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Explosive instability in a linear system with neutrally stable eigenmodes. Part 2. Multi-dimensional disturbances

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We examine the dynamics of a thin film of viscous fluid on the inside surface of a cylinder with horizontal axis, rotating about this axis. Using the so-called lubrication approximation, we derive an asymptotic equation for three-dimensional motion of the film and use this equation to examine its linear stability. It is demonstrated that: (i) there are infinitely many normal modes (harmonic in the axial variable and time), which are all neutrally stable and their eigenfunctions form a complete set; (ii) but the film is nonetheless unstable with respect to non-harmonic disturbances, which develop singularities in a finite time.

1. Introduction

Harmonic solutions (eigenmodes) play an important role in continuum mechanics, and can be used in proving both stability and instability. In the former case, it is sufficient to find a single growing solution, whereas in the latter the task is more difficult. In order to prove that a system is stable with respect to linear disturbances, one has to demonstrate the existence of a complete set of eigenmodes and prove that all of them are stable (bounded in time). Then, an arbitrary initial condition can be represented by a series in eigenmodes; and since all of these are stable, so should be the solution to the initial-value problem.

Some doubt, however, has been cast on this approach in a recent paper by Benilov, O'Brien & Sazonov (2003). A model has been found, where an infinite number of neutrally stable eigenmodes coexist with 'exploding' solutions (which grow infinitely in a finite time). It was further argued that the 'explosion' corresponds to a divergence of the series in eigenmodes.

However, the study of Benilov *et al.* (2003) was not entirely conclusive. Firstly, the eigenmodes were found asymptotically, which is always subject to the criticism that the higher orders of the asymptotic scheme might be inconsistent, and the exact solution might not exist. Secondly, the completeness of the eigenmodes has not been proved. Thus, one could claim that the unstable eigenmodes have simply been missed, which would resolve the paradox without resorting to a new concept. We emphasize, however, that Benilov *et al.* (2003) verified their conclusions by extensive numerical calculations, and it is highly unlikely that they are incorrect. Still, given the importance of the problem, it would be preferable to find a 'clean' example, which would resolve the matter rigorously.

Such example is delivered in the present paper: we consider three-dimensional disturbances in a thin liquid film inside a horizontally rotating cylinder. There are

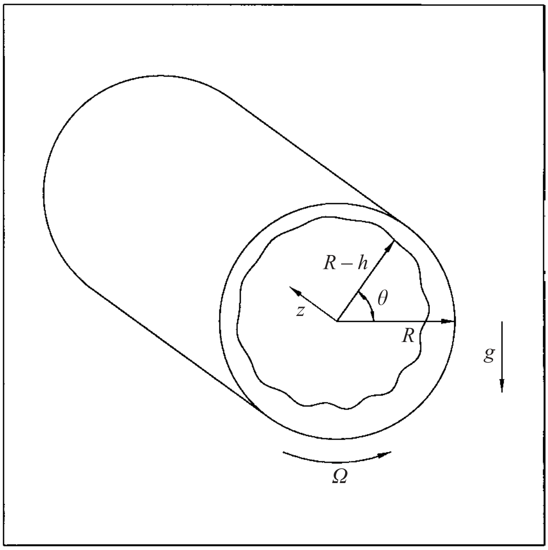

FIGURE 1. Formulation of the problem.

two reasons to examine this particular problem: firstly, it has numerous industrial applications (rotational moulding, coating of fluorescent electric tubes, etc.); secondly, it allows one to find out whether or not exploding solutions exist for multi-dimensional equations (the model of Benilov *et al.* (2003) is based on an equation with one spatial variable).

In § 2 of this paper, we shall derive an asymptotic equation for flows in a rotating cylinder. Two cases will be considered: a relatively simple limit of weak gravity (or strong rotation/viscosity), and the general case. The former is examined first: we shall prove that it admits a complete set of neutrally stable eigenmodes (§ 3) and, on the other hand, exploding solutions (§ 4). In § 5, we extend the results to the general case; and, in § 6, summarize our conclusions and outline further developments and applications.

2. Formulation

2.1. *The governing equation*

Consider a thin film of liquid on the inside surface of an infinitely long cylinder of radius R. Its axis is horizontal, and the cylinder is rotating about this axis with constant angular velocity Ω (see figure 1). We shall use cylindrical coordinates (r, θ, z), so the thickness h of the film depends on the polar angle θ, axial coordinate z, and time t.

The evolution of the film is governed by four non-dimensional parameters. First, we introduce a thickness parameter,

$$\delta = \frac{d}{R}, \tag{2.1}$$

where d is the mean thickness of the film. Second, we define the ratio of the gravitational and centrifugal forces,

$$G = \frac{g}{\Omega^2 R}, \tag{2.2}$$

where g is the acceleration due to gravity. Thirdly, we introduce the Ekman number (which characterizes viscosity relative to rotation),

$$N = \frac{\nu}{\Omega d^2}, \tag{2.3}$$

where ν is the kinematic viscosity. Finally, we define an 'aspect ratio'

$$Z = \frac{L_z}{R}, \tag{2.4}$$

where L_z is the axial scale of disturbances in the film (their characteristic wavelength).

The standard lubrication approximation requires that

$$\delta \ll 1, \quad G \gg 1, \quad N \gg 1, \tag{2.5}$$

i.e. the film is thin, and gravity and viscosity are much stronger than inertia (described by the material derivatives in the Navier–Stokes equations). In order to generalize the lubrication approach for the three-dimensional case, we assume that

$$Z \gg \delta, \tag{2.6}$$

i.e. the axial scale of the flow is much larger than the thickness of the film. Using assumptions (2.5)–(2.6), the Navier–Stokes equations can be reduced to a single asymptotic equation (see Appendix A). Introducing the following non-dimensional variables:

$$\hat{t} = \Omega t, \quad \hat{h} = \left(\frac{g}{\nu \Omega R}\right)^{1/2} h, \quad \hat{z} = \left(\frac{9g}{\nu \Omega R^3}\right)^{1/4} z, \tag{2.7}$$

and omitting the hats, we can represent this equation in the from

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left(h - \tfrac{1}{3}h^3 \cos\theta\right) + \frac{\partial}{\partial z}\left(h^3 \frac{\partial h}{\partial z} \sin\theta\right) = 0. \tag{2.8}$$

It is interesting to compare (2.8) to the corresponding equation of Benilov *et al.* (2003),

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}\left(h - \tfrac{1}{3}h^3 \cos\theta\right) + \varepsilon\left[\frac{\partial}{\partial \theta}\left(\tfrac{1}{3}h^3 \frac{\partial h}{\partial \theta} \sin\theta + \tfrac{1}{2}h^4 \cos\theta - \tfrac{1}{2}h^2\right) - h\frac{\partial h}{\partial t}\right] = 0, \tag{2.9}$$

where

$$\varepsilon = \left(\frac{\nu \Omega}{gR}\right)^{1/2}. \tag{2.10}$$

The terms proportional to ε in equation (2.9) describe weak variation of the hydrostatic pressure. In order to clarify when these terms can be neglected, we compare them with the last term of our equation (2.8). Assuming that the θ-scale of the solution is of order-one, and taking into account (2.7) and (2.10), we obtain

$$Z^2 \ll 1.$$

Thus, if the axial scale L_z is much smaller than the radius of the cylinder (but still much larger than the film's thickness), we should use equation (2.8), whereas the opposite limit corresponds to equation (2.9).

2.2. *Steady states and disturbances*

Let the solution of equation (2.8) be independent of time,

$$h(\theta, t) = \bar{h}(\theta).$$

Then, (2.8) yields

$$\bar{h} - \tfrac{1}{3}\bar{h}^3 \cos\theta = q, \tag{2.11}$$

where q is a constant of integration (physically, q is the non-dimensional mass flux). It has been demonstrated (Johnson 1988) that, for $0 < q < \frac{2}{3}$, equation (2.11) has a smooth unique solution describing a steady-state flow.

In what follows, we shall examine the stability of $\bar{h}$ with respect to small disturbances (O'Brien (2002) tested $\bar{h}$ for stability with respect to small disturbances independent of z, and found it to be neutrally stable). Assuming that

$$h(\theta, z, t) = \bar{h}\,(\theta) + h'(\theta, z, t), \tag{2.12}$$

where h' represents the disturbance, we substitute (2.12) into (2.8) and omit the nonlinear terms. Dropping the primes, we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \theta}[(1 - \bar{h}^2 \cos\theta)\, h] + \bar{h}^3 \sin\theta\, \frac{\partial^2 h}{\partial z^2} = 0. \tag{2.13}$$

Equation (2.13) describes propagation of perturbations in the θ-direction, and their diffusion in the z-direction (the second and third terms, respectively). The propagation speed is

$$C = 1 - \bar{h}^2 \cos\theta$$

and

$$D = -\bar{h}^3 \sin\theta$$

is the effective diffusivity coefficient. Observe that, in the lower half of the cylinder ($-\pi < \theta < 0$), the diffusivity is positive, whereas in the upper half ($0 < \theta < \pi$), it is negative.

Further analysis will be carried out in two steps. First we shall examine the limit of small q, in which case (2.11) yields

$$\bar{h} \to q + \mathrm{O}(q^3) \quad \text{as} \quad q \to 0. \tag{2.14}$$

(Physically, this corresponds to viscous and centrifugal forces dominating gravity. As a result, the asymmetry introduced by gravity is negligible, and $\bar{h}(\theta)$ is almost constant.) Given (2.14), equation (2.13) tends to

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + q^3 \sin\theta\, \frac{\partial^2 h}{\partial z^2} = 0. \tag{2.15}$$

The speed of perturbations in (2.15) is constant, $C = 1$, which makes it much simpler than (2.13). Observe also that the last term in (2.15) includes a small factor, q^3 – which can, however, be eliminated by re-scaling the axial variable, $z \to q^{3/2} z$.

Equation (2.15) will be used to elucidate the general dynamics of systems with sign-variable diffusivity. In the second step, we shall briefly describe how the results obtained in the small-q limit can be generalized for the finite-q equation (2.13).

3. The small-q limit: harmonic disturbances

In this section, we shall examine harmonic (normal) modes of equation (2.15), i.e. solutions of the form

$$h(\theta, z, t) = \phi(\theta)\, \mathrm{e}^{\mathrm{i}(kz - \omega t)}, \tag{3.1}$$

where ω and k are the frequency and axial wavenumber, respectively. Substitution of (3.1) into (2.15) yields a first-order ordinary differential equation(ODE) for $\phi(\theta)$,

$$\frac{d\phi}{d\theta} - (i\omega + \kappa^2 \sin\theta)\phi = 0, \tag{3.2}$$

where

$$\kappa^2 = q^3 k^2.$$

Equation (3.2) and the periodicity condition,

$$\phi(\theta + 2\pi) = \phi(\theta), \tag{3.3}$$

form an eigenvalue problem, where ϕ is the eigenfunction and ω is the eigenvalue. It can be readily shown that (3.2)–(3.3) admit an infinite set of solutions,

$$\phi_n = \exp(in\theta - \kappa^2 \cos\theta), \tag{3.4}$$

$$\omega_n = n, \tag{3.5}$$

where n is an integer. Interestingly, $|\phi_n|$ does not depend on n; note also that the maximum of $|\phi_n|$ occurs at $\theta = \pi$ (see figure 2). The latter feature can be interpreted as follows: a disturbance starts its motion from $\theta = 0$ in the counter-clockwise direction, and grows whilst travelling through the upper half of the cylinder, where the diffusivity is negative. In the lower half, where the diffusivity is positive, the disturbance decays and, having reached $\theta = 2\pi$, its initial amplitude is restored.

It is worth comparing (3.2) with the corresponding equation of Benilov *et al.* (2003),

$$\tfrac{1}{3}\varepsilon q^3 \frac{d}{d\theta}\left(\sin\theta \frac{d\phi}{d\theta}\right) + \frac{d\phi}{d\theta} - i\omega\phi = 0$$

(resulting from the governing equation (2.9) in the small-q limit). Unlike equation (3.2) herein, this is a second-order ODE and, therefore, cannot be solved analytically – which dramatically hampers the analysis. The problem considered here can be advanced much further.

3.1. *Completeness of* $\{\phi_n\}$

In this subsection, we shall prove that the set $\{\phi_n\}$ is complete in the space P' of 2π-periodic functions with absolutely convergent Fourier series. To do so, we shall use a theorem (see Appendix B), according to which the completeness of $\{\phi_n\}$ follows from:

(*a*) the existence of an adjoint set $\{\phi_n^+\}$, such that

$$\int_0^{2\pi} \phi_l(\theta)\, \phi_m^+(\theta)\, d\theta = \delta_{lm}, \tag{3.6}$$

where δ_{ml} is the Kronecker delta;

(*b*) the equality

$$\sum_{n=-\infty}^{\infty} \int_0^{2\pi} \phi_n(\theta)\, e^{-il\theta}\, d\theta \int_0^{2\pi} \phi_n^+(\theta)\, e^{im\theta}\, d\theta = 2\pi\, \delta_{ml}; \tag{3.7}$$

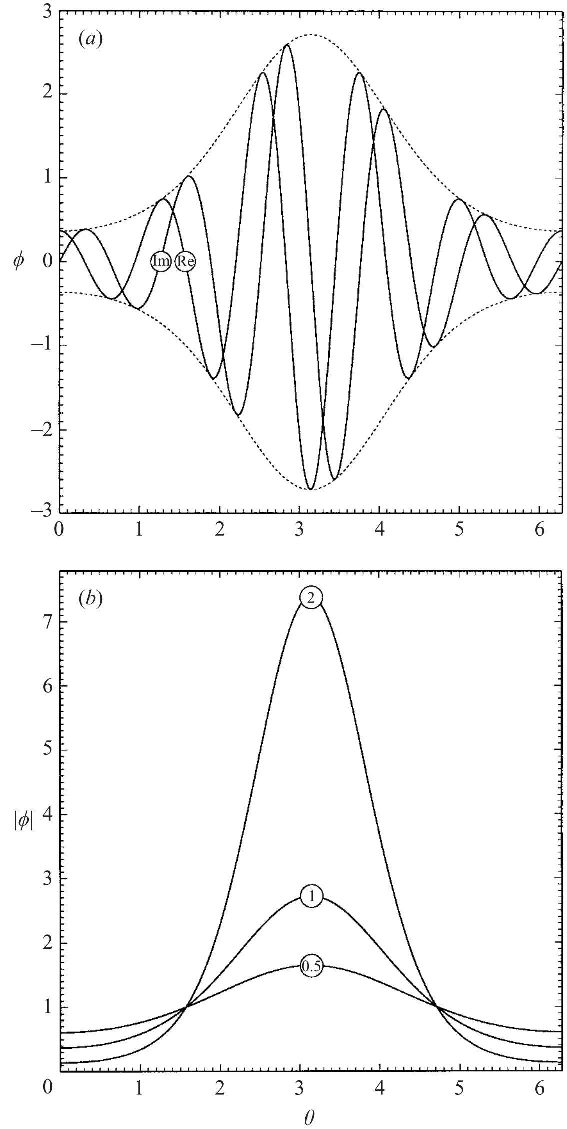

FIGURE 2. Eigenfunctions ϕ_n vs. θ (formula (3.4)). (a) The real and imaginary parts of ϕ_n for $n = 5$ (fifth mode) and $q^{3/2}k^2 = 1$. The dotted line shows $\pm|\phi_n|$. (b) $|\phi_n|$ for various $q^{3/2}k^2$ (the curves are marked by the corresponding values of $q^{3/2}k^2$).

and (c) convergence of the double series

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|b_m|\left|\int_0^{2\pi}\phi_n^+(\theta)\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta\right||\phi_n(\theta)| < \infty, \tag{3.8}$$

where b_m are the Fourier coefficients (defined by (B 4)) of an arbitrary function $f(\theta) \in P'$.

We start by guessing the form of the adjoint functions,

$$\phi_n^+ = \frac{1}{2\pi}\exp(-\mathrm{i}n\theta + \kappa^2\cos\theta), \tag{3.9}$$

and verifying by inspection that they satisfy condition (3.6).

Next, substitute ϕ and ϕ^+ (given by (3.4) and (3.9)) into condition (3.7). Taking into account the integral representation of the modified Bessel function I_n (see (C 1)),† we obtain

$$\text{l.h.s. of (3.7)} = 2\pi \sum_{n=-\infty}^{\infty} I_{n-m}(-\kappa^2)\, I_{l-n}(\kappa^2).$$

This series can be summed using identity (C 2):

$$\text{l.h.s. of (3.7)} = 2\pi\, I_{l-m}(0).$$

Then, taking into account that

$$I_n(0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } |n| \geqslant 1 \end{cases}$$

(which, in turn, follows from (C 3)), one can see that the left-hand side of (3.7) equals its right-hand side, as required.

Finally, substitute (3.4), (3.9) into (3.8), and take into account (C 1),

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |b_m|\, I_{m-n}(\kappa^2) \exp(-\kappa^2 \cos\theta) < \infty.$$

The last factor on the left-hand side of this inequality does not depend on either n or m – hence, it can be omitted. Then, changing the summation index, $n \to m - n$, we split the double series into two non-coupled series

$$\left(\sum_{m=-\infty}^{\infty} |b_m| \right) \left[\sum_{n=-\infty}^{\infty} I_n(\kappa^2) \right] < \infty.$$

Given expression (C 4) for the asymptotics of I_n as $n \to \pm\infty$, the second factor is finite for any κ^2. The first series, in turn, is finite simply because b_m are the Fourier coefficients of a function from P'.

Thus, all three requirements (3.6)–(3.8) are satisfied – hence, the set $\{\phi_n\}$ is complete in P'. It is also worth noting that P' is a sufficiently 'general' space: it includes, for example, all twice-differentiable functions.

3.2. *Solving the initial-value problem by normal modes*

Let the evolutionary equation (2.15) be solved in the domain $(0 \leqslant \theta \leqslant 2\pi, -\infty < z < \infty)$ with an initial condition

$$h = H(\theta, z) \quad \text{at} \quad t = 0. \tag{3.10}$$

First we shall consider the case where H is a 2π-periodic function in z (in addition to the 2π-periodicity in θ). Recalling that $\{\phi_n\}$ is a complete set, (3.10) can be represented by a series in ϕ_n, combined with the usual Fourier series in $\mathrm{e}^{\mathrm{i}kz}$ (where k is an integer),

$$H(\theta, z) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} H_{kn}\, \phi_n(\theta)\, \mathrm{e}^{\mathrm{i}kz}. \tag{3.11}$$

† Here and in what follows, we shall extensively use the properties of I_n – all necessary formulae are listed in Appendix C.

Observe that the n-summation here is to be carried out before the k-summation – which reverses the order in which ϕ_n were obtained (recall that the assumption of harmonic dependence on z preceded finding ϕ_n and, more importantly, the proof of their completeness).

In order to express H_{kn} through $H(\theta, z)$, multiply (3.11) by

$$\frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}kz}\,\phi_n^+,$$

and integrate over $(\theta, z) \in [0, 2\pi] \times [0, 2\pi]$. Taking into account (3.7) and (3.9), we obtain

$$H_{kn} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} H(\theta, z)\,\exp(-\mathrm{i}n\theta + \kappa^2\cos\theta - \mathrm{i}kz)\,\mathrm{d}z\,\mathrm{d}\theta. \tag{3.12}$$

Now, the solution of the initial-value problem (2.15), (3.10) can be obtained by replacing $\phi_n(\theta)$ with $\phi_n(\theta)\,\mathrm{e}^{-\mathrm{i}\omega_n t}$ in (3.11). Taking into account (3.4)–(3.5), we have

$$h(\theta, z, t) = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} H_{kn}\exp[-\kappa^2\cos\theta + \mathrm{i}(n\theta + kz - nt)]. \tag{3.13}$$

At $t = 0$, this double series converges to the initial condition; and each term is bounded for $t > 0$. Following the usual logic, we conclude that the film is neutrally stable.

Observe, however, that the coefficients of the series (3.13) (and, generally, any series in normal modes) depend on t. Thus, even though the series converges at $t = 0$, it may diverge later.† Such divergence would correspond to an 'exploding' solution, which blows up (develops a singularity) in a finite time. To illustrate what can happen, consider a simple example,

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,\mathrm{e}^{\mathrm{i}kt}.$$

If $t = 0$, this series converges to $-\ln 2$ (although not absolutely); and each term is bounded for $t > 0$. Despite that, the series diverges at $t = \pi$.

In conclusion of this section, we shall modify (3.12)–(3.13) for solutions non-periodic in z (periodicity in θ is still implied). Instead of Fourier series, in this case one should use a Fourier integral, which yields

$$H_{kn} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_{-\infty}^{\infty} H(\theta, z)\,\exp(-\mathrm{i}n\theta + \kappa^2\cos\theta - \mathrm{i}kz)\,\mathrm{d}z\,\mathrm{d}\theta \tag{3.14}$$

and

$$h(\theta, z, t) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} H_{nk}\exp[-\kappa^2\cos\theta + \mathrm{i}(n\theta + kz - nt)]\,\mathrm{d}k. \tag{3.15}$$

Again, the integral in (3.15) can diverge, resulting in an 'explosion'.

In the next section, exploding solutions will be found explicitly.

† Using the approach of the previous subsection, one can demonstrate that, for a fairly general $H(\theta, z)$, the n-series in (3.13) converges absolutely. The k-series, however, may, and often does, diverge (see below).

4. The small-q limit: 'exploding' solutions

4.1. *A non-modal approach* (*a particular solution*)

Seek a solution of (2.15) in the form

$$h(\theta, z, t) = A(\theta, t)\, f(\xi), \tag{4.1}$$

where

$$\xi = \frac{z}{W(\theta, t)}$$

and A and W are undetermined functions. Substituting (4.1) into (2.15), we obtain

$$\frac{1}{A}\left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial \theta}\right) = -\frac{1}{W}\left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta}\right), \tag{4.2}$$

$$W\left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta}\right) + q^3 \sin\theta = 0, \tag{4.3}$$

$$f + \xi f_\xi + f_{\xi\xi} = 0. \tag{4.4}$$

Equation (4.4) yields

$$f = \exp\left(-\tfrac{1}{2}\xi^2\right), \tag{4.5}$$

and (4.2) relates A to W,

$$\frac{A}{A_0} = \frac{W_0}{W} + F_A(\theta - t), \tag{4.6}$$

where A_0 and W_0 are arbitrary constants, and F_A is an arbitrary function. Equation (4.3), in turn, yields

$$W^2 = F_W(\theta - t) + 2q^3 \cos\theta, \tag{4.7}$$

where F_W is an arbitrary function. We shall consider the particular case

$$A_0 = 1, \quad F_A(\theta - t) = 0, \quad F_W(\theta - t) = W_0^2 - 2q^3\cos(\theta - t),$$

for which (4.5)–(4.7) yield

$$h = \frac{W_0}{\sqrt{W_0^2 - 2q^3\left[\cos(\theta - t) - \cos\theta\right]}} \exp\left\{-\frac{z^2}{2W_0^2 - 4q^3\left[\cos(\theta - t) - \cos\theta\right]}\right\}. \tag{4.8}$$

This solution is shown in figure 3 – it represents a ring-like disturbance encircling the cylinder from the inside. The z cross-section of the 'ring' is Gaussian and, initially, is independent of θ (i.e. it is of constant width W_0 and unit amplitude). With increasing t, the amplitude and width of the ring begin to change: in the upper half of the cylinder, the ring is becoming larger and narrower. The most important characteristics of this process are the minimum width and maximum amplitude of the ring as functions of t,

$$\min_{0\leqslant\theta\leqslant 2\pi} W(\theta, t) = \sqrt{W_0^2 - 4q^3\left|\sin\tfrac{1}{2}t\right|}, \quad \max_{0\leqslant\theta\leqslant 2\pi} A(\theta, t) = \frac{W_0}{\sqrt{W_0^2 - 4q^3\left|\sin\tfrac{1}{2}t\right|}}.$$

These formulae show that the evolution depends on whether or not the initial width W_0 exceeds the critical value of

$$W_0^{(crit)} = 2q^{3/2}. \tag{4.9}$$

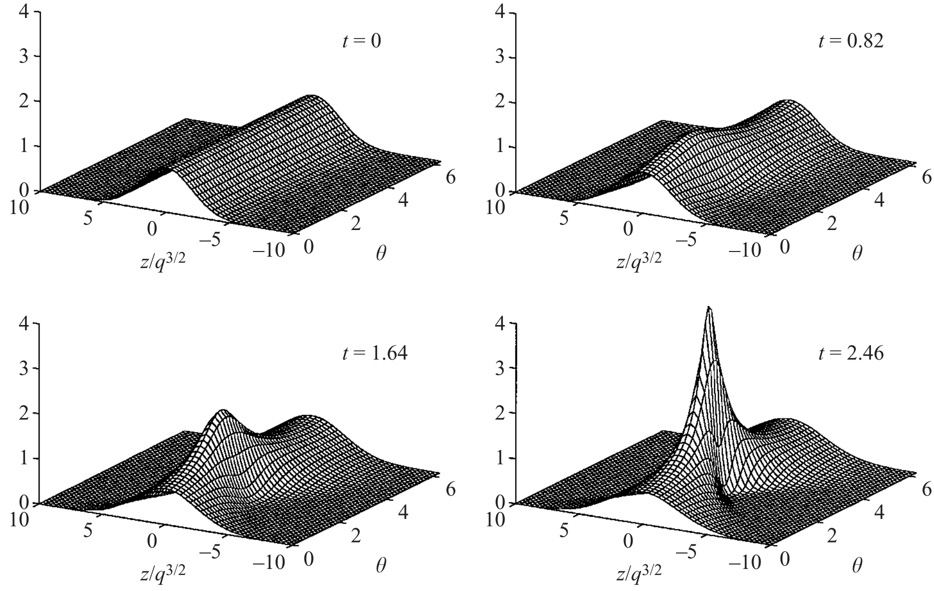

FIGURE 3. The 'exploding' solution (4.8) on the $(\theta, z/q^{3/2})$-plane for $W_0/q^{3/2} = 2$, for various t (the time of explosion is $t_* = \pi$).

(a) If $W_0 > W_0^{(crit)}$, solution (4.8) is periodic in time with a period of 2π, i.e. at $t = 2\pi$ it regains its original form.

(b) If $W_0 \leqslant W_0^{(crit)}$, the minimum width of the ring vanishes (and the maximum amplitude shoots to infinity) at

$$t_* = 2 \arcsin\left(\frac{W_0^2}{4q^3}\right), \tag{4.10}$$

i.e. (4.8) 'explodes'. The explosion occurs at

$$\theta_* = \frac{\pi}{2} + \arcsin\left(\frac{W_0^2}{4q^3}\right), \quad z_* = 0. \tag{4.11}$$

The exploding solution is the main finding of this section.

Note that, even though the diffusivity is negative for $\theta \in (0, \pi)$, explosion never occurs in the first quadrant (see formula (4.11)). To understand this feature, recall that the ring's initial width is uniform (independent of θ) – hence, the explosion should occur for the disturbance that spends most time near the minimum of diffusivity. Then it can be readily seen that such a disturbance should set out from a certain distance before $\theta = \pi/2$ and travel the same distance past it, which effectively means that the explosion occurs in the second quadrant.

To obtain a solution which explodes in the first quadrant, one needs to perturb the initial width of the ring, i.e. modify function F_W. For example, one can add an initial varicose narrowing of the ring – by choosing, say,

$$F_W(\theta - t) = W_0^2 - 2q^3 \cos(\theta - t) - V^2 \exp\left[\frac{\cos(\theta - t - \theta_0) - 1}{L}\right],$$

where V, L, and θ_0, are the amplitude, length, and position of the narrowing. We shall not dwell on this question in more detail, but note only that, by adjusting V,

L, and θ_0, one can move the place of explosion to any given point in the first two quadrants.

4.2. *The normal-mode approach*

It is instructive to derive the exploding solution (4.8) as a solution of the initial-value problem and, thus, clarify what happens with the normal-mode expansion when t approaches the time of explosion.

The initial condition corresponding to (4.8) is

$$H = \exp\left(-\frac{z^2}{2W_0^2}\right).$$

Substituting H into (3.14) and carrying out the integration, we obtain the following expression for the coefficients of the generalized Fourier series:

$$H_{kn} = \frac{W_0}{\sqrt{2\pi}} \exp\left(-\frac{k^2 W_0^2}{2}\right) I_n(q^3 k^2). \tag{4.12}$$

In order to test expansion (3.15), with H_{kn} given by (4.12), for absolute convergence, replace $H_{kn}\phi_n$ with $|H_{kn}\phi_n|$ and require the resulting expression to be finite,

$$\frac{W_0}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{k^2 W_0^2}{2} - q^3 k^2 \cos\theta\right) \sum_{n=-\infty}^{\infty} I_n(q^3 k^2)\,\mathrm{d}k < \infty.$$

The series here converges and can be summed using formula (C 5) (with $a = 0$),

$$\frac{W_0}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{k^2 W_0^2}{2} - q^3 k^2 \cos\theta\right) \exp(q^3 k^2)\,dk < \infty.$$

Hence, the condition of absolute convergence is

$$-\frac{W_0^2}{2} - q^3 \cos\theta + q^3 < 0, \tag{4.13}$$

If $W_0 > 2q^{3/2}$, this inequality holds for all θ, which agrees with the fact that solution (4.8) does not explode if its initial 'width' is sufficiently large.

If, on the other hand, inequality (4.13) does not hold, the normal-mode expansion (3.15), (4.12) may still converge to the correct solution, but only if the n-summation is carried out before the k-integration. To verify this, substitute (4.12) into (3.15),

$$h(\theta, z, t) = \frac{W_0}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{k^2 W_0^2}{2} - q^3 k^2 \cos\theta + \mathrm{i}kz\right) \times \sum_{n=-\infty}^{\infty} I_n(q^3 k^2) \exp[\mathrm{i}n(\theta - t)]\,\mathrm{d}k. \tag{4.14}$$

The series in this expression converges and can be summed using formula (C 5),

$$h(\theta, z, t) = \frac{W_0}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{k^2 W_0^2}{2} - q^3 k^2 \cos\theta + q^3 k^2 \cos(\theta - t)\right] \cos kz\,\mathrm{d}k.$$

Observe that the exponential in this integral is bounded (as $k \to \infty$) only if

$$-\frac{W_0^2}{2} - q^3 \cos\theta + q^3 \cos(\theta - t) < 0, \tag{4.15}$$

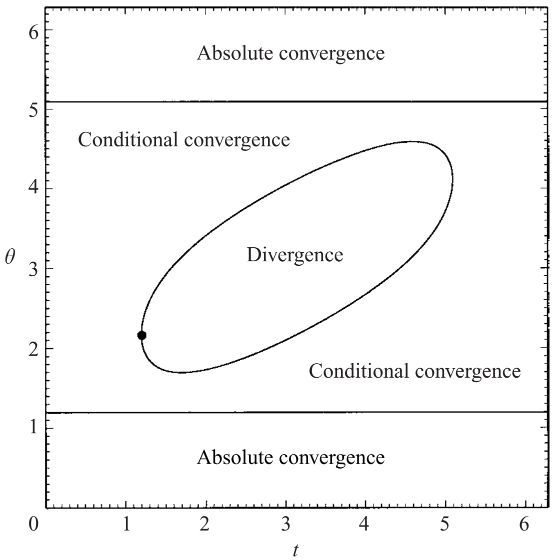

FIGURE 4. The convergence properties of the normal-mode expansion (3.15), (4.12) for an 'exploding' solution ($W_0/q^{3/2} = 1.5$). The place and time of the explosion (indicated by a black circle) are given by (4.10) and (4.11), respectively. The regions of absolute and conditional convergence are determined by (4.13) and (4.15).

which is, essentially, the criterion of conditional convergence of the normal-mode expansion (3.15), (4.12). If (4.15) holds, the expansion converges to solution (4.8) – but, if it does not, the expansion diverges. Interestingly, in the latter case, (4.8) is complex.

The convergence properties of expansion (3.15), (4.12) are summarized in figure 4. One can see that, before the explosion ($t < t_*$), the expansion converges (absolutely or, at least, conditionally). For $t > t_*$, the expansion diverges for some θ and converges for the other values, but this is only of theoretical interest, as the explosion probably makes the whole solution physically irrelevant.

5. The finite-q case

5.1. *Normal modes*

In order to find the normal modes, we substitute (3.1) into (2.13) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}[(1 - \bar{h}^2 \cos\theta)\phi] - (\mathrm{i}\omega + k^2\bar{h}^3 \sin\theta)\phi = 0.$$

This equation and the periodicity condition (3.3) can be readily solved,

$$\phi_n = \frac{1}{1 - \bar{h}^2\cos\theta} \exp\left(\int_0^\theta \frac{\mathrm{i}\omega_n + k^2\bar{h}^3(\theta')\sin\theta'}{1 - \bar{h}^2(\theta')\cos\theta'}\,\mathrm{d}\theta'\right), \quad \omega_n = \frac{2\pi n}{\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - \bar{h}^2(\theta)\cos\theta}},$$

where n is an integer (these formulae are the finite-q analogues of (3.4)–(3.5)). Since $\mathrm{Im}\,\omega_n = 0$, all normal modes are neutrally stable. It should be admitted though that, for finite q, I was unable to prove that the set $\{\phi_n\}$ is complete.

5.2. *Exploding solutions*

Generalization of the exploding solution for finite q is not straightforward, mainly because of the variable speed of perturbations in equation (2.13). The small-q substitution (4.1), however, remains intact, and even the z cross-section of the 'ring' remains Gaussian (described by (4.5)). But, unfortunately, the expressions for the amplitude and width of the ring are much more complicated,

$$\frac{A}{A_0} = \frac{W_0}{W} + F_A\left(\int_0^{\theta} \frac{\mathrm{d}\theta'}{1 - \bar{h}^2(\theta') \cos\theta'} - t \right), \tag{5.1}$$

$$W^2 = F_W\left(\int_0^{\theta} \frac{\mathrm{d}\theta'}{1 - \bar{h}^2(\theta') \cos\theta'} - t \right) - \int_0^{\theta} \frac{2\bar{h}^3(\theta') \sin\theta'}{1 - \bar{h}^2(\theta') \cos\theta'} \,\mathrm{d}\theta', \tag{5.2}$$

where F_A and F_W are arbitrary functions (formulae (5.1)–(5.2) are the finite-q analogues of (4.6)–(4.7)). As before, we seek a solution that, initially, has unit amplitude and constant width W_0. Taking into account these requirements in (5.1)–(5.2), we obtain

$$\left.\begin{aligned} &A_0 = 1, \quad F_A(\theta) = 0, \\ &F_W(\theta) = W_0^2 + \int_0^{\theta} \frac{2\bar{h}^3(\theta') \sin\theta'}{1 - \bar{h}^2(\theta') \cos\theta'} \,\mathrm{d}\theta' \end{aligned}\right\}, \tag{5.3}$$

Unfortunately, formulae (5.2)–(5.3) for W are too complicated to result in analytic expressions for the time and coordinate of the explosion. It is, however, possible to calculate $W_0^{(crit)}$, i.e. the critical initial width of the ring, such that

if $W_0 \leqslant W_0^{(crit)}$	the ring explodes,
if $W_0 > W_0^{(crit)}$	the ring oscillates periodically.

The calculation is based on an obvious fact that, for

$$W_0 = W_0^{(crit)}, \tag{5.4}$$

the explosion occurs at the boundary of the region of negative diffusivity,

$$\theta_*^{(crit)} = \pi \tag{5.5}$$

(for smaller W_0, the explosion would occur inside the region, i.e. at $\theta < \pi$, and for larger W_0 it would not occur at all). It is also clear that the time of explosion for the critical case is given by

$$t_*^{(crit)} = \int_0^{\pi} \frac{\mathrm{d}\theta'}{1 - \bar{h}^2(\theta') \cos\theta'}, \tag{5.6}$$

which is, essentially, the time taken by a perturbation to travel from $\theta = 0$ to $\theta = \pi$. Finally, we recall that, at the time and place of explosion, the width of the ring vanishes,

$$W\left(\theta_*^{(crit)}, t_*^{(crit)}\right) = 0. \tag{5.7}$$

Substituting (5.4)–(5.7) into (5.2)–(5.3), we obtain

$$W_0^{(crit)} = \left[\int_0^{\pi} \frac{2\bar{h}^3(\theta) \sin\theta}{1 - \bar{h}^2(\theta) \cos\theta} \mathrm{d}\theta \right]^{1/2}, \tag{5.8}$$

which generalizes the small-q expression (4.9). Note that, for $q \lesssim 0.6$, the small-q limit approximates the general case sufficiently well (see figure 5). Observe also that,

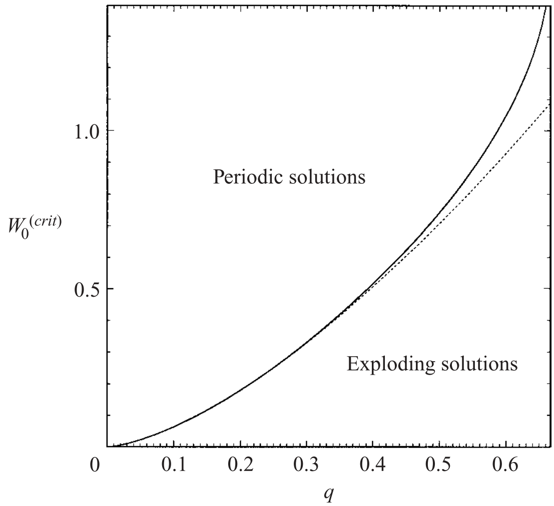

FIGURE 5. The critical initial width $W_0^{(crit)}$ vs. q (formula (5.8)). The dotted line shows the small-q limit of $W_0^{(crit)}$ (formula (4.9)).

as follows from equation (2.11),

$$\bar{h}(0) = \bar{h}(\pi) = 1 \quad \text{for} \quad q = \tfrac{2}{3},$$

and the integral in (5.8) diverges at $\theta = 0, \pi$. Thus,

$$W_0^{(crit)} \to \infty \quad \text{as} \quad q \to \tfrac{2}{3},$$

i.e. if the film has the limiting value of $q = \frac{2}{3}$, all disturbances explode.

6. Summary and concluding remarks

We have examined the stability of a liquid film in a rotating cylinder and obtained the following results:

(i) On the one hand, the system admits an infinite number of neutrally stable normal modes (with real frequencies), such that the corresponding eigenfunctions form a complete set.

(ii) On the other hand, the system admits strongly unstable solutions which develop singularities in a finite time.

The two features can be reconciled if we represent the solution of the initial-value problem by a series in normal modes. Since the frequencies of the modes are real, each term of the series remains bounded for $t > 0$, which, however, does not guarantee convergence. Indeed, the terms depend on t and, even though the series converges at $t = 0$ (to the initial condition), it may diverge later, which would correspond to an 'explosion'. The conclusion to be drawn here is that, in some cases, harmonic solutions are not a reliable indicator of the stability properties of a system.

In fact, we can present an even more paradoxical example than the one examined above. Consider the following modification of equation (2.15):

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + q^3 \sin\theta \, \frac{\partial^2 h}{\partial z^2} = -\gamma h, \tag{6.1}$$

where γ is a positive constant. Equation (6.1) clearly admits exploding solutions (which follows from the simple fact that it can be reduced to (2.15) by the substitution $h \to h\exp(-\gamma t)$). At the same time, all normal modes of (6.1) are not even neutrally, but asymptotically, stable! Another way to introduce asymptotic stability (and, at the same time, keep exploding solutions) is to add a positive component to diffusivity (which, however, should still admit some negative values) – for example,

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta} + (q^3 \sin\theta - \gamma)\frac{\partial^2 h}{\partial z^2} = 0, \tag{6.2}$$

where $q^3 > \gamma > 0$. Unlike (6.1), this equation cannot be reduced to (2.15).

Note also that, mathematically, exploding solutions violate the lubrication approximation under which the original equation (2.15) has been derived. Yet, physically, it is clear that the exact system also admits some kind of singular solution. Indeed, consider a short-scale perturbation on the surface of the film. When the rotation of the cylinder turns it 'upside down', gravity starts increasing its amplitude and shortening its size. Once the perturbation is sufficiently large and short, a drop of fluid detaches itself from the film and falls down. The exploding solution of the linear asymptotic equation captures the initial stage of this process, and the exact model should admit a similar singular solution too.

It appears that the only effect that can prevent explosion is surface tension. If it is introduced into the asymptotic equation, it is likely to transform the explosive instability into the so-called *transient* instability, i.e. disturbances will initially grow, but then decay (see Farrel 1982, Chapman 2002 and references therein).

Finally, note that the experiments show that short-scale instability does occur in flows in a rotating cylinder (e.g. Balmer 1970). So far, this instability has been attributed to the so-called inertial effects (e.g. Hosoi & Mahadevan 1999). The present paper suggests an alternative explanation, and our (explosive) instability appears to be stronger than the inertial one. This issue deserves further investigation.

I am grateful to Stephen O'Brien and Igor Sazonov for many fruitful discussions, and to Sam Howison who drew my attention to equation (6.2).

Appendix A. Derivation of equation (2.15)

A three-dimensional flow of viscous fluid inside a cylinder with horizontal axis (see figure 1) can be characterized by the radial, angular, and axial velocities, $u(r,\theta,z,t)$, $v(r,\theta,z,t)$, and $w(r,\theta,z,t)$, and the pressure $p(r,\theta,z,t)$ (r, θ, z are cylindrical coordinates, and t is the time). In terms of u, v, w, and p, the governing equations are (e.g. Landau & Lifshitz 1995)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial r}$$
$$= -g\sin\theta + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} - u - 2\frac{\partial v}{\partial \theta}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{A 1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\left(\frac{\partial v}{\partial \theta} + u\right) + w\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$= -g\cos\theta + \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 v}{\partial \theta^2} - v + 2\frac{\partial u}{\partial \theta}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{A 2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{\partial p}{\partial z}=\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{A 3}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial \theta}+\frac{\partial}{\partial z}(rw)=0, \tag{A 4}$$

where g is the acceleration due to gravity and ν is the kinematic viscosity. We assume that the cylinder is rotating with constant angular velocity Ω, which corresponds to the following boundary condition:

$$u=0, \quad v=\Omega R, \quad w=0 \quad \text{at} \quad r=R, \tag{A 5}$$

where R is the radius of the cylinder. We shall also need boundary conditions on the free surface of the film, i.e. at $r=R-h(\theta,t)$, where h is the thickness of the film. We shall require that

$$\boldsymbol{\sigma n}=\mathbf{0} \quad \text{at} \quad r=R-h, \tag{A 6}$$

where

$$\boldsymbol{n}=\begin{bmatrix} 1 \\ \dfrac{1}{R-h}\dfrac{\partial h}{\partial \theta} \\ \dfrac{\partial h}{\partial z} \end{bmatrix}$$

is a vector normal to the surface, and

$$\boldsymbol{\sigma}=\begin{bmatrix} 2\nu\dfrac{\partial u}{\partial r}-p & \dfrac{\nu}{r}\left(\dfrac{\partial u}{\partial \theta}-v+r\dfrac{\partial v}{\partial r}\right) & \nu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) \\ \dfrac{\nu}{r}\left(\dfrac{\partial u}{\partial \theta}-v+r\dfrac{\partial v}{\partial r}\right) & \dfrac{2\nu}{r}\left(\dfrac{\partial v}{\partial \theta}+u\right)-p & \nu\left(\dfrac{\partial v}{\partial z}+\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\right) \\ \nu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) & \nu\left(\dfrac{\partial v}{\partial z}+\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\right) & 2\nu\dfrac{\partial w}{\partial z}-p \end{bmatrix}$$

is the stress tensor (see Landau & Lifshitz 1995). Then, (A 6) yields

$$2\frac{\partial u}{\partial r}-\frac{p}{\nu}+\frac{1}{(R-h)^2}\left[\frac{\partial u}{\partial \theta}-v+(R-h)\frac{\partial v}{\partial r}\right]\frac{\partial h}{\partial \theta} + \left(\frac{\partial u}{\partial z}+\frac{\partial u}{\partial \theta}\right)\frac{\partial h}{\partial z}=0 \quad \text{at} \quad r=R-h, \tag{A 7}$$

$$\frac{\partial u}{\partial \theta}-v+(R-h)\frac{\partial v}{\partial r}+\left[\frac{2}{R-h}\left(\frac{\partial v}{\partial \theta}+u\right)-\frac{p}{\nu}\right]\frac{\partial h}{\partial \theta} + \left[(R-h)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial \theta}\right]\frac{\partial h}{\partial z}=0 \quad \text{at} \quad r=R-h, \tag{A 8}$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}+\left(\frac{\partial v}{\partial z}+\frac{1}{R-h}\frac{\partial w}{\partial \theta}\right)\frac{1}{R-h}\frac{\partial h}{\partial \theta}+\left(2\frac{\partial w}{\partial z}-\frac{p}{\nu}\right)\frac{\partial h}{\partial z}=0 \quad \text{at} \quad r=R-h. \tag{A 9}$$

We shall also require that the normal velocity of particles at the surface matches the normal velocity of the surface itself. A straightforward calculation yields

$$\frac{\partial h}{\partial t}+\frac{v}{R-h}\frac{\partial h}{\partial \theta}+w\frac{\partial h}{\partial z}+u=0 \quad \text{at} \quad r=R-h.$$

This condition can be rewritten in a more convenient form by integrating the continuity equation (A 4) with respect to r from $R-h$ to R, and using the boundary condition for u (see (A 5)),

$$-[(R-h)u]_{r=R-h} + \frac{\partial}{\partial\theta}\int_{R-h}^{R} v\,\mathrm{d}r - \frac{\partial h}{\partial\theta}v + \frac{\partial}{\partial z}\int_{R-h}^{R} rw\,\mathrm{d}r - \frac{\partial h}{\partial z}(R-h)\,w = 0.$$

Combining the last two equations, we obtain

$$(R-h)\frac{\partial h}{\partial t} + \frac{\partial}{\partial\theta}\int_{R-h}^{R} v\,\mathrm{d}r + \frac{\partial}{\partial z}\int_{R-h}^{R} rw\,\mathrm{d}r = 0. \tag{A 10}$$

As mentioned in the main body of the paper, the problem at hand is governed by four non-dimensional parameters, δ, G, N, and Z, defined by (2.1)–(2.4). The lubrication approximation corresponds to the assumptions (2.5)–(2.6), in which case the leading-order dynamics is determined by the gravity/viscosity terms in equation (A 2) and pressure/viscosity terms in equation (A 3). The corresponding non-dimensional variables are

$$\widetilde{r} = \frac{1}{\delta}\frac{R-r}{R}, \quad \widetilde{\theta} = \theta, \quad \widetilde{z} = \frac{1}{Z}\frac{z}{R}, \quad \widetilde{t} = \Omega t, \tag{A 11}$$

$$\widetilde{u} = \frac{1}{\delta}\frac{u}{R\Omega}, \quad \widetilde{v} = \frac{v}{R\Omega}, \quad \widetilde{p} = \frac{1}{\delta G}\frac{p}{(R\Omega)^2}, \quad \widetilde{h} = \frac{1}{\delta}\frac{h}{R}. \tag{A 12}$$

In what follows, equations (A 1)–(A 10) will be reduced to a relatively simple asymptotic equation. In order to make sure that all terms in this equation are of the same order, we assume $G \sim N \sim \delta^{-1}$, $Z \sim \delta^{1/2}$ (these assumptions can be verified *a posteriori*, by inspection of the equation derived). Equivalently, we put

$$N = \frac{N'}{\delta}, \quad G = \frac{G'}{\delta}, \quad Z = \delta^{1/2}, \tag{A 13}$$

where N' and G' are constants of order one.

Substituting (A 11)–(A 13) into the governing equations (A 1)–(A 10) and keeping the leading-order terms only, we obtain (tildes omitted)

$$\frac{\partial p}{\partial r} = \sin\theta, \tag{A 14}$$

$$0 = -G'\cos\theta + N'\frac{\partial^2 v}{\partial r^2}, \tag{A 15}$$

$$G'\frac{\partial p}{\partial z} = N'\frac{\partial^2 w}{\partial r^2}, \tag{A 16}$$

$$-\frac{\partial u}{\partial r} + \frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial z} = 0, \tag{A 17}$$

$$u = 0 \quad \text{at} \quad r = 0, \tag{A 18}$$

$$v = 1, \quad w = 0 \quad \text{at} \quad r = 0, \tag{A 19}$$

$$p = 0 \quad \text{at} \quad r = h, \tag{A 20}$$

$$\frac{\partial v}{\partial r} = 0 \quad \text{at} \quad r = h, \tag{A 21}$$

$$-\frac{\partial w}{\partial r} - \frac{G'}{N'}p\frac{\partial h}{\partial z} = 0 \quad \text{at} \quad r = h, \tag{A 22}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial \theta}\left(\int_0^h v\,\mathrm{d}r\right)+\frac{\partial}{\partial z}\left(\int_0^h w\,\mathrm{d}r\right)=0. \tag{A 23}$$

Solving (A 14)–(A 22), we obtain

$$v=1+\frac{G'}{N'}\left(\frac{r^2}{2}-rh\right)\cos\theta, \quad w=\frac{G'}{N'}\left(rh-\frac{r^2}{2}\right)\frac{\partial h}{\partial z}\sin\theta \tag{A 24}$$

(expressions for p and u will not be needed). Next, substitute (A 24) into (A 23),

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial \theta}\left(h-\frac{G'}{3N'}h^3\cos\theta\right)+\frac{\partial}{\partial z}\left(\frac{G'}{3N'}h^3\frac{\partial h}{\partial z}\sin\theta\right)=0.$$

Finally, it is convenient to scale out the constant coefficients in this equation, which can be achieved by rewriting it in terms of

$$\hat{\theta}=\theta, \quad \hat{t}=t, \quad \hat{z}=\left(\frac{9G'}{N'}\right)^{1/4}z, \quad \hat{h}=\left(\frac{G'}{N'}\right)^{1/2}h.$$

Omitting the hats, we obtain the desired asymptotic equation governing $h(\theta, z, t)$,

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial \theta}\left(h-\tfrac{1}{3}h^3\cos\theta\right)+\frac{\partial}{\partial z}\left(h^3\frac{\partial h}{\partial z}\sin\theta\right)=0. \tag{A 25}$$

Note that (A 25) is a particular case of the asymptotic equation derived by Hosoi & Mahadevan (1999). It should be emphasized, however, that in their derivation the term involving the z-derivatives was obtained as a perturbation, i.e. it was assumed to be small. In our derivation, this term is of order-one.

Appendix B. Completeness of a set of functions

In this Appendix, the following theorem will be proved:

Let $\{\phi_n, n=0,\pm 1\ldots\}$ and $\{\phi_n^+, n=0,\pm 1\ldots\}$ be sets of functions in space P (of 2π-periodic functions having convergent Fourier series), such that

$$\int_0^{2\pi}\phi_l(\theta)\,\phi_m^+(\theta)\,\mathrm{d}\theta=\delta_{lm}, \tag{B 1}$$

where δ_{lm} is the Kronecker delta ($\{\phi_n^+\}$ will be referred to as the adjoint *set to $\{\phi_n\}$).*

Then, if

$$\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\phi_n(\theta)\,\mathrm{e}^{-\mathrm{i}l\theta}\,\mathrm{d}\theta\int_0^{2\pi}\phi_n^+(\theta)\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta=2\pi\,\delta_{lm}, \tag{B 2}$$

the set $\{\phi_n, n=0,\pm 1\ldots\}$ is complete in the subspace $P'\subset P$ defined by

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|b_m|\left|\int_0^{2\pi}\phi_n^+(\theta)\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta\right||\phi_n(\theta)|<\infty, \tag{B 3}$$

where

$$b_m=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta \tag{B 4}$$

are the Fourier coefficients of an arbitrary function $f(\theta)\in P'$.

To prove this theorem, observe that the Fourier set, $\{\mathrm{e}^{\mathrm{i}m\theta}, l=0,\pm 1\ldots\}$, is complete in any subspace of P. Hence, if every function of the Fourier set can be represented by a convergent series in ϕ_n,

$$\mathrm{e}^{\mathrm{i}m\theta} = \sum_{n=-\infty}^{\infty} a_{mn}\phi_n, \tag{B 5}$$

the set $\{\phi_n\}$ should also be complete. Indeed, consider an arbitrary $f(\theta) \in P$,

$$f(\theta) = \sum_{m=-\infty}^{\infty} b_m\, \mathrm{e}^{\mathrm{i}m\theta}.$$

Taking into account (B 5), we obtain

$$f(\theta) = \sum_{m=-\infty}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_{mn}\phi_n. \tag{B 6}$$

Then, changing the order of summation (the legitimacy of which will be justified below), we have

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n\phi_n, \tag{B 7}$$

where

$$c_n = \sum_{m=-\infty}^{\infty} b_m a_{mn}$$

are the coefficients of the series in ϕ_n. Evidently, equality (B 7) proves that the set $\{\phi_n\}$ is complete.

In order to finish the proof, it remains to

(*a*) transform (B 5) into (B 2),

(*b*) justify the change of the order of summation, which was necessary to derive (B 7) from (B 6).

B.1. *Transformation of* (*B 5*) *into* (*B 2*)

Rewrite (B 5) in terms of the Fourier coefficients of its right- and left-hand sides (if two continuous functions are equal, their Fourier coefficients are also equal, and vice versa). To do so, multiply (B 5) by $\mathrm{e}^{-\mathrm{i}l\theta}$ and integrate over $0 \leqslant \theta \leqslant 2\pi$:

$$2\pi\, \delta_{lm} = \sum_{n=-\infty}^{\infty} a_{mn} \int_0^{2\pi} \phi_n\, \mathrm{e}^{-\mathrm{i}l\theta}\, \mathrm{d}\theta. \tag{B 8}$$

To derive an expression for a_{mn}, multiply (B 5) by ϕ_l^+ and, again, integrate over $0 \leqslant \theta \leqslant 2\pi$. Taking into account (B 1), we have

$$a_{ml} = \int_0^{2\pi} \phi_l^+\, \mathrm{e}^{\mathrm{i}m\theta}\, \mathrm{d}\theta. \tag{B 9}$$

Having changed l to n, substitute (B 9) into (B 8), which results in (B 2), as required.

B.2. *Change of the order of summation in (B 6)*

To justify the change of the order of summation in a double series, it is sufficient to prove that the series converges absolutely. In application to (B 6), this implies

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |b_m|\,|a_{mn}|\,|\phi_n| < \infty. \tag{B 10}$$

Then, taking into account (B 9) (with l changed to n), we see that (B 10) is equivalent to one of the conditions of our theorems, namely, (B 3).

Appendix C. Properties of I_n

In this Appendix, we shall list the formulae used in the main body of the paper and involving the modified Bessel function $I_n(x)$.

(*a*) Abramowitz & Stegun (1964) (formula 9.6.19):

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)\cos(n\theta)\,\mathrm{d}\theta. \tag{C 1}$$

(*b*) Prudnikov, Brychkov & Marichev (1981) (volume 2, formula 5.8.6.1):

$$\sum_{n=-\infty}^{\infty} I_n(x)\, I_{l-n}(y) = I_l(x+y). \tag{C 2}$$

(*c*) Abramowitz & Stegun (1964) (formula 9.6.10):

$$I_n(x) = \sum_{j=0}^{\infty} \frac{x^{n+2j}}{2^{n+2j}\, j!\,(n+j)!} \quad \text{if} \quad n > 0,$$

Abramowitz & Stegun (1964) (formula 9.6.6):

$$I_n(x) = I_{-n}(x).$$

It follows from these formulae that

$$I_n(x) = \frac{1}{|n|!}\left(\frac{x}{2}\right)^{|n|}[1 + \mathrm{O}(x^2)] \quad \text{as} \quad x \to 0, \quad n = \text{const}, \tag{C 3}$$

and

$$I_n(x) = \frac{1}{|n|!}\left(\frac{x}{2}\right)^{|n|}\left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] \quad \text{as} \quad n \to \pm\infty, \quad x = \text{const}. \tag{C 4}$$

(*d*) Prudnikov *et al.* (1981) (volume 2, formula 5.8.4.4):

$$\sum_{n=-\infty}^{\infty} \cos(na)\, I_n(x) = \exp(x\cos a). \tag{C 5}$$

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