surfaces are circular, i.e.

$$\alpha(l) = \frac{l}{R} \tag{4.22}$$

(in a three-dimensional case, the steady shape would be spherical). Substitution of (4.22) into (4.4)–(4.5) yields a circular trajectory of radius R,

$$x(l) = R\sin\frac{l}{R}, \quad z(l) = -R\cos\frac{l}{R}. \tag{4.23a,b}$$

Together with (4.19), these equalities describe a two-dimensional free-falling 'liquid ring'. It is probably unstable with respect to three-dimensional (axial) perturbations, but this is unimportant. What matters is that the convex/concave capillary pressure difference can support a steady curved flow.

The same pressure difference can support upwards-bending curtains (if it is sufficiently strong, of course).

(4) Perhaps the most convincing argument supporting the present results follows from the fact that similar solutions have been also found for slender jets with a nearly circular cross-section by Wallwork (2001). It was shown that, under the assumption of a plug flow, the condition of existence of steady upwards-bending jets is

$$\gamma > Ru^2, \tag{4.24}$$

where $R(l)$ is the jet's radius. Unfortunately, this condition also makes the Plateau–Rayleigh instability so strong that the jet breaks up just outside the outlet, at a distance comparable to the jet's radius. This is probably why upwards-bending jets have never been observed in experiments.

Note that the Plateau–Rayleigh instability affects only flows with cylindrical geometry, so liquid curtains are not subject to this kind of instability.

5. Steady curtains: the low-viscosity limit

Assume that $\mu \ll 1$, so that the viscosity term in (3.35) can be omitted. This assumption will be referred to as the low-viscosity limit.

Even though the flow's cross-stream profile is not homogeneous in this case (due to the sheared boundary condition (3.46) at the outlet), the governing equation can still be reduced to a simple set of ordinary differential equations in l.

To do so, one should use the so-called semi-Lagrangian variables. They were put forward by Odulo (1979) in his study of shallow-water equations with rotation, and later used for various shallow-water/slender-flow problems (e.g. Zakharov 1981; Benilov 1995, 2015). These variables are called 'semi' because only the cross-stream Eulerian coordinate is replaced with a Lagrangian marker, whereas the streamwise coordinate remains Eulerian.

5.1. *A semi-Lagrangian form of the asymptotic equations*

Introduce new variables (l_{new}, ξ, t_{new}) related to the Eulerian coordinates (l, τ, t) by

$$l = l_{new}, \quad \tau = \tau(l_{new}, \xi, t_{new}), \quad t = t_{new}, \tag{5.1a–c}$$

where $\tau(l_{new}, \xi, t_{new})$ satisfies (the subscript *new* omitted)

$$\frac{\partial \tau}{\partial t} + u_l \frac{\partial \tau}{\partial x} - u_\tau = 0, \tag{5.2}$$

$$\tau = \tau_\pm \quad \text{if } \xi = \pm 1. \tag{5.3}$$

Physically, ξ is the Lagrangian marker corresponding to the Eulerian coordinate τ. Note that conditions (5.3) map the curtain's free boundaries onto fixed straight lines $\xi = \pm 1$, and that (3.38) follows from (5.2) evaluated at $\xi = \pm 1$ – so the former can be omitted.

Introduce also

$$\eta = \frac{\partial \tau}{\partial \xi}, \tag{5.4}$$

which describes the liquid's stretching in the cross-stream direction.

Further technical details will be skipped (but can be looked up in Benilov 2015). Omitting the subscripts *new* and l, one can rewrite (3.35), (3.37), (3.39) and (3.41)–(3.42) in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial l} = \frac{\mu}{\eta}\frac{\partial}{\partial \xi}\left(\frac{1}{\eta}\frac{\partial u}{\partial \xi}\right) - \sin\alpha - \frac{\partial^2 x}{\partial t^2}\cos\alpha - \frac{\partial^2 z}{\partial t^2}\sin\alpha, \tag{5.5}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial\,(u\eta)}{\partial l} = 0, \tag{5.6}$$

$$\frac{\partial u}{\partial \xi} = 0 \quad \text{if } \xi = \pm 1, \tag{5.7}$$

$$\frac{\partial x}{\partial l} = \cos\alpha, \quad \frac{\partial z}{\partial l} = \sin\alpha, \tag{5.8a,b}$$

$$2\frac{\partial \alpha}{\partial t}\int_{-1}^{1} u\eta\,\mathrm{d}\xi + \frac{\partial \alpha}{\partial l}\left(\int_{-1}^{1} u^2\eta\,\mathrm{d}\xi - 2\gamma\right) = W\left(-\cos\alpha + \frac{\partial^2 x}{\partial t^2}\sin\alpha - \frac{\partial^2 z}{\partial t^2}\cos\alpha\right), \tag{5.9}$$

where

$$W = \int_{-1}^{1} \eta\,\mathrm{d}\xi \tag{5.10}$$

is, as before, the curtain's thickness. Observe that the cross-stream velocity u_τ has been eliminated, or rather replaced with η.

Note that set (5.5)–(5.10) can only be used if the flow does not have stagnation points (where the Jacobian of the Eulerian-to-semi-Lagrangian transformation becomes infinite).

To formulate a boundary condition for η, let the Lagrangian markers of the particles passing through the outlet coincide with their cross-stream coordinates, i.e. $\xi = \tau$ at $l = 0$, so definition (5.4) of η yields

$$\eta = 1 \quad \text{at } l = 0. \tag{5.11}$$

The boundary conditions for x, z and α remain as given by (4.7) and (4.9), whereas that for u is given by (3.46) with subscript l omitted and τ changed to ξ, i.e.

$$u = \tfrac{3}{2}(1 - \xi^2) \quad \text{at } l = 0. \tag{5.12}$$

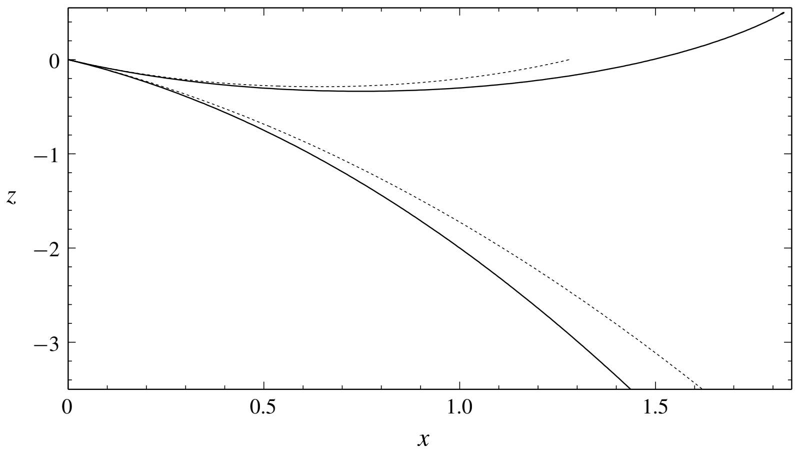

FIGURE 3. Comparison of steady curtains under high (solid line) and low (dotted line) viscosity. The downwards- and upwards-bending curtains correspond to $\gamma = 0$ and $\gamma = 2$, respectively. In all cases, $\alpha_0 = -(1/4)\pi$.

5.2. *Steady curtains*

With the viscous term omitted, equations (5.5)–(5.10), (4.7)–(4.9), (5.11)–(5.12) admit the following steady solution:

$$u = \sqrt{\tfrac{9}{4}(1-\xi^2)^2 - 2z}, \quad \eta = \frac{\tfrac{3}{2}(1-\xi^2)}{u}, \tag{5.13a,b}$$

and $\alpha(l)$ satisfies

$$\frac{\mathrm{d}\alpha}{\mathrm{d}l} = F(z)\cos\alpha, \tag{5.14}$$

where

$$F(z) = \frac{\displaystyle\int_{-1}^{1} \frac{\tfrac{3}{2}(1-\xi^2)}{\sqrt{\tfrac{9}{4}(1-\xi^2)^2 - 2z}}\,\mathrm{d}\xi}{\displaystyle 2\gamma - \int_{-1}^{1} \tfrac{3}{2}(1-\xi^2)\sqrt{\tfrac{9}{4}(1-\xi^2)^2 - 2z}\,\mathrm{d}\xi}. \tag{5.15}$$

The downwards-bending curtains described by the low-viscosity set (5.8), (5.14)–(5.15) are qualitatively similar to their high-viscosity counterparts, except that the latter bend steeper, and thus fall shorter, than the former (see figure 3).

If $\gamma > 6/5$, low-viscosity curtains bend upwards. Unlike their high-viscosity counterparts, they all 'terminate' at $z = 0$ (see figure 3). This is due to the fact that low-viscosity curtains are sheared, with the slowest particles (at the curtain's boundaries) having zero velocity at the point of ejection. As a result, these particles cannot climb to a greater height than that of the outlet, and the corresponding streamlines end with stagnation points located at $z = 0$.

Mathematically, expression (5.15) for $F(z)$ becomes complex for $z > 0$, as the semi-Lagrangian variables fail due to the stagnation points at $z = 0$. This failure is of little importance, however, as stagnation points make the flow locally unstable (e.g. Friedlander & Vishik 1991; Lifschitz 1991; Leblanc 1997). As a result, inviscid upwards-bending curtains break up at $z = 0$ and do not reach higher.

5.3. *Discussion*

(1) Thus, curtains with high and low viscosity differ only by the cross-sectional profile of the velocity: a plug flow for the former and Poiseuille flow for the latter. Interestingly, neither kind of curtains is affected by viscosity: due to the (almost homogeneous) velocity field in the former and the (weak) viscosity in the latter – different reasons, but the same result.

Given the above, it may be sensible to rename the high- and low-viscosity limits as the plug-flow and Poiseuille-flow limits, respectively.

(2) Note that, even though the analysis of Tillett (1968) suggests that the homogenisation length L_h is proportional to $H\,Re$, the coefficient of proportionality can be small. For jets, for example, the surface velocity reaches 50 % of the centreline velocity after a distance of only $L_h \sim 0.04\,H\,Re$ (see Goren 1966; Sevilla 2011, figure 2*b*).

If the same is true for curtains, this would severely reduce the region of applicability of the Poiseuille-flow limit and, at the same time, expand that of the plug-flow one.

6. Summary and concluding remarks

The two main results of this paper are as follows. First, an asymptotic set (3.35), (3.37)–(3.39), (3.41*a*,*b*)–(3.42) has been derived for two-dimensional liquid curtains with a large Froude number. It was shown that, depending on how strong is the viscosity, the set is to be solved with either the plug-flow or Poiseuille-flow boundary conditions (see § 3.5).

Second, the equations derived were used to examine steady curtains in the limits of high and low viscosity. For both cases, similar results were obtained: if the reciprocal of the Weber number exceeds a certain threshold, the curtain bends upwards, i.e. against gravity. In dimensional terms, this occurs when the surface tension exceeds half of the momentum flux passing through the outlet, i.e.

$$\sigma > \frac{\rho}{2}\int_{-H}^{H} u^2(z)\,\mathrm{d}z. \tag{6.1}$$

For any given σ, one can force (6.1) to hold by reducing the outlet width and keeping the ejection velocity fixed. This way, one can also ensure that the Froude number is large (which is the condition under which (6.1) was derived).

A possible experimental setup for testing the existence of upwards-bending curtains may consist of a rectangular tank with a thin horizontal slot cut through one of its walls. Once the tank is filled with water, a liquid curtain will emerge through the slot (outlet). Assuming the plug-flow velocity profile at the outlet and using the Bernoulli expression for the ejection velocity,

$$u \approx \sqrt{2gD}, \tag{6.2}$$

where D is the outlet's depth, one obtains

$$\frac{\rho}{2}\int_{-H}^{H} u^2(z)\,\mathrm{d}z \approx \rho g D H. \tag{6.3}$$

Let the outlet's depth and width be $D = 1$ cm and $2H = 1$ mm, respectively, in which case

$$\frac{\rho}{2}\int_{-H}^{H} u^2(z)\,\mathrm{d}z \approx 49\ \mathrm{mN\ m^{-1}}. \tag{6.4}$$

With the surface tension of water at room temperature being $\sigma \approx 72.5$ mN m^{-1}, condition (6.1) holds – hence, curtains with such parameters should bend upwards. Note also that the corresponding Froude number is indeed large:

$$\frac{U^2}{gH} \approx \frac{2D}{H} = 40. \tag{6.5}$$

Still, to make sure that upwards-bending curtains can be observed in an experiment, one should verify that they are stable. The simplest way to do this is to employ the three-dimensional extension of the asymptotic model proposed here.

Note, however, that vertical curtains have been shown to be stable with respect to small perturbations – both experimentally (Finnicum *et al.* 1993; Li 1993, 1994; Roche *et al.* 2006; Lhuissier *et al.* 2016) and theoretically (Benilov *et al.* 2016). This implies that at least near-vertical curtains are also stable, so the effect predicted in this paper could be observed for these. Upwards-bending curtains can also be observed if $\gamma - 1 \ll 1$ (i.e. criterion (6.1) holds, but only just): such curtains are almost flat and, thus, stable.

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Appendix A. Material derivatives in terms of time-dependent curvilinear coordinates

To derive formulae (2.6)–(2.7), let

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\boldsymbol{\cdot}\nabla)\boldsymbol{u}. \tag{A 1}$$

The second term on the right-hand side of (A 1) is not affected by the dependence of the curvilinear coordinates on t and, thus, can be found in any text of fluid dynamics (e.g. Kochin *et al.* 1964):

$$[(\boldsymbol{u}\boldsymbol{\cdot}\nabla)\boldsymbol{u}]_l = \frac{u_l}{h_l}\frac{\partial u_l}{\partial l} + \frac{u_\tau}{h_\tau}\frac{\partial u_l}{\partial \tau} + \frac{u_l u_\tau}{h_l h_\tau}\frac{\partial h_l}{\partial \tau} - \frac{u_\tau^2}{h_l h_\tau}\frac{\partial h_\tau}{\partial l}, \tag{A 2}$$

$$[(\boldsymbol{u}\boldsymbol{\cdot}\nabla)\boldsymbol{u}]_\tau = \frac{u_l}{h_l}\frac{\partial u_\tau}{\partial l} + \frac{u_\tau}{h_\tau}\frac{\partial u_\tau}{\partial \tau} + \frac{u_l u_\tau}{h_l h_\tau}\frac{\partial h_\tau}{\partial l} - \frac{u_l^2}{h_l h_\tau}\frac{\partial h_l}{\partial \tau}. \tag{A 3}$$

To calculate the first term in (A 1), re-denote

$$(x, z) = (x_1, x_2), \qquad (l, \tau) = (q_1, q_2). \tag{A 4a,b}$$

It is convenient to use two different notations for the time variable: if used with x_i (or q_i), it will be denoted by t (or t'), e.g.

$$x_1 = x_1(q_1, q_2, t'), \qquad x_1 = x_1(q_1, q_2, t'), \qquad t = t'. \tag{A 5a–c}$$

Let the matrix $\partial x_k/\partial q_i$ be orthogonal, i.e.

$$\sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} = h_i^2 \delta_{ij}, \tag{A 6}$$

where h_i are the Lamé coefficients, δ_{ij} is the Kronecker delta and the summation with respect to repeated indices is not implied unless stated explicitly. It can be readily shown that (A 6) entails

$$\frac{\partial q_j}{\partial x_i} = h_j^{-2} \frac{\partial x_i}{\partial q_j}. \tag{A 7}$$

Now, denoting the Cartesian components of the velocity by u_i, one obtains

$$\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial t'} - \sum_k \sum_n \frac{\partial u_i}{\partial q_n} \frac{\partial q_n}{\partial x_k} \frac{\partial x_k}{\partial t'}, \tag{A 8}$$

which can be rearranged using (A 7),

$$\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial t'} - \sum_k \sum_n \frac{\partial u_i}{\partial q_n} h_n^{-2} \frac{\partial x_k}{\partial q_n} \frac{\partial x_k}{\partial t'}. \tag{A 9}$$

The Navier–Stokes equations need to be rewritten in terms of the velocity components with respect to the curvilinear coordinates, i.e.

$$v_j = \hat{\boldsymbol{q}}_j \cdot \boldsymbol{u}, \tag{A 10}$$

where $\hat{\boldsymbol{q}}_j$ is the unit vector corresponding to the coordinate q_j. Its Cartesian coordinates are

$$(\hat{\boldsymbol{q}}_j)_i = h_j^{-1} \frac{\partial x_i}{\partial q_j}, \tag{A 11}$$

so (A 9) yields

$$\hat{\boldsymbol{q}}_j \cdot \frac{\partial \boldsymbol{u}}{\partial t} = h_j^{-1} \sum_i \frac{\partial x_i}{\partial q_j} \frac{\partial u_i}{\partial t'} - h_j^{-1} \sum_i \frac{\partial x_i}{\partial q_j} \sum_k \sum_n \frac{\partial u_i}{\partial q_n} h_n^{-2} \frac{\partial x_k}{\partial q_n} \frac{\partial x_k}{\partial t'}. \tag{A 12}$$

Expressing the Cartesian components of the velocity through v_j,

$$u_i = \sum_n \frac{\partial x_i}{\partial q_n} h_n^{-1} v_n, \tag{A 13}$$

one can rewrite (A 12) in the form

$$\hat{\boldsymbol{q}}_j \cdot \frac{\partial \boldsymbol{u}}{\partial t} = h_j \frac{\partial (h_j^{-1} v_j)}{\partial t'} + h_j^{-1} \sum_m a_{jm} h_m^{-1} v_m - \sum_n h_n^{-2} b_n \left[h_j \frac{\partial (h_j^{-1} v_j)}{\partial q_n} + h_j^{-1} \sum_m c_{jnm} h_m^{-1} v_m \right], \tag{A 14}$$

where

$$a_{jm} = \sum_i \frac{\partial x_i}{\partial q_j} \frac{\partial^2 x_i}{\partial t' \partial q_m}, \quad b_n = \sum_k \frac{\partial x_k}{\partial q_n} \frac{\partial x_k}{\partial t'}, \quad c_{jnm} = \sum_i \frac{\partial x_i}{\partial q_j} \frac{\partial^2 x_i}{\partial q_n \partial q_m}. \tag{A 15a–c}$$

Expressions (A 14)–(A 15) are written in dimension-independent form. Adapting them to the two-dimensional problem at hand and returning to the (l, τ) notation, one obtains, after routine algebra,

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} \right)_l = \frac{\partial u_l}{\partial t'} + \frac{u_s}{h_l h_s} \left(\frac{\partial x}{\partial l} \frac{\partial^2 x}{\partial t' \partial s} + \frac{\partial z}{\partial l} \frac{\partial^2 z}{\partial t' \partial s} \right)$$

$$
\begin{aligned}
&-\frac{1}{h_l^2}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t'}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t'}\right)\left(\frac{\partial u_l}{\partial l}+\frac{u_s}{h_s}\frac{\partial h_l}{\partial s}\right)\\
&-\frac{1}{h_s^2}\left(\frac{\partial x}{\partial s}\frac{\partial x}{\partial t'}+\frac{\partial z}{\partial s}\frac{\partial z}{\partial t'}\right)\left[\frac{\partial u_l}{\partial s}+\frac{u_s}{h_l h_s}\left(\frac{\partial x}{\partial l}\frac{\partial^2 x}{\partial s^2}+\frac{\partial z}{\partial l}\frac{\partial^2 z}{\partial s^2}\right)\right], \qquad \text{(A 16)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_s = {}& \frac{\partial u_s}{\partial t'}+\frac{u_l}{h_l h_s}\left(\frac{\partial x}{\partial s}\frac{\partial^2 x}{\partial t'\partial l}+\frac{\partial z}{\partial s}\frac{\partial^2 z}{\partial t'\partial l}\right)\\
&-\frac{1}{h_l^2}\left(\frac{\partial x}{\partial l}\frac{\partial x}{\partial t'}+\frac{\partial z}{\partial l}\frac{\partial z}{\partial t'}\right)\left[\frac{\partial u_s}{\partial l}+\frac{u_l}{h_l h_s}\left(\frac{\partial x}{\partial s}\frac{\partial^2 x}{\partial l^2}+\frac{\partial z}{\partial s}\frac{\partial^2 z}{\partial l^2}\right)\right]\\
&-\frac{1}{h_s^2}\left(\frac{\partial x}{\partial s}\frac{\partial x}{\partial t'}+\frac{\partial z}{\partial s}\frac{\partial z}{\partial t'}\right)\left(\frac{\partial u_s}{\partial s}+\frac{u_l}{h_l}\frac{\partial h_s}{\partial l}\right). \qquad \text{(A 17)}
\end{aligned}
$$

Substituting these expressions and (A 2)–(A 3) into (A 1) and omitting the prime from t', one can obtain formulae (2.6)–(2.7) as required.

Appendix B. Can steady oblique curtains change curvature?

In mathematical terms, the title question of this appendix amounts to whether or not $\mathrm{d}\alpha/\mathrm{d}l$ can change sign for $\alpha_0 \neq \pm(1/2)\pi$. When answering it, one can confine oneself to the case $\gamma > 1$ (if $\gamma < 1$, it can be readily shown that the curtain bends with gravity, which is not an interesting scenario).

As follows from (4.14), $\mathrm{d}\alpha/\mathrm{d}l$ can change sign only if $\gamma - \sqrt{1-2z}$ changes sign, which happens only if z assumes the value of

$$z_0 = -\tfrac{1}{2}(\gamma^2 - 1). \qquad \text{(B 1)}$$

Considering (4.5) and (4.14) as a set for $z(l)$ and $\alpha(l)$, one can readily verify that they admit a first integral of the form

$$F(z)\cos\alpha = \text{const.}, \qquad \text{(B 2)}$$

where $F(z)$ satisfies

$$\frac{\mathrm{d}F}{\mathrm{d}z} = \frac{F}{(\gamma - \sqrt{1-2z})\sqrt{1-2z}}. \qquad \text{(B 3)}$$

In principle, this equation can be integrated exactly, but one only needs to observe that

$$F = (z - z_0) + O[(z - z_0)^2] \quad \text{as } z \to z_0. \qquad \text{(B 4)}$$

Since vertical curtains have been excluded (hence, $\cos\alpha > 0$) and since the curtain 'comes' from the region where $z > z_0$, it follows from (B 2)–(B 4) that

$$\text{const.} > 0. \qquad \text{(B 5)}$$

Subject to this inequality, equations (B 2)–(B 4) imply that z never equals z_0 – hence, $\gamma - \sqrt{1-2z}$ never changes sign – hence, the right-hand side of (4.14) is always positive, and so is $\mathrm{d}\alpha/\mathrm{d}l$.

REFERENCES

BENILOV, E. S. 1995 Stability of large-amplitude geostrophic flows localized in a thin layer. *J. Fluid Mech.* **288**, 157–174.

BENILOV, E. S. 2015 Hydraulic jumps in a shallow flow down a slightly inclined substrate. *J. Fluid Mech.* **782**, 5–24.

BENILOV, E. S., BARROS, R. & O'BRIEN, S. B. G. 2016 Stability of thin liquid curtains. *Phys. Rev.* E **94**, 043110.

BENILOV, E. S., LAPIN, V. N. & O'BRIEN, S. B. G. 2011 On rimming flows with shocks. *J. Engng Maths* **75**, 49–62.

BROWN, D. R. 1961 A study of the behaviour of a thin sheet of moving liquid. *J. Fluid Mech.* **10**, 297–305.

DECENT, S. P., KING, A. C. & WALLWORK, I. M. 2002 Free jets spun from a prilling tower. *J. Engng Maths* **42**, 265–282.

DECENT, S. P., PĂRĂU, E. I., SIMMONS, M. J. H. & UDDIN, J. 2018 On mathematical approaches to modelling slender liquid jets with a curved trajectory. *J. Fluid Mech.* **844**, 905–916.

DYSON, R. J., BRANDER, J., BREWARD, C. J. W. & HOWELL, P. D. 2009 Long-wavelength stability of an unsupported multilayer liquid film falling under gravity. *J. Engng Maths* **64**, 237–250.

ENTOV, V. M. & YARIN, A. L. 1984 The dynamics of thin liquid jets in air. *J. Fluid Mech.* **140**, 91.

FINNICUM, D. S., WEINSTEIN, S. J. & RUSCHAK, K. J. 1993 The effect of applied pressure on the shape of a two-dimensional liquid curtain falling under the influence of gravity. *J. Fluid Mech.* **255**, 647–665.

FRIEDLANDER, S. & VISHIK, M. M. 1991 Instability criteria for the flow of an inviscid incompressible fluid. *Phys. Rev. Lett.* **66**, 2204–2206.

GOREN, S. L. 1966 Development of the boundary layer at a free surface from a uniform shear flow. *J. Fluid Mech.* **25**, 87–95.

KHAYAT, R. E. 2014 Free-surface jet flow of a shear-thinning power-law fluid near the channel exit. *J. Fluid Mech.* **748**, 580–617.

KOCHIN, N. E., KIBEL, I. A. & ROZE, N. V. 1964 *Theoretical Hydromechanics*. Wiley.

LEBLANC, S. 1997 Stability of stagnation points in rotating flows. *Phys. Fluids* **9**, 3566–3569.

LHUISSIER, H., BRUNET, P. & DORBOLO, S. 2016 Blowing a liquid curtain. *J. Fluid Mech.* **795**, 784–807.

LI, X. 1993 Spatial instability of plane liquid sheets. *Chem. Engng Sci.* **48**, 2973–2981.

LI, X. 1994 On the instability of plane liquid sheets in two gas streams of unequal velocities. *Acta Mech.* **106**, 137–156.

LIFSCHITZ, A. 1991 Short wavelength instabilities of incompressible three-dimensional flows and generation of vorticity. *Phys. Lett.* A **157**, 481–487.

ODULO, A. B. 1979 Long non-linear waves in the rotating ocean of variable depth. *Dokl. Akad. Nauk SSSR* **248**, 1439–1442.

RAMOS, J. I. 1996 Planar liquid sheets at low Reynolds numbers. *Intl J. Numer. Meth. Fluids* **22**, 961–978.

ROCHE, J. S., GRAND, N. L., BRUNET, P., LEBON, L. & LIMAT, L. 2006 Perturbations on a liquid curtain near break-up: wakes and free edges. *Phys. Fluids* **18**, 082101.

SEVILLA, A. 2011 The effect of viscous relaxation on the spatiotemporal stability of capillary jets. *J. Fluid Mech.* **684**, 204–226.

SHIKHMURZAEV, Y. D. & SISOEV, G. M. 2017 Spiralling liquid jets: verifiable mathematical framework, trajectories and peristaltic waves. *J. Fluid Mech.* **819**, 352–400.

TILLETT, J. P. K. 1968 On the laminar flow in a free jet of liquid at high Reynolds numbers. *J. Fluid Mech.* **32** (02), 273.

WALLWORK, I. M. 2001 The trajectory and stability of a spiralling liquid jet. PhD thesis, University of Birmingham.

WALLWORK, I. M., DECENT, S. P., KING, A. C. & SCHULKES, R. M. S. M. 2002 The trajectory and stability of a spiralling liquid jet. Part 1. Inviscid theory. *J. Fluid Mech.* **459**, 43–65.

ZAKHAROV, V. E. 1981 On the Benney equations. *Physica* D **3**, 193–202.