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A NOTE ON THE STABILITY OF ONE-LAYER GEOSTROPHIC FRONTS

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It is shown that all one-layer geostrophic isolated fronts are stable regardless of their profiles.

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1. FORMULATION OF THE PROBLEM

Equations, describing barotropic motion in a layer of an ideal fluid on the β-plane, are

$$\left.\begin{aligned} u_t + uu_x + vu_y + gh_x &= (\Omega + \beta y)v, \\ v_t + uv_x + vv_y + gh_y &= -(\Omega + \beta y)u, \\ h_t + (uh)_x + (vh)_y &= 0, \end{aligned}\right\} \tag{1}$$

where (x, y) are the eastward and northward Cartesian coordinates, t is the time, (u, v) are the depth-averaged horizontal velocity components, h is the thickness of the layer, $(\Omega + \beta y)$ is the (variable) Coriolis parameter and g is the acceleration due to gravity. System (1) with the boundary conditions

$$h \to h_{\pm}, \quad u \to 0, \quad v \to 0 \qquad \text{as} \qquad y \to \pm\infty, \tag{2}$$

can be used as the simplest model of atmospheric fronts. If h is a monotonic function of y, the corresponding front is said to be *isolated* (or single).

Equations (1) have also been used for the description of *two*-layer fronts in the ocean (e.g. Griffiths *et al.*, 1982; Paldor, 1983a,b; Williams and Yamagata, 1984; Cushman-Roisin, 1986; Paldor and Ghil, 1990; etc.). In this case u, v and h describe the velocity and depth variations of the *upper* layer and g should be substituted for $g' = g\varepsilon$, where $\varepsilon = (\rho_2 - \rho_1)/\rho_2$ is the relative difference in the layers' densities. Physical relevance of system (1) will not be discussed here in detail, we only remark that (1) is believed to adequately describe two-layer fluids if the depth of the bottom layer is much greater than the depth of the upper layer and the velocity in the former is negligible. In the oceanographic context (1) can be supplemented with either boundary conditions (2) (including the cases $h_+ = 0$ and $h_\pm = 0$), or

$$\left.\begin{array}{lll} h \to h_+, & u, v \to 0, & \text{as} \quad y \to \infty, \\ v = 0 & & \text{at} \quad y = 0; \end{array}\right\} \tag{3}$$

which corresponds to a *coastal* front.

Equations (1) have an exact solution describing a steady zonal front:

$$\left.\begin{array}{ll} h = h(y), & \\ u = -\dfrac{g}{\Omega + \beta y} h_y(y), & v = 0, \end{array}\right\} \tag{4}$$

where $h(y)$ is an arbitrary function describing profile of the front. The stability of solution (4) was investigated mostly for the case of constant Coriolis parameter

$$\beta = 0 \tag{5a}$$

and for fronts of constant potential vorticity:

$$(\Omega - u_y + v_x)/h = \text{const.} \tag{5b}$$

It was shown (Paldor, 1983a) that the following isolated front with constant potential vorticity

$$h(y) = \begin{cases} h_+[1 - \exp(-y/l)] & \text{if} \quad y \geqslant 0, \\ 0 & \text{if} \quad y \leqslant 0 \end{cases} \tag{6a}$$

(where h_+ and l are constants) is stable.

Coastal fronts were analyzed within the framework of assumptions (5) also. Specifically, the front of zero potential vorticity

$$h(y) = \begin{cases} \Omega^2/2g(y^2 - l^2) & \text{if} \quad y \leqslant l, \\ 0 & \text{if} \quad y \geqslant l, \end{cases} \tag{6b}$$

was shown (Paldor, 1983b) to be stable provided the mean velocity of the flow exceeds Ωl.

It should be emphasized, however, that most of the major fronts in the ocean (except maybe Gulf Stream) are all characterized by *low* values of the Rossby number

$$Ro = U/\Omega L,$$

where U and L are the characteristics scales of u and y, and therefore cannot be described by solution (6b). Indeed, for the zero-potential-vorticity fronts (6b) $U \sim \Omega l$, $L \sim l$ and $Ro \sim 1$.

The present paper is devoted to the investigation of stability of geostrophic fronts (i.e. fronts with $Ro \ll 1$). Within the framework of geostrophic approximation the original equations (1) can be considerably simplified, which enables us to give up constraints (5). This approach has been used earlier by Cushman-Roisin (1986) to prove the stability of wedge shaped front

$$h(y) = \begin{cases} \alpha y & \text{if} \quad y \geqslant 0, \\ 0 & \text{if} \quad y \leqslant 0. \end{cases}$$

In this paper, we consider geostrophic isolated fronts with *arbitrary* profiles.

2. BASIC EQUATIONS

Non-dimensional variables and parameters can be introduced as follows:

$$x' = \Omega(gH)^{-1/2}x, \qquad y'\ \Omega(gH)^{-1/2}y, \qquad t' = \Omega t,$$

$$u' = (gH)^{-1/2}u, \qquad v' = (gH)^{-1/2}v, \qquad h' = H^{-1}h,$$

$$\beta' = \Omega^2(gH)^{1/2}\beta;$$

where H is the characteristic scale of h. The equation, governing geostrophic motion, can be easily derived from system (1) (cf. Williams and Yamagata, 1984; Cushman-Roisin, 1986). In terms of the new variables (with primes omitted), this equation is

$$h_t - \nabla\cdot[hJ(h,\nabla h)] - \beta hh_x = 0 \tag{7}$$

where $J(h,p) = h_x p_y - h_y p_x$ is the Jacobian operator. Boundary conditions (2) are

$$h \to h_{\pm} \qquad \text{as} \qquad y \to \pm\infty. \tag{8a}$$

The non-dimensional velocity components are determined by standard geostrophic formulae:

$$u \approx -h_y, \qquad v \approx h_x;$$

and the "coastal-front" boundary conditions can be rewritten as

$$\left.\begin{aligned} h &\to h_+ && \text{as} \quad y \to \infty, \\ h &= h_0 && \text{at} \quad y = 0; \end{aligned}\right\} \tag{8b}$$

where h_0 is a constant. The solution, describing small perturbations superimposed on a steady front, is

$$h(x,y,t) = h(y) + \Phi(x,y,t), \qquad |\Phi| \ll h. \tag{9}$$

Substituting (9) into (7) and linearizing the latter, we obtain

$$\Phi_t - (hh_{yy}\Phi_x) + \nabla \cdot (hh_y \nabla \Phi) - \beta h \Phi_x = 0. \tag{10}$$

We consider harmonic perturbations

$$\Phi(x,y,t) = \phi(y) \exp[ik(ct - x)], \tag{11}$$

where k and c are the wave number and phase velocity of the perturbations. Substitution of (11) into (10) yields

$$c\phi - (hh_y\theta_y)_y + (hh_{yy}\phi)_y + k^2 hh_y \phi + \beta h \phi = 0. \tag{12}$$

In terms of ϕ, boundary conditions (8a,b) can be written in the form of a single equation:

$$\phi = 0 \qquad \text{at} \qquad y = y_{\pm}, \tag{13}$$

where ($y_- = -\infty$, $y_+ = \infty$) or ($y_- = 0$, $y_+ = \infty$). Boundary-value problem (12), (13) determines the eigenfunction ϕ and the eigenvalue c.

3. STABILITY OF FRONTS

In order to prove that fronts are stable, it is necessary to show that all eigenvalues of boundary-value problem (12), (13) are real.

In terms of a new variable $\psi = \phi/h_y$, equations (12), (13) become

$$h_y(c + \beta h)\psi = [h(h_y)^2 \psi_y]_y - k^2 h(h_y)^2 \psi, \tag{14a}$$

$$\psi = 0 \qquad \text{at} \qquad y = y_{\pm}. \tag{14b}$$

Then, multiplying (14a) by ψ^* (where the asterisk denotes complex conjugate), we integrate it by parts with respect to y over the interval (y_-, y_+). Taking into account boundary conditions (14b), we obtain

$$c\int_{y_-}^{y_+} h_y |\psi|^2 \, dy = -\int_{y_-}^{y_+} [h(h_y)^2(|\psi_y|^2 + k^2|\psi|^2) - \beta h h_y |\psi|^2] dy, \tag{15}$$

which clearly demonstrates that, if

$$\int_{y_-}^{y_+} h_y |\psi|^2 \, dy \neq 0,$$

c may not be complex. The stability of an isolated (with a monotonic profile) geostrophic front is proven.

Equation (15) shows that for westward flows ($h_y \geqslant 0$), c is negative and the perturbations go downstream. For the case of eastward flows the sign of c is unclear: it depends on balance of the inertial force and β-effect. Equation (15) also gives an approximate estimate of the magnitude of c:

$$c \sim k^2 \int_{y_-}^{y_+} h(h_y)^2 \, dy \bigg/ \int_{y_-}^{y_+} h_y \, dy, \qquad (16)$$

where the integrals in the numerator and denominator are the momentum and kinetic energy of the front, respectively. Formula (16) corresponds to the expression for the meander propagation speed established by Cushman-Roisin *et al.* (1992).

4. CONCLUSIONS

Thus, it has been proven that all isolated one-layer geostrophic fronts are stable with respect to small harmonic perturbations. It should be emphasized that fronts with non-monotonic profile (e.g. *coupled* fronts) are not included in the proven theorem and therefore may be unstable (Pavia 1992).

It should be emphasized that the result obtained in the present paper does not prove that *real* atmospheric or oceanic fronts are stable (we know that this is not so—e.g. Killworth *et al.*, 1984); it rather indicates that the instability occurs due to *stratification* of the atmosphere and ocean. It should be emphasized that the fronts considered would not be found stable just because of the geostrophic approximation: for example, *two*-layer *geostrophic* fronts are unstable (cf. Benilov, 1992). One can conclude that one-layer models have serious shortcomings and should not be used for description of unstable flows.

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