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Key Points:

- A wide class of stable quasigeostropic vortices in a continuously stratified ocean is presented
- The stable vortices explain the observed longevity of oceanic mesoscale eddies

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Can Large Oceanic Vortices Be Stable?

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Abstract Observations show that radii of oceanic eddies often exceed the Rossby radius of deformation, whereas theoretical studies suggest that such vortices should be unstable. The present paper resolves this paradox by presenting a wide class of large geostrophic vortices with a sign-definite gradient of potential vorticity (which makes them stable), in an ocean where the density gradient is mostly confined to a thin near-surface layer (which is indeed the case in the real ocean). The condition of a thin "active" layer is what makes the present work different from the previous theoretical studies and is of utmost importance. It turns out that without it, the joint requirement that a vortex be large and have a sign-definite potential vorticity gradient trivializes the problem by eliminating all vortices except nearly barotropic ones.

1. Introduction

Mesoscale eddies play an important role in oceanic circulation, as they transport significant amounts of water a long way across the ocean. Satellite imagery shows that eddies exist in a highly variable field, and yet they retain their characteristics over distances of thousands kilometers.

The paradox associated with oceanic eddies was identified more than 30 years ago: observations show that mesoscale eddies exist for years (Chelton et al., 2011; Lai & Richardson, 1977), whereas theoretical studies suggest that they are unstable and should disintegrate within weeks. Stable vortices with typical oceanic parameters have been found only in fluids that are unbounded (Dritschel, 1988; Dritschel et al., 2005; Tsang & Dritschel, 2014) or layered (Benilov, 2004; Benilov & Flanagan, 2008; Dewar & Killworth, 1995; Katsman et al., 2003), whereas more realistic models—with rigid/free boundaries and continuous stratification—have all indicated instability (e.g., Mahdinia et al., 2017; Nguyen et al., 2012; Yim et al., 2016, and references therein). The mechanism of the instability is as follows: since the radii of eddies may exceed the Rossby radius of deformation by a factor of 2–4 (Chelton et al., 2011; Olson, 1991), the horizontal shear in the vortex is too weak to suppress the baroclinic instability caused by the vertical shear (the role of horizontal shear in supressing baroclinic instability was discussed by Benilov, 2003, see p. 320, paragraph 2).

One cannot help but wondering, however, why the paradox cannot be resolved by extending the quasi-geostrophic (QG) criterion of Dritschel (1988) to bounded fluids and finding (stable) vortices that satisfy it. This appears to be a straightforward path, but no one has followed it to the end. There have been only four attempts to exploit this idea. The theoretical results of Sutyrin (1989), however, have never been tested for the "real" ocean, whereas the eddies examined by Sutyrin and Radko (2016) and Radko and Sisti (2017) were relatively small (in both cases, the radius of the maximum swirl velocity was 21 km—i.e., comparable to, or smaller than, typical values of R_d). Finally, Benilov (2017) examined eddies using an asymptotic model which neglects (potentially unstable) short-wave disturbances.

To understand why the stability criterion of Dritschel (1988) has been underused, note that it requires that the radial gradient of potential vorticity (PV) be sign definite. Then, as shown below, large vortices with sign-definite PV gradient cannot generally have a sizable vertical shear, that is, they are nearly barotropic—whereas mesoscale oceanic eddies are strongly baroclinic.

There seems to be only one exception to this rule: the large size, sign-definite PV gradient, and baroclinicity *can* coexist in the same vortex—but only if the ocean's density stratification is confined to a thin near-surface layer (sometimes referred to as "active").

To express the above claim in mathematical terms, introduce the radius R of the vortex, the Rossby radius L, the Burger number $\varepsilon = L^2/R^2$, and the active-to-passive depth ratio δ. Then, if $\varepsilon \to 0$ (the large-vortex limit), the condition of sign-definite PV gradient and baroclinicity are consistent only if $\delta \to 0$, under the additional condition

$$\frac{\delta}{\varepsilon} \to \text{const.}$$

This result is sufficient for explaining the available observations, according to which $\varepsilon \sim \delta \sim 0.1$.

This paper has the following structure: section 2 briefly reviews the stability criterion of Dritschel (1988), and section 3 presents examples of large vortices that satisfy it. The results obtained are summarized and discussed in section 4.

2. Formulation of the Problem

2.1. The Governing Equations

Consider a rotating, density-stratified ocean characterized by the Coriolis parameter f and the Väisälä-Brunt frequency $N(z)$, where z is the vertical coordinate of the cylindrical set (also comprising the horizontal radial variable r and angle θ). Let the flow be described by the QG equation for the stream function ψ (e.g., Pedlosky, 1987),

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] + J\left[\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] = 0, \tag{1}$$

where t is the time variable, and the horizontal Laplacian and Jacobian are

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}, \qquad J[\psi, Q] = \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial Q}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial Q}{\partial r}\right).$$

Under the rigid-lid approximation, the boundary conditions at the ocean's surface and bottom are

$$\frac{\partial^2\psi}{\partial t\,\partial z} + J\left[\psi, \frac{\partial\psi}{\partial z}\right] = 0 \qquad \text{at} \qquad z = 0, -H, \tag{2}$$

where H is the ocean's depth.

Equations (1)–(2) admit a class of steady axisymmetric solutions, $\psi = \Psi(r, z)$, such that

$$\frac{\partial\Psi}{\partial r} = 0 \qquad \text{if} \qquad r = 0, \tag{3}$$

$$\Psi \to 0 \qquad \text{as} \qquad r \to \infty, \tag{4}$$

describing a localized vortex. To examine its stability, let

$$\psi = \Psi(r, z) + \tilde\psi(r, \theta, z, t).$$

Substituting this expression into equations (1)–(2) and linearizing them, one obtains

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial\theta}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\tilde\psi}{\partial\theta^2} + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\tilde\psi}{\partial z}\right)\right] - \frac{\partial\tilde\psi}{\partial\theta}\frac{\partial Q}{\partial r} = 0, \tag{5}$$

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial\theta}\right)\frac{\partial\tilde\psi}{\partial z} - \frac{\partial\tilde\psi}{\partial\theta}\frac{\partial^2\Psi}{\partial r\,\partial z} = 0 \qquad \text{at} \qquad z = 0, -H, \tag{6}$$

where the swirl velocity V and PV Q are

$$V = \frac{\partial\Psi}{\partial r}, \tag{7}$$

$$Q(r, z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\Psi}{\partial z}\right). \tag{8}$$

This paper is confined to normal-mode disturbances, that is, solutions of the form

$$\tilde\psi(r, \theta, z, t) = \hat\psi(r, z)\,\mathrm{e}^{\mathrm{i}k(\theta - ct)}, \tag{9}$$

where k is the azimuthal wave number and c is the phase speed. Substituting (9) into (5)–(6), one obtains

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat\psi}{\partial r}\right) + \frac{k^2}{r^2}\hat\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\hat\psi}{\partial z}\right) - \frac{1}{V - c}\frac{\partial Q}{\partial r}\hat\psi = 0, \tag{10}$$

$$\frac{\partial\hat\psi}{\partial z} - \frac{1}{V - c}\frac{\partial^2\Psi}{\partial r\,\partial z}\hat\psi = 0 \qquad \text{at} \qquad z = 0, -H. \tag{11}$$

Disturbances should be regular at $r = 0$ and localized near the vortex, that is,

$$\hat{\psi} \to 0 \qquad \text{as} \qquad r \to 0, \infty. \tag{12}$$

Equation (10) and boundary conditions (11)–(12) form an eigenvalue problem, where $\hat{\psi}(r, z)$ is the eigenfunction and c is the eigenvalue. If one or more solutions exists such that $\operatorname{Im} c > 0$, the vortex under consideration is unstable.

2.2. A Stability Criterion for QG Vortices

Multiply equation (10) by $r\hat{\psi}^*$ where the asterisks denote complex conjugate and integrate with respect to r and z over the region $(0, \infty) \times (-H, 0)$. Integrating the first and third terms by parts, taking into account (11)–(12), and separating the imaginary part of the resulting equality, one obtains

$$(\operatorname{Im} c) \int_0^\infty \left[\int_{-H}^0 \frac{r\,|\hat{\psi}|^2}{|V - c|^2} \frac{\partial Q}{\partial r} \mathrm{d}z - \int_0^\infty \left(\frac{rf^2\,|\hat{\psi}|^2}{N^2\,|V - c|^2} \frac{\partial^2 \Psi}{\partial r\,\partial z} \right)_{z=0} + \int_0^\infty \left(\frac{rf^2\,|\hat{\psi}|^2}{N^2\,|V - c|^2} \frac{\partial^2 \Psi}{\partial r\,\partial z} \right)_{z=-H} \right] \mathrm{d}r = 0.$$

As usual, stability can be guaranteed only if the integrand in the above expression is sign definite regardless of the specific form of $\hat{\psi}(r, z)$ – which, in turn, can be guaranteed only if

1. $\partial Q / \partial r$ is sign definite, and
2. $$\left(\frac{\partial^2 \Psi}{\partial r\,\partial z} \right)_{z=0} \frac{\partial Q}{\partial r} \leq 0, \qquad \left(\frac{\partial^2 \Psi}{\partial r\,\partial z} \right)_{z=-H} \frac{\partial Q}{\partial r} \geq 0. \tag{13}$$

For simplicity, conditions (13) will be reduced to

$$\frac{\partial \Psi}{\partial z} = 0 \qquad \text{if} \qquad z = -H, 0. \tag{14}$$

Note that all the results obtained below for (14) can be readily reproduced for the full conditions (13).

Thus, to find a stable vortex, one should choose a function $Q(r, z)$ with a sign-definite radial gradient, and then find $\Psi(r, z)$ from the boundary-value problem (8), (14).

Finally, to accommodate simplified models with piecewise constant Väisälä-Brunt frequency (to be used later), $N(z)$ should be allowed to have a jump. Assuming it to be located at $z = -H_a$, one should require there the continuity of the pressure and isopycnal displacement,

$$(\Psi)_{z=-H_a-0} = (\Psi)_{z=-H_a+0}, \qquad \left(\frac{1}{N^2} \frac{\partial \Psi}{\partial z} \right)_{z=-H_a-0} = \left(\frac{1}{N^2} \frac{\partial \Psi}{\partial z} \right)_{z=-H_a+0}. \tag{15}$$

3. Large Vortices in an Ocean With Thin Active Layer

3.1. The Scaling

Assume that the ocean can be subdivided into an upper "active" layer of depth H_a and a lower "passive" layer of depth $H_p = H - H_a$. The two layers differ by their scales, N_a and N_p, of the Väisälä-Brunt frequency, such that $N_a \gg N_p$. Introduce also the active-layer PV scale Q_a, and assume that the PV field does not penetrate the passive layer (this assumption is not essential for the analysis to come but does simplify the calculations).

Introduce the following nondimensional variables:

$$\Psi_{nd} = \frac{f^2 \Psi}{H_a^2 N_a^2 Q_a}, \qquad Q_{nd} = \frac{Q}{Q_a}, \qquad N_{nd} = \frac{N}{N_a},$$

$$r_{nd} = \frac{r}{R}, \qquad z_{nd} = \begin{cases} \frac{z+H_a}{H_a} & \text{if} \qquad z \in \left(0, -H_a\right), \\ \frac{z+H_a}{H_p} & \text{if} \qquad z \in \left(-H_a, -H\right), \end{cases}$$

where R is the radius of the vortex (say, the distance between its center and the maximum of the swirl velocity). Observe that in terms of the nondimensional variables, the active and passive layers correspond to $z_{nd} \in (0, 1)$ and $z_{nd} \in (-1, 0)$, respectively.

Rewriting (8) and (14)–(15) in terms of the nondimensional variables and omitting the subscript $_{nd}$, one obtains

$$\frac{\partial\Psi}{\partial z}=0 \qquad \text{if} \qquad z=1, \tag{16}$$

$$\frac{\varepsilon_a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\Psi}{\partial z}\right)=Q(r,z) \qquad \text{if} \qquad z\in(0,1), \tag{17}$$

$$(\Psi)_{z=-0}=(\Psi)_{z=+0}, \qquad \left(\frac{1}{N^2}\frac{\partial\Psi}{\partial z}\right)_{z=-0}=\delta\frac{\varepsilon_p}{\varepsilon_a}\left(\frac{1}{N^2}\frac{\partial\Psi}{\partial z}\right)_{z=+0}, \tag{18}$$

$$\frac{\varepsilon_p}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\Psi}{\partial z}\right)=0 \qquad \text{if} \qquad z\in(-1,0), \tag{19}$$

$$\frac{\partial\Psi}{\partial z}=0 \qquad \text{if} \qquad z=-1, \tag{20}$$

where the equations are arranged in the order corresponding to the vertical structure of the ocean, and the active-to-passive depth ratio and the layers' Burger numbers are

$$\delta=\frac{H_a}{H_p}, \qquad \varepsilon_a=\frac{N_a^2H_a^2}{f^2R^2}, \qquad \varepsilon_p=\frac{N_p^2H_p^2}{f^2R^2},$$

Given a specific PV field $Q(r,z)$, one can solve set (16)–(20) with the boundary conditions (3)–(4) (whose nondimensional versions look exactly the same as the dimensional ones) and thus find the stream function $\Psi(r,z)$.

3.2. The Asymptotic Analysis

Everywhere in this work, it is assumed that

$$\varepsilon_a \ll \varepsilon_p, \qquad \delta\sim\varepsilon_a.$$

The former assumption is based on the fact that the passive layer's stratification is much weaker than the active layer's one. The latter assumption is justified by the "real-ocean" estimate $\delta\sim 0.1$, plus our interest in vortices whose radii are no more than 4 times greater than the active-layer Rossby radius.

Employing a simple iterative procedure, one can seek the solution of (16)–(17) and (19)–(20) in the form of series in ε_a and ε_p, respectively, substitute the series into the matching conditions (18), and obtain

$$\Psi=A(r)-\int_0^z N^2(z')\int_{z'}^1 Q\left(r,z''\right)\,\mathrm{d}z''\mathrm{d}z'+\mathcal{O}(\varepsilon_a) \qquad \text{if} \qquad z\in(0,1), \tag{21}$$

$$\Psi=A(r)-\frac{\varepsilon_p}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right]\int_{-1}^z N^2(z')\left(z'+1\right)\mathrm{d}z'+\mathcal{O}(\varepsilon_p^2) \qquad \text{if} \qquad z\in(-1,0), \tag{22}$$

where $A(r)$ satisfies

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}A(r)}{\mathrm{d}r}\right]=\frac{\delta}{\varepsilon_a}\int_0^1 Q\left(r,z\right)\,\mathrm{d}z. \tag{23}$$

Observe that the passive-layer expansion (22) includes two terms, whereas its active-layer counterpart includes only one. This has been done for a reason: if the former were truncated at the leading order, equation (23) would have zero left-hand side.

For sign-definite $\partial Q/\partial r$, the above asymptotic solution does generate stable vortices. It is not clear, however, whether or not the velocity field described by (21)–(23) fits the structure of the observed mesoscale eddies.

This issue will be addressed in the next section.

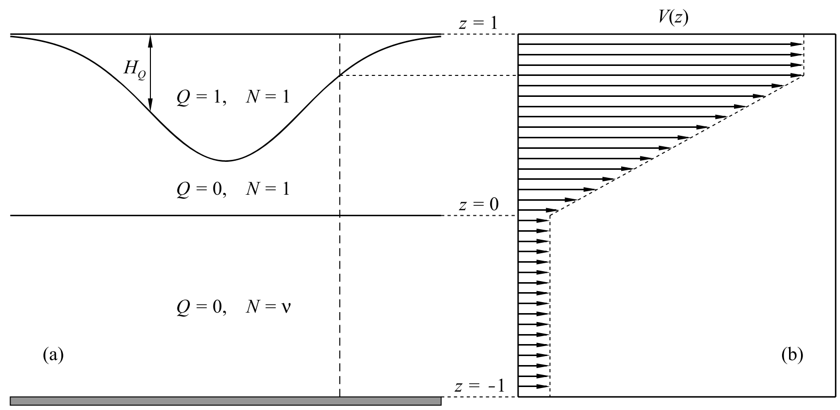

Figure 1. A vortex in an ocean with piecewise constant stratification and potential vorticity. (a) The cross section of a vortex; (b) the velocity profile at the location indicated by the vertical dashed line in panel (a).

3.3. Can Vortices Described by (21)–(23) Be Baroclinic?

It is instructive to consider vortices with a uniform-PV core, in an ocean with piecewise constant N (see Figure 1a),

$$Q = \begin{cases} 1 \text{ if} & z \in \left(1 - H_Q, 1\right), \\ 0 \text{ if} & z \in \left(-1, 1 - H_Q\right), \end{cases} \qquad N = \begin{cases} 1 \text{ if} & z \in (0, 1), \\ \nu \text{ if} & z \in (-1, 0), \end{cases} \tag{24}$$

where $\nu \ll 1$ and $H_Q(r) < 1$ (the latter condition guarantees that the core is fully within the active layer). Since, with increasing r and fixed z, the PV field $Q(r, z)$ either remains constant or decreases (with a jump), such a vortex is stable.

To leading order, substitution of (24) into (21)– (22) yields

$$V = \begin{cases} \frac{\mathrm{d}A}{\mathrm{d}r} - \frac{\mathrm{d}H_Q}{\mathrm{d}r}\left(1 - H_Q\right) & \text{if} \quad z \in \left(1 - H_Q, 1\right), \\ \frac{\mathrm{d}A}{\mathrm{d}r} - \frac{\mathrm{d}H_Q}{\mathrm{d}r} z & \text{if} \quad z \in \left(0, 1 - H_Q\right), \\ \frac{\mathrm{d}A}{\mathrm{d}r} & \text{if} \quad z \in (-1, 0), \end{cases} \tag{25}$$

where V is the swirl velocity (see (7)). Expression (25) describes the vertical structure of the vortex and is illustrated in Figure 1b. Note that out of the two "corners" in the velocity profile, only the upper one is accompanied by a PV jump.

Since the term $\mathrm{d}A/\mathrm{d}r$ appears in all three layers, $A(r)$ can be interpreted as the vortex's *barotropic* component, whereas $H_Q(r)$ characterizes the *baroclinic* one. To interrelate the two, substitute (24) into (23), which yields

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}A}{\mathrm{d}r}\right) = \frac{\delta}{\varepsilon_a} H_Q.$$

This equality shows that unless $\delta \lesssim \varepsilon_a$, the vortex's barotropic component is larger than the baroclinic one.

In other words, large stable vortices with a uniform-PV core are baroclinic only if the ocean's active layer is sufficiently *thin* ($\delta \lesssim \varepsilon_a$). If the active layer is *thick* ($\delta/\varepsilon_a \gg 1$), then $A \gg H_Q$, and such vortices are nearly barotropic.

To show the same for nonuniform cores, observe that the vortex's baroclinic component is described by the second term in expression (21). Since this term is a nonsingular integral involving order-one (scaled) quantities, the baroclinic component is order-one too. As a result, it is the *barotropic* component that determines the vortex's structure: if $A \gg 1$, the vortex is nearly barotropic, whereas $A \sim 1$ makes it baroclinic.

Now, consider a large vortex with a nonuniform core, in an ocean with a thick active layer—such that

$$\frac{\delta}{\varepsilon_a} \gg 1. \tag{26}$$

When searching for baroclinic vortices, one should assume that there exists a substantial range of r (including, say, $r = r_0$) where

$$A \sim 1. \tag{27}$$

Clearly, conditions (26)–(27) are consistent with equation (23) only if

$$\int_0^1 Q\left(r, z_0\right)\, dz \ll 1.$$

Since $Q \sim 1$, the above inequality can hold only if $Q(r_0, z)$ changes sign—that is, there exist z_1 and z_2 such that

$$Q(z_1, r_0) > 0, \qquad Q(z_2, r_0) < 0.$$

Finally, since $Q(r, z) \to 0$ as $r \to \infty$, a value r_1 exists between r_0 and ∞ such that

$$\left(\frac{\partial Q}{\partial r}\right)_{z=z_1} < 0, \qquad \left(\frac{\partial Q}{\partial r}\right)_{z=z_2} > 0 \qquad \text{at} \qquad r = r_1,$$

that is, the PV gradient is not sign definite.

4. Summary and Concluding Remarks

The main result of this paper is a class of QG vortices described by (21)–(23), where it is implied that the radial gradient of the PV distribution $Q(r, z)$ is sign definite. If the Burger numbers ε_a and ε_p of the ocean's active and passive layers satisfy

$$\varepsilon_p \ll \varepsilon_a \ll 1,$$

all such vortices are as follows:

1. definitely stable,
2. and large (i.e., their radii exceed the active-layer Rossby radius).

It is also shown that if ε_a is much smaller than the active-to-passive depth ratio δ, vortices (21)–(23) become nearly barotropic and are not suitable for modeling oceanic eddies. However, since in the "real" ocean $\delta \sim \varepsilon_a \sim 0.1$, the results obtained are sufficient for explaining the observed longevity of mesoscale eddies.

Finally, the following remarks are in order:

1. The stability of eddies localized in a thin active layer has been previously examined by Benilov (2017) through an asymptotic model assuming the eddy's radius, *as well as the wavelength of the disturbances*, to exceed the deformation radius. As a result of the latter assumption, the (apparently stable) eddies described by Benilov (2017) can still be *un*stable with respect to short disturbances.
   The present work, in turn, does *not* restrict the allowable disturbances, although the assumptions of large eddies and a thin active layer are used in both papers. It is also worth mentioning that Benilov (2017) has overlooked the connection between the ocean's density stratification and the baroclinicity of large stable eddies (which is the main result of the present work).
2. Even though stable nearly barotropic vortices have nothing to do with mesoscale eddies, they are still of interest in the context of so-called "columnar" vortices (e.g., Dritschel & de la Torre Juárez, 1996).
3. All present results have been obtained using the QG approximation, which only applies to vortices with a small Rossby number and small displacement of isopycnal surfaces. This is not much of a restriction, however, as the QG approximation holds fairly accurately for most oceanic eddies (see Chelton et al., 2011, Figures 14 and 16). One might also mention that in many cases, the stability properties of quasigeostrophic (Benilov, 2003, 2004) and ageostrophic (Benilov, 2005; Benilov & Flanagan, 2008) vortices are remarkably similar.
4. Mesoscale eddies are often examined using simplified models with *constant* Väisälä-Brunt frequency, with a hope that the dynamics will be qualitatively correct. Such models, however, do not involve a passive layer, the presence of which can strongly affect the results (as the present work illustrates).

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References

Benilov, E. S. (2003). Instability of quasi-geostrophic vortices in a two-layer ocean with a thin upper layer. *Journal of Fluid Mechanics*, *475*, 303–331. https://doi.org/10.1017/s0022112002002823

Benilov, E. S. (2004). Stability of vortices in a two-layer ocean with uniform potential vorticity in the lower layer. *Journal of Fluid Mechanics*, *502*, 207–232. https://doi.org/10.1017/s0022112003007547

Benilov, E. S. (2005). The effect of ageostrophy on the stability of thin oceanic vortices. *Journal of Fluid Mechanics*, *39*, 211–226. https://doi.org/10.1016/j.dynatmoce.2005.01.001

Benilov, E. S. (2017). Stable vortices in a continuously stratified ocean with a thin active layer. *Fluids*, *2*, 43. https://doi.org/10.3390/fluids2030043

Benilov, E. S., & Flanagan, J. D. (2008). The effect of ageostrophy on the stability of vortices in a two-layer ocean. *Ocean Modelling*, *23*, 49–58. https://doi.org/10.1016/j.ocemod.2008.03.004

Chelton, D. B., Schlax, M. G., & Samelson, R. M. (2011). Global observations of nonlinear mesoscale eddies. *Progress in Oceanography*, *91*, 167–216. https://doi.org/10.1016/j.pocean.2011.01.002

Dewar, W. K., & Killworth, P. D. (1995). On the stability of oceanic rings. *Journal of Physical Oceanography*, *25*, 1467–1487.

Dritschel, D. G. (1988). Nonlinear stability bounds for inviscid, two-dimensional, parallel or circular flows with monotonic vorticity, and the analogous three-dimensional quasi-geostrophic flows. *Journal of Fluid Mechanics*, *191*, 575–581. https://doi.org/10.1017/s0022112088001715

Dritschel, D. G., & de la Torre Juárez, M. (1996). The instability and breakdown of tall columnar vortices in a quasi-geostrophic fluid. *Journal of Fluid Mechanics*, *328*, 129–160. https://doi.org/10.1017/s0022112096008671

Dritschel, D. G., Scott, R. K., & Reinaud, J. N. (2005). The stability of quasi-geostrophic ellipsoidal vortices. *Journal of Fluid Mechanics*, *536*, 401–421. https://doi.org/10.1017/s0022112005004921

Katsman, C. A., van der Vaart, P. C. F., Dijkstra, H. A., & de Ruijter, W. P. M. (2003). Stability of multi-layer ocean vortices: A parameter study including realistic gulf stream and agulhas rings. *Journal of Physical Oceanography*, *33*, 1197–1218. https://doi.org/10.1175/1520-0485(2003)033<1197:SOMOVA>2.0.CO;2

Lai, D. Y., & Richardson, P. L. (1977). Distribution and movement of gulf stream rings. *Journal of Physical Oceanography*, *7*, 670–683. https://doi.org/10.1175/1520-0485(1977)007<0670:DAMOGS>2.0.CO;2

Mahdinia, M., Hassanzadeh, P., Marcus, P. S., & Jiang, C.-H. (2017). Stability of three-dimensional Gaussian vortices in an unbounded, rotating, vertically stratified, Boussinesq flow: Linear analysis. *Journal of Fluid Mechanics*, *824*, 97–134. https://doi.org/10.1017/jfm.2017.303

Nguyen, H. Y., Hua, B. L., Schopp, R., & Carton, X. (2012). Slow quasigeostrophic unstable modes of a lens vortex in a continuously stratified flow. *Geophysical and Astrophysical Fluid Dynamics*, *106*, 305–319. https://doi.org/10.1080/03091929.2011.620568

Olson, D. B. (1991). Rings in the ocean. *Annual Review of Earth and Planetary Sciences*, *19*, 283–311. https://doi.org/10.1146/annurev.ea.19.050191.001435

Pedlosky, J. (1987). *Geophysical fluid dynamics* (2nd ed.). New York: Springer.

Radko, T., & Sisti, C. (2017). Life and demise of intrathermocline mesoscale vortices. *Journal of Physical Oceanography*, *47*(12), 3087–3103. https://doi.org/10.1175/jpo-d-17-0044.1

Sutyrin, G. G. (1989). The structure of a monopole baroclinic eddy. *Oceanology*, *29*, 139–144.

Sutyrin, G. G., & Radko, T. (2016). Stabilization of isolated vortices in a rotating stratified fluid. *Fluids*, *1*, 26. https://doi.org/10.3390/fluids1030026

Tsang, Y.-K., & Dritschel, D. G. (2014). Ellipsoidal vortices in rotating stratified fluids: Beyond the quasi-geostrophic approximation. *Journal of Fluid Mechanics*, *762*, 196–231. https://doi.org/10.1017/jfm.2014.630

Yim, E., Billant, P., & Ménesguen, C. (2016). Stability of an isolated pancake vortex in continuously stratified-rotating fluids. *Journal of Fluid Mechanics*, *801*, 508–553. https://doi.org/10.1017/jfm.2016.402