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Nonlinear interaction of a zonal jet and barotropic Rossby-wave turbulence: the problem of turbulent friction

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ABSTRACT

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Interaction of a zonal jet and small-amplitude Rossby-wave turbulence is studied within the framework of the barotropic β-plane model. It is demonstrated that turbulent–laminar interaction in this case transfers energy from the wave turbulence to the laminar flow (the effect of negative friction). We derive a conclusion that, as the geophysical turbulence is determined partly by wave turbulence and none of the traditional heuristic models can adequately describe the effect of negative friction associated with wave turbulence, the application of these models to the 'real' ocean and atmosphere is unreliable.

It is also demonstrated that, as they are affected by the turbulence, all westward jets slowly expand without strengthening. Each jet has a core, within the limits of which the velocity of the fluid is constant. In some cases, the core expands faster than the jet periphery, resulting in jumps on the profile of the flow. All eastward jets are steady irrespective of their profiles.

1. INTRODUCTION

There exist two common idealizations in theoretical studies of geophysical turbulence. The term strong turbulence usually describes a cascade of large-amplitude strongly nonlinear vortices, and weak turbulence implies random motion induced by a stochastic spectrum of weakly nonlinear Rossby waves. In some respects the two types of turbulence are similar (e.g. both cause irreversible stretching of material lines (Benilov and Wolanski, 1992)), but there are also some important differences. In the atmosphere,

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both observations and numerical models indicate that the strong component of the turbulence is likely to dominate weak-wave interaction; however, experimental data on space–time structure of oceanic turbulence does not allow us to assess the contributions of strong and weak components to the total energy balance.

We are concerned with the interaction of laminar and turbulent motions and, as an example of such, consider the interaction of weak Rossby-wave turbulence and a zonal jet in the barotropic ocean. The influence of strong turbulence is neglected, which allows us to transfer our problem into the framework of wave–flow interaction and divide it into three minor parts: (1) study of the scattering of each wave by the jet; (2) study of the reverse collective effect of the waves upon the jet; (3) study of the interaction of Rossby waves with each other.

All three problems have been considered earlier; e.g. a detailed study of monochromatic Rossby-wave scattering by a given zonal jet (1) was made by Dickinson (1970). The problem of self-consistent evolution of the wave–jet system has also been studied for the case of a monochromatic wave (a particular case of (2) + (3)) by many authors (see Killworth and McIntyre (1985) and references therein). Thus, to develop a realistic model of the oceanic weak turbulence, we should generalize the results obtained earlier for the case of a wide spectrum of Rossby waves. An analogous problem in the context of internal-gravity-wave turbulence was considered by Lindzen and Holton (1968) and Tsimring (1982). However, these authors did not take into account the nonlinear wave interaction (3), which in the case of internal waves appears to be slow and therefore can be neglected. (The characteristic time of Rossby-wave 'three-wave' interaction is proportional to ϵ^{-2} (where ϵ is the wave amplitude), whereas that for internal waves is 'four-wave' and, consequently, proportional to ϵ^{-4} (Zakharov, 1974).) Thus in the case of Rossby waves, the wave interaction is relatively fast and must be taken into consideration. It can be described within the framework of a kinetic approach (e.g. Reznik, 1980).

2. BASIC EQUATIONS

The equations governing barotropic motion on the β-plane are

$$\left.\begin{aligned} &u_t + (u^2)_x + (vu)_y + p_x = (\Omega + \beta y)v \\ &v_t + (uv)_x + (v^2)_y + p_y = -(\Omega + \beta y)u \\ &u_x + v_y = 0 \end{aligned}\right\} \tag{1}$$

where (x, y) are the spatial variables (the x-axis has the eastward direction), t is the time, (u, v) are the zonal and the meridional components of the velocity of the fluid, p is the pressure and $(\Omega + \beta y)$ is the (varying) Coriolis parameter. We seek a solution in the form

$$u = U(y, t) + \epsilon u'(x, y, t)$$
$$v = \epsilon v'(x, y, t)$$
$$p = P(y, t) + \epsilon p'(x, y, t)$$

Here (U, P) and (u', v', p') describe a zonal jet and the wave turbulence, respectively; ϵ is the parameter characterizing the wave amplitude. Clearly, mean values of random quantities are equal to zero: $\langle u' \rangle = \langle v' \rangle = 0$, $\langle p' \rangle = 0$ (here $\langle \cdot \rangle$ denotes the average over the ensemble of realizations).

Separating mean and random variables and introducing the stream function ψ:

$$u' = -\psi_y, \; v' = \psi_x$$

Describing the wave-induced velocity only, we have

$$U_t = \epsilon^2 \langle \psi_x \psi_y \rangle_y \tag{2a}$$

$$\Delta\psi_t + U\Delta\psi_x + (\beta - U_{yy})\psi_x - \epsilon\left[\langle \psi_x \psi_y \rangle_{yy} - \psi_x \Delta\psi_y + \psi_y \Delta\psi_x\right] = 0 \tag{2b}$$

(in obtaining (2), we used the statistical homogeneity of our system in the east–west direction: $\langle u'^2 \rangle_x = \langle v'^2 \rangle_x = \langle u'v' \rangle_x = 0$). This system is an exact consequence of the original eqns. (1).

We assume the amplitudes of Rossby waves to be small ($\epsilon \ll 1$), the effect of waves upon the jet to be weak and the evolution of the latter to be slow (in comparison with characteristic periods of the waves). Correspondingly, we can 'freeze' $U(y, t)$, i.e. treat it as a function of slow time variable $T = \epsilon^2 t$: $U = U(y, T) \neq U(y, t, T)$. At the same time, Rossby-wave field ψ depends on T and the fast time t. Substituting in (3) $\partial/\partial t$ for $((\partial/\partial t) + \epsilon^2(\partial/\partial T))$ (i.e. treating t and T as independent variables), we can drop in (2b) small terms of the order of ϵ, ϵ^2:

$$U_T = \epsilon^2 \langle \psi_x \psi_y \rangle_y \tag{3a}$$

$$\Delta\psi_t + U\Delta\psi_x + (\beta - U_{yy})\psi_x = 0 \tag{3b}$$

The right-hand-side term in (3a) describes the Reynolds stress. Using eqn. (3b), it can be calculated as a functional of $U(y, T)$ and spectrum of the turbulence. Equation (3a) will then turn into a closed-form equation governing $U(y, T)$.

It should be noted that, during the interaction with a zonal flow, each (linear) Rossby wave conserves the zonal wavenumber k and the frequency ω. The meridional wavenumber l, in its turn, does vary and can be fixed only far from the jet (at $y \to \pm\infty$), where the waves are free and their parameters satisfy the Rossby-wave dispersion relationship

$$\omega = -\beta k/(k^2 + l^2) \tag{4a}$$

The solution to the (linear) eqn. (3b) can be represented in the form of a Fourier (spectral) integral. The parameters 'numbering' the Rossby-wave spectrum are k and the ∞-value of l:

$$\psi = \int\int \alpha(k, l, T)\phi(k, l, y, T)\, \mathrm{e}^{i\omega(k,l)t - ikx}\, \mathrm{d}k\mathrm{d}l \tag{4b}$$

where $\phi(k, l, y, T)$ describes the scattering of the wave with the parameters (k, l) by the jet $U(y, T)$, and $\alpha(k, l, T)$ can be interpreted as the spectrum of the turbulence far from the jet (at $y \to \pm\infty$). Clearly, ψ is real only if

$$\alpha(-k, -l, T) = \alpha^*(k, l, T) \tag{5a}$$

$$\phi(-k, -l, y, T) = \phi^*(k, l, y, T) \tag{5b}$$

where the asterisk denotes the complex conjugate. Statistical homogeneity of the turbulence in the absence of mean flow entails the equality

$$\langle \alpha(k, l, T)\alpha(k', l', T)\rangle = (k^2 + l^2)^{-1} S(k, l, T)\delta(k + k')\delta(l + l') \tag{6}$$

where $\delta(k)$ is the Dirac delta-function ((6) can also be treated as a formal definition of the energy spectral density $S(k, l, ,T)$). As S characterizes the structure of the turbulence far from the jet, its dependence on T is determined by some external factors (this question will be discussed in more detail below). It should be noted also that (5a) and (6) entail

$$S(-k, -l, T) = S(k, l, T)$$

$$S(k, l, T) \geqslant 0$$

which should be understood as constraints limiting the allowed values of $S(k, l, T)$.

Substitution of (4) into (3b) yields an equation for ϕ:

$$\phi_{yy} - \left[\frac{k(\beta - U_{yy})}{\omega(k, l) + i0 - kU} + k^2\right]\phi = 0 \tag{7a}$$

where the term $i0$ corresponds, as usual, to the infinitesimal wave dissipation. It regularizes the behaviour of ϕ at the so-called critical layers, where the zonal component of the wave phase speed $c_{\mathrm{ph}}^{(x)} = \omega/k$ coincides with local velocity of the flow:

$$U(y_n) = c_{\mathrm{ph}}^{(x)} \quad n = 1, 2, \ldots N$$

(here N is the total number of critical layers for a given value of $c_{\mathrm{ph}}^{(x)}$). For

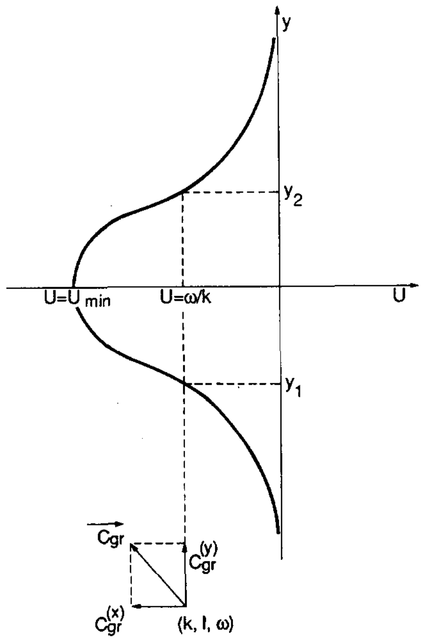

Fig. 1. Statement of the problem. $y_{1,2}$ are the critical layers of a Rossby wave with wave vector (k, l) and frequency ω; $U(y)$ is the velocity profile of the jet.

example, in the case of single-maximum jets, $N = 2$ if $U_{max} > c_{ph}^{(x)} > U_{min}$, and $N = 0$ if $c_{ph}^{(x)} > U_{max}$ or $c_{ph}^{(x)} < U_{min}$ (see Fig. 1). Obviously, as the Rossby-wave phase speed is always directed westwards, all eastward jets have no critical layers.

The viscous '$i0$'-regularization of the critical layer is not unique; for example, one can take into account infinitesimal nonlinear effects, after which the solution also becomes regular. Viscous, nonlinear and some other approaches to critical-layer regularization have been discussed by Killworth and McIntyre (1985).

Equation (7a) should be supplemented by boundary conditions at $y \to \pm\infty$, (where U tends to zero—we consider jet-like flows only). Rossby waves are free there, propagating at their group speeds:

$$c_{gr}^{(x)} = \frac{\partial \omega}{\partial k} = \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}$$

$$c_{gr}^{(y)} = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2 + l^2)^2}$$

One can see that, if at $y \to \pm\infty$ the group velocity of a wave is directed towards the jet, the corresponding wave is incident and, consequently, has an undisturbed amplitude. In terms of ϕ, this condition yields

$$\left.\begin{array}{lll} \phi \to \mathrm{e}^{-ily} + r\,\mathrm{e}^{ily} & \text{at} & y \to -\left[\operatorname{sign} c_{\mathrm{gr}}^{(y)}\right]\infty \\ \phi \to d\,\mathrm{e}^{-ily} & \text{at} & y \to \left[\operatorname{sign} c_{\mathrm{gr}}^{(y)}\right]\infty \end{array}\right\} \tag{7b}$$

where the scattering coefficients $r(k, l, T)$ and $d(k, l, T)$ characterize the amplitudes of the reflected and the penetrating waves, respectively. The boundary-value problem (7) determines r and d together with the eigenfunction ϕ.

Substituting (4) into (3a), making the integrand symmetrical and taking into account (6) and (5b), we have

$$U_T = \int\int k(k^2 + l^2)^{-1} S(k, l, T) W_y(k, l, y, T)\,\mathrm{d}k\mathrm{d}l \tag{8}$$

where the Wronskian $W = i/[2(\phi^*\phi_y - \phi\phi_y^*)]$ is a piecewise constant function and has jumps only at singular points of eqn. (7a), i.e. at the critical layers $y = y_n$. To calculate W_y, we construct the following combination of the eigenfunction ϕ, eqn. (7a) and their complex conjugates:

$$i/\{2[(7a)\cdot\phi^* - (7a)^*\cdot\phi]\} \equiv$$

$$W_y + k(\beta - U_{yy})\,|\phi|^2\,\mathrm{Im}\frac{1}{\omega + i0 - kU} = 0 \tag{9}$$

Substituting (9) into (8) and using the formula $\mathrm{Im}(\omega + i0)^{-1} = -\pi\delta(\omega)$, we obtain

$$U_T = -\pi(\beta - U_{yy})\int\int k^2(k^2 + l^2)^{-1} S\,|\phi|^2\delta(\omega - kU)\,\mathrm{d}k\mathrm{d}l \tag{10}$$

Equation (10) and boundary-value problem (7) form a closed-form system governing the slow evolution of the jet. Within the framework of this system the spectrum of the 'incident' turbulence $S(k, l, T)$ should be treated as an external parameter.

3. THE EVOLUTION OF A ZONAL JET

Critical layers play crucial role in wave–flow interaction. For example, the delta-function in (10) indicates that, within the framework of linear theory, each Rossby wave affects the mean flow only in its critical layers (the integral in (10) sums up contributions of all waves that have critical layers at a given point). On the other hand, critical layers determine the scattering properties of the mean flow:

$$1 - |r|^2 - |d|^2 = \frac{\pi}{|l|}\sum_{n=1}^{N}\left(\frac{\beta - U_{yy}}{|U_y|}|\phi|^2\right)\Bigg|_{y=y_n} \tag{11}$$

(see Appendix A). The left-hand side of (11) can be interpreted as the absorption coefficient of a Rossby wave (k, l) by the zonal jet $U(y)$. For example, eastward flows do not have critical layers at all, and the Rossby-wave scattering in this case is conservative:

$$|r|^2 - |d|^2 = 1$$

In contrast to eastward flows, stable $(\beta > U_{yy})$ westward jets do absorb waves in critical layers:

$$|r|^2 - |d|^2 < 1$$

The 'absorbing' properties of zonal jets entirely determine their time evolution. In particular, all eastward flows are steady irrespective of their profiles. Indeed, if $U > 0$, the argument of the delta-function in (10b) cannot turn into zero, and (10a) yields $U_T = 0$. (It should be emphasized, however, that this conclusion is 'asymptotic', i.e. valid within the framework of our leading-order asymptotic theory.)

An important conclusion can be derived from the sign of the right-hand side of eqn. (10) for the case of westward jets with $\beta > U_{yy}$:

$$U_T < 0 \text{ for } U < 0.$$

At $T > 0$, $|U|$ begins to grow; hence, the turbulent–laminar interaction transfers energy from the Rossby-wave turbulence to laminar westward motion. This phenomenon can be interpreted as the effect of negative anisotropic friction.

Unfortunately, any detailed description of the evolution of westward jets can be obtained only by using some additional assumption. Indeed, although the asymptotic system (7) and (10) is simpler than the original equations (1), it cannot be solved analytically in the general case (the main difficulty here is the calculation of $|\phi|^2$ at critical layers). Therefore, we shall consider the case of short-wave turbulence; this allows us to obtain $|\phi|^2|_{y=y_n}$ with the help of formula (11).

We consider a westward jet with the minimum value of its velocity profile located at $y = 0$ (Fig. 1). It is convenient to number the critical layers as follows:

$$y_1\text{: } \operatorname{sign}\left[y_1 c_{\text{gr}}^{(y)}\right] < 0$$

$$y_2\text{: } \operatorname{sign}\left[y_2 c_{\text{gr}}^{(y)}\right] > 0$$

(see Fig. 1). If the incident wave is much shorter than the width of the jet, the wave energy is almost completely absorbed in the first critical layer:

$$r \approx 0,\ d \approx 0,\ \phi|_{y=y_2} \approx 0$$

(Dickinson, 1970). Correspondingly, (11) yields

$$|\phi|^2|_{y=y_1} \approx \frac{|l|}{\pi}\left(\frac{|U_y|}{\beta - U_{yy}}\right)\bigg|_{y=y_1} \tag{12}$$

(One of the referees of this paper pointed out that the above formula can also be obtained as a consequence of the pseudo-momentum conservation law (Held, 1985; Held and Phillips, 1987).) Substituting (12) into (7), we obtain

$$U_T + G(U, T)|U_y| = 0 \tag{13a}$$

where

$$G = \int |l| k^2 (k^2 + l^2)^{-1} S\delta(\omega - kU) H\left[-yc_{\mathrm{gr}}^{(y)}\right] \,\mathrm{d}k\,\mathrm{d}l$$

(here $H(c)$ is the Heaviside step function: $H(c > 0) = 1$, $H(c < 0) = 0$). Introducing the polar variables,

$$k = q \cos \vartheta, \; l = q \sin \vartheta$$

and making use of the formula $\delta[f(q)] = |\mathrm{d}f/\mathrm{d}q|^{-1}\delta(q - q_0)$, where q_0 is the root of the function $f(q)$ ($f(q_0) = 0$), we have

$$\left.\begin{aligned} G(U, T, y) &= \tfrac{1}{4}U^{-2}\int_0^{2\pi} S\left[(-\beta/U)^{\frac{1}{2}}, \vartheta, T\right] |\sin 2\vartheta| \cdot H(-y \cdot \sin 2\vartheta) \,\mathrm{d}\vartheta \\ &\qquad \text{if } U < 0 \\ G(U, T, y) &= 0 \quad \text{if } U > 0 \end{aligned}\right\} \tag{13b}$$

Equation (13) is of hyperbolic type and can be easily solved by means of the method of characteristics.

We let the initial condition

$$U|_{T=0} = U_0(y) \tag{14}$$

be a negative function with its minimum value located at $y = 0$: $U_0(0) = U_{\min} < 0$ (see Fig. 1). Then, the solution of the Cauchy problem (13), (14) can be written in the parametric form:

$$y = \begin{cases} \xi + \int_0^T G^+[U_0(\xi), T'] \,\mathrm{d}T' & \text{if } \xi > 0 \\ \xi - \int_0^T G^-[U_0(\xi), T'] \,\mathrm{d}T' & \text{if } \xi < 0 \end{cases} \tag{15a}$$

$$U = U_0(\xi); \tag{15b}$$

where

$$\left.\begin{aligned} G^{+} &= \tfrac{1}{2}\beta U^{-2}\int_{\pi/2}^{\pi} S\left[(-\beta/U)^{\frac{1}{2}}, \vartheta, T\right] |\sin 2\vartheta|\, d\vartheta \\ G^{-} &= \tfrac{1}{2}\beta U^{-2}\int_{\pi}^{3\pi/2} S\left[(-\beta/U)^{\frac{1}{2}}, \vartheta, T\right] |\sin 2\vartheta|\, d\vartheta \end{aligned}\right\} \tag{16}$$

It should be noted that G^+ and G^- differ from each other only in their limits of integration, which 'cut out' the waves being absorbed at the northern and the southern 'slopes' of the jet, respectively.

Evidently, parameterization (15a) does not cover the whole y-axis: there is a gap at

$$-\int_0^T G^-[U_{\min}, T']\, dT' < y < \int_0^T G^+[U_{\min}, T']\, dT' \tag{17a}$$

where U is constant:

$$U = U_{\min} \tag{17b}$$

Thus, each westward jet has a constant-velocity core which expands at a varying speed (equal to $G^-[U_{\min}, T] + G^+[U_{\min}, T]$). It should be noted, also, that the fluid velocity in the jet does not grow.

The evolution of jet periphery (16) strongly depends on the behaviour of $G^{\pm}$ at $U \to 0$, which, in its turn, depends on the asymptotic behaviour of the energy spectral density $S(q, \vartheta, T)$ at $q \to \infty$. Indeed, we let S be a slowly decreasing function of q:

$$S > \text{const} \cdot q^{-4}$$

A typical sketch of $G^{\pm}(U)$ for this case is shown in Fig. 2a: one can see that characteristics (15a) with smaller values of the velocity U move along the y-axis at greater speeds $G^{\pm}(U)$. Correspondingly, any jet decreasing at $y \to \pm\infty$ velocity profile, remains smooth and 'gently sloping' for all $T > 0$ (see Fig. 3). For example, the equilibrium isotropic spectrum

$$S = \frac{1}{\text{const}_1 + \text{const}_2 \cdot q^2}$$

(Reznik, 1984) is of the slowly decreasing type and corresponds to the smooth pattern of jet behaviour.

In the case of rapidly decreasing S, $G^{\pm}(U)$ are not monotonic functions (see Fig. 2b), characteristics (15a) intersect and low values of U overtake the higher ones. This phenomenon is typical for nonlinear hyperbolic systems—from the physical point of view it means that the jet core expands

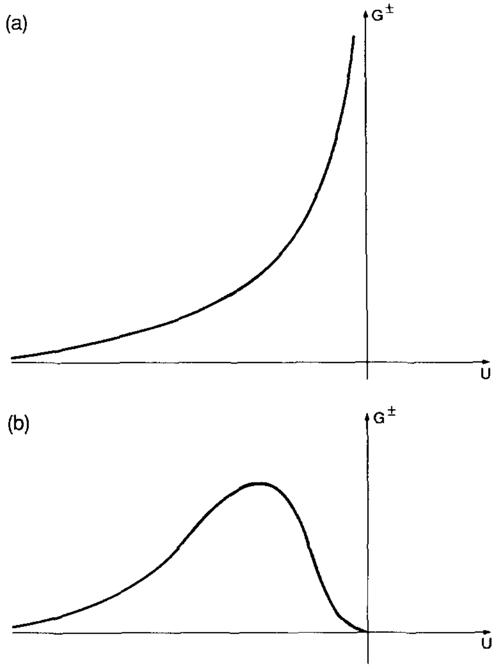

Fig. 2. Sketch of $G^{\pm}$ vs. U (eqns. (16)). (a) the case of slowly decreasing $S(k, l)$; (b) the case of rapidly decreasing $S(k, l)$.

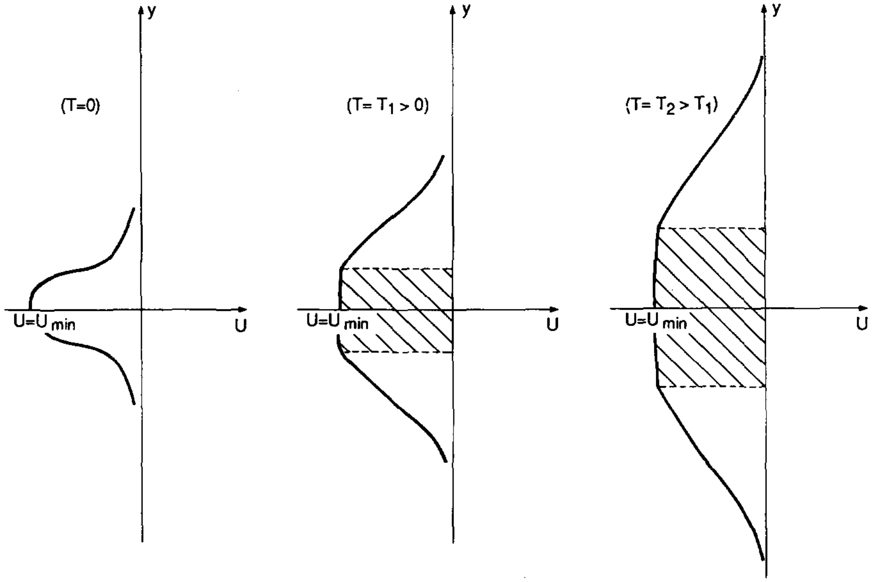

Fig. 3. The evolution of a westward jet affected by turbulence with slowly decreasing energy spectral density (eqns. (15) and (17)). The core of the jet is shaded. The jet expands without strengthening and becomes increasingly 'gently sloping'.

faster than its periphery. The solution cannot remain smooth, and a tangential jump appears on the jet velocity profile. The short-wave approximation of the turbulence fails before the jump appears, but it fails only for the waves that have their critical layers in the vicinity of the jump that appears. All other waves keep on being absorbed by the flow in front of and behind the jump—thus, the steepening of the jet profile also continues. In the end, the criterion $\beta > U_{yy}$ is violated and the jet becomes unstable. The evolution of such locally supercritical flows was investigated by Shepherd (1988).

4. HOW DOES THE JET AFFECT THE WAVE TURBULENCE?

It is clear that the asymptotic system (7) and (10) does not describe the interaction of Rossby waves with each other. Indeed, as the wave amplitudes are small (of the order of ϵ), their nonlinear interaction is slow and cannot affect the waves significantly as they cross the jet. However, waves do interact outside the jet, for the scattering by mean flow changes the spectral composition of the turbulence, violates its statistical equilibrium and, consequently, initiates the nonlinear wave interaction (in other words, the reflected waves interact with the incident ones). As a result, the jet is 'dipped' into a domain of non-relaxed turbulence, where the waves are free and their evolution can be described using the so-called kinetic equation. Though the Rossby-wave kinetic equation was derived for spatially homogeneous fields (e.g. Reznik, 1980), it can be easily generalized for the inhomogeneous case (as it has been for surface gravity waves (Hasselmann, 1968), plasma waves (Vedenov and Rudakov, 1964), etc.):

$$E_T + c_{\mathrm{gr}}^{(y)} E_Y = St[E] \tag{18}$$

where $Y = \epsilon^2 y$ is the slow meridional variable, $E(k, l, T, Y)$ is the local energy spectral density and the functional $St[E]$ is the so-called collision integral (see Appendix B). The evolution of turbulence is slow both in space and time, as the domain of non-relaxed turbulence is wide (approximately ϵ^{-2}) in comparison with the width of the jet. One should understand system (7) and (10) and the kinetic eqn. (18) as the 'inner' and 'outer' expansions of the original problem, respectively. As always, the $(y = \infty)$ value of the inner expansion must be matched with the $(y = 0)$ value of the outer one:

$$S(k, l, T) = \begin{cases} E(k, l, T, Y)|_{Y=-0} & \text{if } c_{\mathrm{gr}}^{(y)} > 0 \\ E(k, l, T, Y)|_{Y=+0} & \text{if } c_{\mathrm{gr}}^{(y)} < 0 \end{cases} \tag{19a}$$

The kinetic equation determines the 'incident' spectrum of wave turbulence; and conversely, wave scattering by the jet determines boundary conditions for the kinetic equation:

$$\left.\begin{aligned} &E(k,l)|_{Y=+0} = |d(k,l)|^2 E(k,l)|_{Y=-0} + |r(k,-l)|^2 E(k,l)|_{Y=+0} \\ &\quad \text{if } c_{\mathrm{gr}}^{(y)}(k,l) > 0 \\ &E(k,l)|_{Y=-0} = |d(k,l)|^2 E(k,l)|_{Y=+0} + |r(k,-l)|^2 E(k,l)|_{Y=-0} \\ &\quad \text{if } c_{\mathrm{gr}}^{(y)}(k,l) < 0 \end{aligned}\right\} \tag{19b}$$

(as above, we treat the jet as a straight line $Y=0$). Equations (7), (10), (18) and (19) constitute a closed-form system describing the interaction of a zonal jet with Rossby-wave turbulence and containing deterministic quantities only. The latter is its main advantage in comparison with the original equations (1).

It should be noted, also, that the wave turbulence is governed by an evolutionary equation, and as E and S depend on their initial values, one laminar flow can correspond to multiple distributions of the turbulence. Thus, the effect of turbulent friction cannot be 'parameterized' within the framework of turbulent–laminar interaction only (i.e. without knowledge of the source of the turbulence). In the case where the waves are generated by baroclinic instability, one might still hope to parameterize the structure of the turbulence using the instability theory.

5. CONCLUSION

Thus, we have studied the interaction of a zonal laminar jet with weak Rossby-wave turbulence. In our opinion, this particular problem deserves to be considered because: (1) the weak turbulence bears some of the characteristic features inherent to any turbulent motion in fluid; (2) the weak turbulence does exist in the ocean and atmosphere and deserves as much consideration as the strong one.

The asymptotic system, governing the wave–jet interaction and containing deterministic quantities only, was derived. The system consists of: (1) a boundary-value problem (7), describing the scattering of each Rossby wave by the jet; (2) eqn. (10), describing the reverse collective effect of the waves upon the jet; (3) a boundary-value problem for the kinetic eqns. (18) and (19), describing the interaction of Rossby waves with each other and determining the spectrum of the 'incident' wave turbulence.

Two important conclusions can be derived directly from the form of the equations obtained: (1) the effect of turbulent friction is nonlinear, anisotropic and negative; (2) as the spectrum of the turbulence is governed by an evolutionary equation, this effect cannot be parameterized without knowledge of the sources of the turbulence (e.g. baroclinic instability).

For the case of short-wave turbulence, the solution, which describes the evolution of a zonal jet, has been constructed. This solution demonstrates that all westward jets slowly expand without strengthening. Each jet has a 'core', within the limits of which the velocity of the fluid is constant. In some cases, the core of the jet expands faster than its periphery, resulting in jumps on the profile of the jet. All eastward jets (within the framework of the theory presented) are steady irrespective of their profiles.

It should be noted, however, that in the terrestrial atmosphere (and almost certainly in that of Jupiter as well), there are significant Reynolds-stress divergences where the zonal flow is eastward, apparently contradicting the results obtained. Hypothetical mechanisms that may be responsible for wave–flow interaction in those cases were discussed by Feldstein and Held (1989) and Fyfe and Held (1990). Also, in the present model the waves are created in a region of zero mean flow (and thus must have westward phase speed), whereas in the real atmosphere waves can be created by baroclinic instability of an eastward flow, and thus possess eastward (Doppler-shifted) phase speed. Apparently, critical layers produced in eastward jets by such 'eastward-flow-created' waves can explain the discrepancy between our results and atmospheric observations.

It is worth noting that none of the traditional diffusive-type models of turbulent–laminar interaction (see Pedlosky, 1979) can provide a satisfactory description of the turbulent friction in the example considered. On the other hand, there are numerous manifestations of the negative friction effect in the ocean and the atmosphere (e.g. Lorenz, 1967; Ivchenko and Klepikov, 1985). It seems probable that this phenomenon can be explained only within the framework of the wave-turbulence concept.

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REFERENCES

Benilov, E.S. and Wolanski, E., 1992. A note on the diffusion in fluids with wave induced random motion. Phys. Fluids., in press.

Dickinson, R.E., 1970. Development of a Rossby wave critical level. J. Atmos. Sci., 27: 627–633.

Feldstein, S.B. and Held, I.M., 1989. Barotropic decay of baroclinic waves in two-layer beta-plane model. J. Atmos. Sci., 46: 3416–3430.
Fyfe, J. and Held, I.M., 1990. The two-fifths and one-fifth rules for Rossby wave breaking in the WKB limit. J. Atmos. Sci., 47: 697–706.
Hasselmann, K., 1968. Weak interaction theory of ocean waves. In: M. Holt (Editor), Basic Developments in Fluid Dynamics, Vol. 2: Academic Press, New York, pp. 117–281.
Held, I.M., 1985. Pseudomomentum and the orthogonality of modes in shear flows. J. Atmos. Sci., 42: 2280–2288.
Held, I.M. and Phillips, J.P., 1987. Linear and nonlinear barotropic decay on the sphere. J. Atmos. Sci., 44: 200–207.
Ivchenko, V.O. and Klepikov, A.V., 1985. Eddy flux of momentum, heat and vorticity in the atmosphere and ocean and their parameterization in the general circulation. Oceanology, 25: 5–18.
Killworth, P.D. and McIntyre, M.E., 1985. Do Rossby-wave critical layers absorb, reflect or over-reflect? J. Fluid Mech., 161: 449–492.
Lindzen, R.S. and Holton, J.R., 1968. A theory of the quasi-biennial oscillation. J. Atmos. Sci., 25: 1095–1107.
Lorenz, E.N., 1967. The Nature and Theory of the General Circulation of the Atmosphere. World Meteorological Organization, Geneva, 161 pp.
Pedlosky, J., 1979. Geophysical Fluid Dynamics. Springer, Heidelberg, 624 pp.
Reznik, G.M., 1980. On the energy transfer equation for weakly interacting waves. J. Non-Linear Mech., 19: 95–113.
Reznik, G.M., 1984. On the features of the equilibrium spectra of the weakly nonlinear Rossby waves. Oceanology, 24: 869–873.
Shepherd, T.G., 1988. Rigorous bounds on the nonlinear saturation of instabilities to parallel shear flows. J. Fluid Mech., 196: 291–322.
Tsimring, L.Sh., 1982. Destabilization of an initially stable shear flow by internal waves. Oceanology, 22: 396–399.
Vedenov, A.A. and Rudakov, L.I., 1964. On the weak interaction in fluids. Sov. Phys. Dokl., 9: 1073–1075 (Russian version 159: 767–770).
Zakharov, V.E., 1974. The Hamiltonian formalism for waves in nonlinear media having dispersion. Radiophys. Quantum Electron., 17: 326–343 (Russian version 17: 431–453).

APPENDIX A: DERIVATION OF EQN. (11)

Integrating (9) with respect to y over the interval $(-\infty, \infty)$, we have

$$W|_{y\to\infty} - W|_{y\to-\infty} = -\pi(\text{sign } k)\sum_{n=1}^{N}\left(\frac{\beta - U_{yy}}{|U_y|}|\phi|^2\right)\bigg|_{y=y_n} \tag{A1}$$

On the other hand, with the help of boundary conditions (7b), we can see that

$$W|_{y\to\infty} - W|_{y\to-\infty} = -(\text{sign } k)|l|(1 - |r|^2 - |d|^2) \tag{A2}$$

Equating the right-hand sides of (A1) and (A2), we obtain the desired formula for the absorption coefficient [eqn. (11)].

APPENDIX B: KINETIC EQUATION FOR WEAK TURBULENCE

The kinetic equation for spatially homogeneous Rossby-wave turbulence (e.g. Reznik, 1980) is

$$E_T(\boldsymbol{q}) = St[E(\boldsymbol{q})] \tag{B1}$$

Here $T = \epsilon^2 t$ is the slow time, $E(\boldsymbol{q}, T)$ is the energy spectral density, $\boldsymbol{q} = (k, l)$ is the wave vector and St is the so-called collision integral:

$$\begin{aligned} St = 4\pi \int\int I(\boldsymbol{q}, \boldsymbol{q}_1, \boldsymbol{q}_2)[I(\boldsymbol{q}, \boldsymbol{q}_1, \boldsymbol{q}_2)E(\boldsymbol{q}_1)E(\boldsymbol{q}_2) + I(\boldsymbol{q}_2, \boldsymbol{q}, \boldsymbol{q}_1)E(\boldsymbol{q})E(\boldsymbol{q}_1) \\ + I(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q})E(\boldsymbol{q}_2)E(\boldsymbol{q})] \\ \cdot\ \delta(\boldsymbol{q} - \boldsymbol{q}_1 - \boldsymbol{q}_2)\delta[\omega(\boldsymbol{q}) - \omega(\boldsymbol{q}_1) - \omega(\boldsymbol{q}_2)]\ \mathrm{d}\boldsymbol{q}_1\ \mathrm{d}\boldsymbol{q}_2 \end{aligned}$$

where

$$I(\boldsymbol{q}, \boldsymbol{q}_1, \boldsymbol{q}_2) = (l_1 k_2 - k_1 l_2)(q_1^2 - q_2^2)/(q q_1 q_2)$$

$$\omega(\boldsymbol{q}) = -\beta k/q^2$$

are the coefficient of nonlinear interaction and the frequency of Rossby waves.

In another particular case, where the nonlinear wave interaction is negligible, but the wave field is smoothly modulated in space, the equation governing weak turbulence is

$$E_T(\boldsymbol{q}) + c_{\mathrm{gr}}^{(x)}(\boldsymbol{q})E_x(\boldsymbol{q}) + c_{\mathrm{gr}}^{(y)}(\boldsymbol{q})E_Y(\boldsymbol{q}) = 0 \tag{B2}$$

where $\boldsymbol{c}_{\mathrm{gr}} = (c_{\mathrm{gr}}^{(x)}, c_{\mathrm{gr}}^{(y)})$ is the group velocity of Rossby waves, $(X, Y) = (\epsilon^2 x, \epsilon^2 y)$ and ϵ^{-2} determines the spatial scale of the modulation. Apparently, this equation is applicable to all types of dispersive waves, for example, surface water waves (Hasselmann, 1968) and plasma waves (Vedenov and Rudakov, 1964).

An equation describing both nonlinear interaction and group transfer, has to be a combination of eqns. (B1) and (B2):

$$E_T(\boldsymbol{q}) + c_{\mathrm{gr}}^{(x)}(\boldsymbol{q})E_X(\boldsymbol{q}) + c_{\mathrm{gr}}^{(y)}(\boldsymbol{q})E_Y(\boldsymbol{q}) = St[E(\boldsymbol{q})] \tag{B3}$$

This heuristic derivation of the kinetic equation for spatially inhomogeneous Rossby-wave turbulence is not rigorous. A more straightforward procedure starting from the original system (1) could be developed similarly to the case of surface waves (Hasselmann, 1968). Obviously, it would yield exactly the same equation.

Equation (18) used in the present paper follows from (B3) in the case of homogeneity in the X-direction.