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LARGE-AMPLITUDE GEOSTROPHIC DYNAMICS: THE TWO-LAYER MODEL

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This paper is concerned with large-amplitude flows of a two-layer fluid on the β-plane. The Rossby number ε is small, while the displacement of the interface and the depth of the upper layer are both of the order of the total depth of the fluid. Two systems of equations are derived, corresponding to two asymptotic ranges of the parameter β/ε (where β is the ratio of the deformation radius to the earth's radius). With the help of the equations derived, the stability of parallel density-driven flows is examined. It is shown that all flows are unstable with respect to the perturbations with wave length being of the order of the deformation radius.

KEY WORDS: Two-layer fluid, β-plane, Rossly waves, stability.

1. INTRODUCTION

Shallow-water equations, governing motion of a two-layer fluid on the β-plane, are

$$\left(\frac{\partial}{\partial t}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)u_1+\frac{\partial\eta}{\partial x}=(1+\beta y)v_1, \tag{1a}$$

$$\left(\frac{\partial}{\partial t}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)v_1+\frac{\partial\eta}{\partial y}=-(1+\beta y)u_1, \tag{1b}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu_1)+\frac{\partial}{\partial y}(hv_1)=0. \tag{1c}$$

$$\left(\frac{\partial}{\partial t}+u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}\right)u_2+\frac{\partial}{\partial x}(\eta-h)=(1+\beta y)v_2, \tag{2a}$$

$$\left(\frac{\partial}{\partial t}+u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}\right)v_2+\frac{\partial}{\partial y}(\eta-h)=-(1+\beta y)u_2, \tag{2b}$$

$$-\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}[(1-h)u_2]+\frac{\partial}{\partial y}[(1-h)v_2]=0. \tag{2c}$$

Here we have introduced the dimensionless variables

$$\left.\begin{aligned} h = \tilde{h}/H, \quad \eta = \tilde{\eta}/(\mu H), \quad u_{1,2} = \tilde{u}_{1,2}/(L_d f), \quad v_{1,2} = \tilde{v}_{1,2}/(L_d f); \\ t = f\tilde{t}, \qquad x = \tilde{x}/L_d, \qquad y = \tilde{y}/L_d; \end{aligned}\right\} \tag{3}$$

where the dimensional variables are marked with tilds: $\tilde{t}$, $\tilde{x}$, $\tilde{y}$ are the time and the horizontal spatial variables; $\tilde{h}$ is the depth of the interface; $\tilde{\eta}$ is the elevation of the free surface[1]; $\tilde{u}_{1,2}$, $\tilde{v}_{1,2}$ are the horizontal components of fluid velocity in the layers (1 indicates the upper layer); H is the total depth of the ocean; $f = 2\Omega \sin\phi$ is the local Coriolis parameter (Ω is the frequency of the earth rotation, ϕ is the latitude), $L_d = (\mu g H)^{1/2} f^{-1}$ is the internal deformation radius (g is the acceleration due to gravity, $\mu = (\rho_2 - \rho_1)/\rho_2$ is the relative difference in densities) and $\beta = (L_d/R)\cot\phi$ is the non-dimensional gradient of the Coriolis parameter (R is the earth's radius).

System (1), (2) consists of six nonlinear equations, and its analysis is a rather complicated problem. It can be simplified, however, if we

(1) take into account that most of real oceanic currents (except, maybe, Gulf Stream) are geostrophic and/or
(2) assume that the displacement of the interface is much smaller than the depths of the layers.

The implication of both assumptions results in the well-known Rossby-wave equations. Unfortunately, these relatively simple equations are not applicable to the important case of *large*-amplitude density-driven flows where the variations of the depth of the upper layer are comparable with its mean value. Traditionally, such flows were examined within the framework of *one-layer reduced-gravity model*, based on the equality

$$\eta \approx h, \tag{4}$$

which closes the "upper-layer" Equations (1) and allows them to be solved apart from (2). System (1), (4) describes the evolution of the upper layer as if it is influenced by reduced gravity $g' = \mu g$ with no bottom layer at all. Assumption (4) is supposed to be valid when the upper layer is much thinner than the bottom layer and the fluid velocity in the latter is negligible (e.g. Killworth *et al.*, 1984; Cushman-Roisin, 1986; Paldor and Ghil, 1990; Chassignet and Cushman-Roisin, 1991; Cushman-Roisin *et al.*, 1992).

Note, however, that all major currents in the real ocean can hardly be approximated by a two-layer system with *thin* upper layer, and the one-layer reduced-gravity model (1), (4) cannot be applied to them. Moreover, even when the upper layer *is* thin, the direct comparison of one-layer and two-layer results (Killworth, 1983; Killworth *et al.*, 1984) demonstrates that system (1), (4) is adequate only when the ratio of the depth of the upper layer to the total depth of the fluid is less than 0.01.

[1] If there was a real lid on the surface of the ocean, $\tilde{\eta}$ would be proportional to the pressure of fluid on the lid.

The present paper is devoted to the investigation of dynamics of geostrophic density-driven flows. Two asymptotic systems of equations are derived, describing a "thick"-upper-layer flow under the influence of "strong" or "weak" β-effect (Section 3). Physical aspects of these systems are discussed in Secton 4, the stability of their zonal-flow solutions is studied and discussed in Sections 5 and 6.

2. BASIC EQUATIONS AND FORMULATIONS OF THE PROBLEM

We shall need the vorticity equation, which can be derived according to the following recipe:

$$\left[\frac{\partial}{\partial x}(1b) - \frac{\partial}{\partial y}(1a)\right]h + \left[\frac{\partial}{\partial x}(2b) - \frac{\partial}{\partial y}(2a)\right](1-h) - (1+\beta y)[(1c)+(2c)],$$

yielding

$$h\left[\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right) + \left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)\left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right)\right]$$

$$+ (1-h)\left[\frac{\partial}{\partial t}\left(\frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y}\right) + \left(u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right)\left(\frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y}\right)\right]$$

$$= (1+\beta y)\left[\left(u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y}\right)h + \left(u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\right)(1-h)\right] - \beta[hv_1 + (1-h)v_2]. \tag{5}$$

Equation (5) will be used in the original system instead of (2c).

Large-amplitude density-driven oceanic motion is governed by three non-dimensional parameters:

non-dimensional depth of the upper layer	h_0,
the β-effect number	β,
the Rossby number	$\varepsilon = \tilde{U}/(\tilde{L}f)$;

where $\tilde{U}$ is the characteristic value of fluid velocity and $\tilde{L}$ is the spatial scale of the motion. Since the internal deformation radius is always much smaller than the earth's radius, the parameter $\beta \sim L_d/R$ is small. Then h_0, in its turn, is usually of the order of unity ($\sim 1/3 - 1/5$ for major oceanic currents), while the Rossby number varies within the limits ($0.01 - 0.3$) and for most of large-scale oceanic flows can be assumed small.

3. ASYMPTOTIC EQUATIONS

Since we are interested in flows where the displacement of interface and the depth of the upper layer are of the order of the total depth of the fluid, h and η should be scaled by unity:

$$h = h', \qquad \eta = \eta'. \tag{6a}$$

At the same time, the Rossby number is small, thus we have to assume that the horizontal scale of motion is much bigger than the deformation radius:

$$x = \varepsilon^{-1/2} x', \qquad y = \varepsilon^{-1/2} y'; \tag{6b}$$

where the powers of ε correspond to its physical meaning of the Rossby number. The fluid velocity is supposed to be geostrophic and should be scaled as follows:

$$u_{1,2} = \varepsilon^{1/2} u'_{1,2}, \qquad v_{1,2} = \varepsilon^{1/2} v'_{1,2}. \tag{6c}$$

As will be seen later, the time scale of motion is very sensitive to the ratio β/ε: in particular, there is a significant difference between the cases of "weak" and "strong" β-effect, i.e. $\beta \sim \varepsilon^{3/2}$ and $\beta \sim \varepsilon$, respectively (in the former case the β-effect is of the order of the first ageostrophic corrections, while in the latter case it is of the order of the main term of geostrophic dynamics). Accordingly, the time t should be scaled separately for different specific cases of the β-effect.

The case of *weak* β-effect will be considered first.

3.1) We set

$$\beta = \varepsilon^{3/2} \alpha. \tag{7a}$$

As will be justified by the result obtained, t should be scaled as follows:

$$t = \varepsilon^{-1} t'. \tag{7b}$$

In terms of the new variables (with primes omitted), Equations (1), (2a,b) and (5) are

$$\left.\begin{aligned}
\varepsilon\left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y}\right) u_1 - \alpha y v_1\right] + \frac{\partial \eta}{\partial x} &= v_1, \\
\varepsilon\left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y}\right) v_1 + \alpha y u_1\right] + \frac{\partial \eta}{\partial y} &= -u_1, \\
\varepsilon\left[\left(\frac{\partial}{\partial t} + u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}\right) u_2 - \alpha y v_2\right] + \frac{\partial}{\partial x}(\eta - h) &= v_2, \\
\varepsilon\left[\left(\frac{\partial}{\partial t} + u_2 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}\right) v_2 + \alpha y u_2\right] + \frac{\partial}{\partial y}(\eta - h) &= -u_2,
\end{aligned}\right\} \tag{8a}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu_1)+\frac{\partial}{\partial y}(hv_1)=0, \tag{8b}$$

$$\begin{aligned}&\varepsilon h\left[\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)+\left(u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}+\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)\right]\\&+\varepsilon(1-h)\left[\frac{\partial}{\partial t}\left(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\right)+\left(u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)\left(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\right)\right]\\&=(1+\varepsilon\alpha y)\left[\left(u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)h+\left(u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}\right)(1-h)-\varepsilon\alpha[hv_1+(1-h)v_2].\right.\end{aligned} \tag{8c}$$

With the help of (8a), $u_{1,2}$ and $v_{1,2}$ can be expressed in the form of a "quasigeostrophic" series. Taking into account terms O(1) and O(ε), we have:

$$\left.\begin{aligned}u_1&=-(1-\varepsilon\alpha y)\eta_y-\varepsilon[\eta_{xt}+J(\eta,\eta_x)],\\v_1&=(1-\varepsilon\alpha y)\eta_x-\varepsilon[\eta_{yt}+J(\eta,\eta_y)];\\u_2&=-(1-\varepsilon\alpha y)(\eta-h)_y-\varepsilon\{(\eta-h)_{xt}+J[(\eta-h),(\eta-h)_x]\},\\v_2&=(1-\varepsilon\alpha y)(\eta-h)_x-\varepsilon\{(\eta-h)_{yt}+J[(\eta-h),(\eta-h)_y]\};\end{aligned}\right\} \tag{9}$$

where the subscripts x, y and t denote derivatives and $J(\eta,h)=\eta_xh_y-\eta_yh_x$ is the Jacobian operator. Substitution of (9) into (8b,c) yields two equations governing the evolution of η and h (small terms $\sim\varepsilon$ dropped):

$$h_t+J(\eta,h)=0,$$

$$\begin{aligned}&h[\Delta\eta_t+J(\eta,\Delta\eta)+\alpha\eta_x]+(1-h)\{\Delta(\eta-h)_t+J[(\eta-h),\Delta(\eta-h)]+\alpha(\eta-h)_x\}\\&\qquad+\nabla h\cdot\{\nabla h_t+J(\eta,\nabla\eta)-J[(\eta-h),\nabla(\eta-h)]\}=0.\end{aligned}$$

In terms of a new variable

$$\eta\to\Psi=\eta-h+\tfrac{1}{2}h^2, \tag{10}$$

this system can be written as

$$h_t+J(\Psi,h)=0, \tag{11a}$$

$$\Delta\Psi_t + \alpha\Psi_x - (\Delta h)h_t - \nabla h \cdot \nabla h_t$$

$$+ h\{J(h, \Delta h) + J[h, \Delta(\Psi - \tfrac{1}{2}h^2)] + J[\Psi - \tfrac{1}{2}h^2), \Delta h]\} + J[(\Psi - \tfrac{1}{2}h^2), \Delta(\Psi - \tfrac{1}{2}h^2)]$$

$$+ \nabla h \cdot \{J(h, \nabla h) + J[\nabla(\Psi - \tfrac{1}{2}h^2)] + J[(\Psi - \tfrac{1}{2}h^2), \nabla h]\} = 0. \tag{11b}$$

After straightforward, but rather cumbersome calculations [including the use of (11a) to substitute for h_t], most of the nonlinear terms cancel out, and system (11) turns into

$$h_t + J(\Psi, h) = 0, \tag{12a}$$

$$\Delta\Psi_t + J(\Psi, \Delta\Psi) + \alpha\Psi_x + \nabla \cdot [h(1 - h)J(h, \nabla h)] = 0. \tag{12b}$$

Equation (12b) indicates that Ψ is the amplitude of *barotropic* component of motion, while Equation (12a) shows that the *baroclinic* mode does not contribute to the motion of the interface.

3.2) In the case of *strong* β-effect

$$\beta = \varepsilon\alpha, \tag{13a}$$

the motion is much slower:

$$t = \varepsilon^{-3/2}t'. \tag{13b}$$

Formula (10) should be scaled as follows:

$$\varepsilon^{1/2}\Psi = \eta - h + \tfrac{1}{2}h^2. \tag{13c}$$

[(13c) inter alia shows that, in the zeroth order, motions in the layers are related *algebraically*: $\eta \approx h - \frac{1}{2}h^2$. From a physical viewpoint, this should be interpreted as *a vertical nonlinear mode.*]

Substitution of (6) and (13) into (1), (2a,b) and (5), after some straightforward calculations, yields (primes omitted):

$$h_t + J(\Psi, h) = \alpha h(1 - h)h_x, \tag{14a}$$

$$\alpha\Psi_x + \nabla \cdot [h(1 - h)J(h, \nabla h)] = 0. \tag{14b}$$

Note that terms, describing β-effect and the geostrophic motion of the interface in (14a), are of the same order.

4. DISCUSSION

In this section, physical aspects of (12) and (14) are briefly discussed.

4.1) Dimensional parameters of flows, described by the equations derived, are represented in the following table [cf. (6) and (7), (6) and (13), and (3)]:

	(12)	(14)
horizontal spatial scale$/L_d$	$\gtrsim \varepsilon^{-1/2}$	$\gtrsim \varepsilon^{-1/2}$
time scale $\cdot f$	$\gtrsim \varepsilon^{-1}$	$\gtrsim \varepsilon^{-3/2}$
depth of the upper layer$/H$	~ 1	~ 1
velocity in the 1st layer$/(fL_d)$	$\lesssim \varepsilon^{1/2}$	$\lesssim \varepsilon^{1/2}$
velocity in the 2nd layer$/(fL_d)$	$\lesssim \varepsilon^{1/2}$	$\lesssim \varepsilon^{1/2}$
β	$\ll \varepsilon$	$\gg \varepsilon^{3/2}$

Note that *both* systems (12) and (14) are valid at $\varepsilon^{3/2} \ll \beta \ll \varepsilon$. Accordingly, the following "transitional" system of equation

$$\left.\begin{aligned} h_t + J(\Psi, h) = 0, \\ \alpha\Psi_x + \nabla\cdot[h(1-h)J(h, \nabla h)] = 0; \end{aligned}\right\} \tag{15}$$

can be obtained as ($\alpha \to \infty$)-limit of (12) or ($\alpha \to 0$)-limit of (14).

4.2) As well as the original two-layer equations, both systems (12) and (14) have a solution describing parallel zonal flows:

$$h = h(y), \qquad \Psi = \Psi(y), \tag{16}$$

where $h(y)$ and $\Psi(y)$ are arbitrary functions.

In addition to solution (16), system (12) has

a) a solution describing steady *non*-zonal flows

$$h = h(x\sin\gamma + y\cos\gamma), \qquad \Psi \equiv 0, \tag{17a}$$

(γ is the angle between the flow and the eastward direction) and

b) a radially-symmetric steady solution

$$h = h(x^2 + y^2), \qquad \Psi \equiv 0. \tag{17b}$$

Both these solutions are rather unusual for a system, taking into account β-effect. In what follows, we discuss their physical meaning.

Solutions (17) describe a current in the upper layer and a counter-current in the bottom layer, *the mass flux* $\mathbf{I} = h\mathbf{u}_1 + (1-h)\mathbf{u}_2$ *being equal to zero.* Indeed, with the help of (9) and (10), one can obtain the following expressions for the velocities:

$$u_1 \approx -(\Psi + h - \tfrac{1}{2}h^2)_y, \qquad v_1 \approx (\Psi + h - \tfrac{1}{2}h^2)_x;$$

$$u_2 \approx -(\Psi - \tfrac{1}{2}h^2)_y, \qquad v_2 \approx (\Psi - \tfrac{1}{2}h^2)_x;$$

and then verify the equality

$$\mathbf{I} = (-\Psi_y, \Psi_x).$$

Accordingly, the solutions with $\Psi = 0$ do not transfer mass across the β-plane. Of course, solutions (17) are not steady within the framework of the original Equations (1), (2); but the time of their relaxation is abnormally long ($\gg \varepsilon^{-1}$).

4.3) System (12) conserves the following invariant:

$$\text{Inv} = \iint [h(1-h)|\nabla h|^2 - |\nabla\Psi|^2]\, dxdy, \tag{18}$$

which is sign-indefinite and indicates the *elliptical* type of system (12). The type of a system, in its turn, is closely connected with the stability properties of its steady solutions—as it will be seen in the next section, all steady solutions of system (12) are unstable.

System (14) also conserves an invariant analogous to (18):

$$\text{Inv} = \iint h(1-h)|\nabla h|^2\, dxdy \geqslant 0,$$

which is, however, strictly positive; indicating the existence of *stable* steady solutions.

4.4) It should also be emphasized that, although system (12) looks very "quasigeostrophic", it can be used as a model of cross-frontal mixing. Indeed, coefficients of equations (12) [as well as those of (14)] are regular and finite at the point $h = 0$, and the corresponding validity criteria are not exceeded. From a physical viewpoint this means that systems (12) and (14) do describe eddies which detach from their parent water mass and cross over to the other side of the front—this can be seen from the fact that the "h"-equations in (12) and (14) are of the first order.

5. STABILITY OF ZONAL FLOWS

In this section, the stability of zonal-flow solution (16) will be investigated within the framework of systems (12) and (14).

5.1) *The case of weak β-effect.*

In terms of a new variable

$$\Phi = \int_0^h \sqrt{h'(1-h')}\, dh', \tag{19}$$

Equations (12) are

$$\left.\begin{aligned} \Phi_t + J(\Psi, \Phi) = 0, \\ \Delta\Psi_t + J(\Psi, \Delta\Psi) + \alpha\Psi_x + J(\Phi, \Delta\Phi) = 0. \end{aligned}\right\} \tag{20}$$

Linearizing (20) against the background of the zonal-flow solution

$$\left.\begin{aligned} \Psi(x,y,t) = \Psi(y) + \psi(y)\exp(i\omega t - ikx), \\ \Phi(x,y,t) = \Phi(y) + \phi(y)\exp(i\omega t - ikx); \end{aligned}\right\} \tag{21}$$

we obtain

$$\left.\begin{aligned} (\omega + k\Psi_y)\phi - k\Phi_y\psi = 0, \\ (\omega + k\Psi_y)(\psi_{yy} - k^2\psi) - k\Psi_{yyy}\psi - \alpha k\psi + k\Phi_y(\phi_{yy} - k^2\phi) - k\Phi_{yyy}\phi = 0. \end{aligned}\right\} \tag{22}$$

As will be seen later, the most unstable are *short* perturbations:

$$|\Psi_y|k^2 \gg \alpha. \tag{23}$$

Omitting, correspondingly, the term $|\alpha k\psi| \ll |k\Psi_y||k^2\psi|$ and introducing a new variable χ according to the formulae:

$$\psi = (\omega + k\Psi_y)\chi, \tag{24a}$$

$$\phi = k\Phi_y\chi; \tag{24b}$$

we can reduce (22) to

$$(\mathscr{S}\chi_y)_y - k^2\mathscr{S}\chi = 0, \tag{25}$$

where

$$\mathscr{S} = (\omega + k\Psi_y)_2 + (k\Phi_y)^2.$$

Equation (25) and the boundary condition:

$$\chi \to 0 \qquad \text{as} \qquad y \to \pm\infty$$

constitute a boundary-value problem, determining the eigenfunction χ and the eigenvalue ω. If we multiply (25) by χ^* (the asterisk denotes complex conjugate) and then integrate it with respect to y, we obtain the equality

$$\int_{-\infty}^{\infty} \mathscr{S}(|\chi_y|^2 + k^2|\chi|)dy = 0,$$

which clearly demonstrates that ω is complex. Both $\operatorname{Re}\mathscr{S}(y)$ and $\operatorname{Im}\mathscr{S}(y)$ must change their signs (say, at $y = y_r, y_i$):

$$\omega = k\Psi_y(y_i) \pm ik\sqrt{[\Psi_y(y_i) - \Psi_y(y_r)]^2 + [\Phi_y(y_r)]^2}. \tag{26}$$

Thus, *all zonal geostrophic flows are unstable regardless of their profiles.*

It should be emphasized, however, that formula (26) is valid in a rather narrow spectral interval:

$$\varepsilon^{-1} \gg k^2 \gg \alpha/U_{bt}, \tag{27}$$

where $U_{bt} = \max\{\Psi_y\}$. The first part of condition (27) arises because the (dimensional) spatial scale of the motion within the framework of system (12) should be much greater than the internal deformation radius L_d (cf. the table in Section 4), while the second part just follows from (23).

It is worth noting that if

$$\alpha \ll U_{bt}/L^2,$$

where L is the (non-dimensional) width of the flow, the above analysis is valid for *long*-wave perturbations as well. Indeed, in this case

$$k\Psi_{yyy}\psi \gg \alpha k\psi$$

and (22) can be reduced to clearly unstable (25) without short-wave assumption (23).

5.2) *The case of strong β-effect.*

Exactly the same procedure

$$(h,\Psi) \xrightarrow{(19)} (\Phi,\Psi) \xrightarrow{(21)} (\phi,\psi) \xrightarrow{(24b)} \chi$$

yields

$$\alpha[\omega + k\Psi_y + \alpha k f(\Phi)]\chi = k\Phi_y(\Phi_y\chi_{yy} + 2\Phi_{yy}\chi - k^2\Phi_y\chi), \tag{28}$$

where $f = h(\Phi)[1 - h(\Phi)]$ and h is related to Φ by (19). Multiplying (28) by χ^*

and integrating over $(-\infty < y < \infty)$, we obtain

$$\int_{-\infty}^{\infty} \alpha[\omega + k\Psi_y + \alpha k f(\Phi)]\,|\chi|^2\,dy = -\int_{-\infty}^{\infty} (\Phi_y)^2(|\chi_y|^2 + k^2|\chi|^2)\,dy.$$

Since this equality is linear with respect to ω and does not contain any complex quantities, ω is real; which corresponds to *the stability of all geostrophic flows regardless of their profiles.*

6. DISCUSSION

Thus, we have obtained a very robust criterion: if β-effect is *strong*, all flows are *stable*; if β-effect is *weak*, all flows are *unstable*. In order to understand its significance and physical meaning, we should place it in the broader context of results obtained earlier.

6.1) The existence of a strong universal instability of zonal flows is confirmed by analytic results obtained by Killworth *et al.* (1984) (hereinafter referred to as KPS). In spite of a number of differences in formulation of the problem:

	KPS	present paper
width of the flow	$\sim L_d$	$\gg L_d$
velocity in the bottom layer	$=0$	$\neq 0$
wavelength of perturbations	$\gg L_d$	$\gg L_d$
ratio of the upper layer's depth to the bottom layer's depth	$\ll 1$	~ 1
β	$=0$	$\neq 0$

both KPS and the present paper arrive to the same conclusion: *all zonal flows are unstable with respect to long perturbations* (where "long" means "long compared to the deformation radius"). At the same time, experimental data (Griffiths *et al.*, 1982), as well as numerical results (KPS, Paldor and Killworth, 1987; Paldor and Ghil, 1990), point out that much of the growth takes place at wavelengths *comparable* to L_d.

In order to clarify this contradiction, we note that short disturbances with wave numbers $k \gg 1/L_d$ have an internal-wave, rather than planetary-wave, nature and therefore are stable (internal waves propagating in a large-scale flow have been examined by many authors, e.g. Voronovich, 1976). Thus,

growth rate of perturbations with $k \gg 1/L_d$ is equal to zero.

On the other hand, results obtained in KPS and the present paper (formula (26)) show that

growth rate of perturbations with $k \ll 1/L_d$ grows with k.

Summing up both statements, we can conclude that the growth rate has its maximum at $k \sim 1/L_d$—as it should do in accordance with numerical and experimental evidence.

6.2) Another contradiction is associated with flows in the transitional domain $\varepsilon^{3/2} \ll \beta \ll \varepsilon$: those can be described by either system (12) or (14) and, accordingly, seem to be stable and unstable at the same time. In order to clarify this question, one should consider the transitional system (15) as the ($\alpha \to \infty$)-limit of (12). As the β-effect grows stronger, the marginal wave number of the unstable perturbations

$$k_m = (\alpha/U)^{1/2}$$

(cf. (27)) *tends to infinity*. Accordingly, if we expand an arbitrary initial condition in terms of the eigenfunctions of boundary-value problem (22), the Fourier amplitudes of the unstable short-wave harmonics, will be *exponentially* small[2]. As a result, the instability disappears from the transitional system (15), as well as from the strong β-effect system (14).

Thus, as $\beta \to \infty$

growth rate of the instability $\to$ const,

marginal wave number $\to \infty$.

6.3) Finally, setting $\varepsilon = 1$, we can construct a "mixed" system

$$\left.\begin{aligned} h_t + J(\Psi, h) &= \beta h(1-h)h_x, \\ \Delta\Psi_t + J(\Psi, \Delta\Psi) + \beta\Psi_x + \operatorname{div}[h(1-h)J(h, \nabla h)] &= 0; \end{aligned}\right\} \tag{29}$$

which apparently describes the evolution of geostrophic flows for all values of β. System (29) conserves sign-indefinite invariant (18) indicating the short-wave instability which can also be confirmed by direct asymptotic analysis of the corresponding boundary-value problem.

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