

# Propagation of nonlinear waves in a fluctuating medium

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1. The scattering of nonlinear waves in a weakly dispersive medium with random inhomogeneities is an important problem in the general theory of waves. Powerful general methods have been developed for the linear case (see Ref. 1 and the literature cited there), but in the nonlinear case it is non-trivial to take into account inhomogeneities and far fewer results have been obtained in this case. There are three different approaches to the problem of nonlinear waves in a fluctuating medium: the Born approximation, perturbation theory for systems which are almost integrable, and the mean-field method. The Born approximation<sup>2,3</sup> is applicable only to small space-time intervals, while use of the second approach<sup>4</sup> assumes that the "homogeneous" form of the problem is integrable by means of the methods used to solve inverse-scattering problems. The mean-field method<sup>5</sup> has been widely used in different physical problems. Although it assumes that the nonlinearity and inhomogeneity of the problem are additive, this assumption was later shown to be incorrect.<sup>6</sup>

The present paper is devoted to a modification of the mean-field method, which allows a rigorous asymptotic justification. The modification is based on a transformation to a reference frame which moves with the fluctuating velocity of the wave. In this frame the phase fluctuations can be filtered out and the evolution of the wave profile can be described. We will show that because of this transformation to a new reference frame, secular terms are absent in all orders of perturbation theory.

2. We consider the standard equation describing the evolution of sound waves of small, but finite, amplitude in a one-dimensional medium with small fluctuations in the speed of sound:

$$u_{tt} - (1 + \epsilon\alpha)^2 u_{xx} = \epsilon^2 (u^2)_{xx}, \quad (1)$$

where  $\epsilon$  is a small positive number, and  $\alpha(x, t)$  is a stationary random function with a zero mean. We first consider purely temporal fluctuations:  $\alpha = \alpha(t)$ . We transform to a reference frame moving with the fluctuating velocity (which is to be determined):

$$x' = x - \int c(t) dt, \quad t' = t. \quad (2)$$

In terms of these new variables Eq. (1) takes the form (the primes are omitted)

$$u_{tt} - 2cu_{tx} - c_t u_x + [c^2 - (1 + \epsilon\alpha)^2] u_{xx} = 0. \quad (3)$$

In analyzing (3) we shall use the asymptotic method of multiple time scales; i.e., in addition to the "fast" time  $t$ , we introduce a hierarchy of "slow" times:  $T = \epsilon^2 t$ ,  $T_1 = \epsilon t$ , and so on. The time derivatives in (3) can then be transformed to

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T} + \epsilon^3 \frac{\partial}{\partial T_1} + \dots$$

The solution of Eq. (3) is written in the form of asymptotic series

$$u(x, t, T, \dots) = u^{(0)}(x, T, \dots) + \epsilon u^{(1)}(x, t, T, \dots) + \dots, \quad (4)$$

$$c(t, T, \dots) = 1 + \epsilon c^{(1)}(t, T, \dots) + \dots$$

The lowest-order terms in (4) correspond to waves travelling to the right with a velocity close to the speed of sound. To zero order in  $\epsilon$  the perturbation-theory equations are satisfied automatically. To first order we obtain an equation for the scattered field  $u^{(1)}$ :

$$u_{tt}^{(1)} - 2u_{tx}^{(1)} = c_t^{(1)} u_x^{(0)} + 2(\alpha - c^{(1)}) u_{xx}^{(0)}. \quad (5.1)$$

Equation (5.1) should be supplemented with the condition that there be no reflected waves in the limit  $t \rightarrow -\infty$ :

$$u^{(1)} \Big|_{t \rightarrow -\infty}^{x = \text{const}} = u^{(1)} \Big|_{x \rightarrow -\infty}^{x + 2t = \text{const}} = 0. \quad (5.2)$$

Here it is implicitly assumed that the fluctuation is switched on adiabatically; i.e., Eq. (5) should be solved by replacing  $\alpha(t) \rightarrow \alpha(t) e^{\nu t}$ ,  $0 < \nu \ll 1$  and then by taking the limit  $\nu = 0$ . This device ensures that the improper integrals arising below converge. The problem (5) can be trivially integrated:

$$u^{(1)} = \int_0^{\infty} u_x(x + 2\tau, T) \alpha(t - \tau) d\tau - u_x^{(0)}(x, T) \int_{-\infty}^t [\alpha(\tau) - c^{(1)}(\tau)] d\tau. \quad (6)$$

Because of the arbitrariness in the choice of  $c(t)$ , we assume

$$c^{(1)} = \alpha(t), \quad (7)$$

which will be justified below.

To second order in  $\epsilon$  we have an inhomogeneous linear equation for  $u^{(2)}$

$$\frac{\partial}{\partial t} (u_t^{(2)} - 2u_x^{(2)}) = F(x, t, T),$$

$$F = 2u_{Tx}^{(0)} + [(u^{(0)})^2]_{xx} + 2\alpha u_{tx}^{(1)} + \alpha_t u_x^{(1)} + 2c^{(2)} u_{xx}^{(0)} + c_t^{(2)} u_x^{(0)}, \quad (8)$$

Because of the stationary nature of  $\alpha(t)$ , the finiteness of  $u^{(2)}$  can be ensured by requiring that

$$\langle F \rangle \equiv \lim_{\Delta \rightarrow \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} F(x, t, T) dt = 0, \quad (9)$$

and, taking the special case  $\langle c^{(2)} \rangle = 0$ , we obtain with the help of (6)-(8) an equation describing the slow evolution of the wave [the index (0) is omitted]:

$$u_T + \frac{\sigma^2}{2} u_x + uu_x + \int_0^{\infty} u_{xx}(x + 2\tau, T) W(\tau) d\tau = 0, \quad (10)$$

where  $W(\tau) = \langle \alpha(t)\alpha(t + \tau) \rangle$  is the correlation function of the speed of sound fluctuations,<sup>1)</sup>  $\sigma^2 = W(0)$ . Equation (10) involves only determined coefficients, which is an advantage over the original equation (1). Since the field is averaged in a reference frame moving with the local speed of sound and "following" the phase fluctuations of the wave,  $u(x, T)$  describes the mean wave form and not the mean field. We note also that the fast-time average in our method arises as a natural condition for the absence of secular terms and in this sense our method differs in an essential way from the ensemble average technique ordinarily used in problems of this kind. Our averaging technique better corresponds to the experimental situation, where one typically averages over time.

We emphasize that we have nowhere used the hypothesis that the distribution of the speed of sound fluctuations is a Gaussian; this is also true of the higher-order approximations of perturbation theory, which are constructed in a similar way.

3. To justify condition (7) and to compare our results with the mean-field method, we return to expression (6) for  $u^{(1)}$  and assume, as is customary in the mean-field method, that  $c^{(1)} = 0$  (i.e., we transform to an inertial reference frame moving with a constant velocity). In this case the first term in (6) describes the scattered wave, while the second term describes a small shift of the incident wave as a whole (this shift is due to the fluctuations in the propagation velocity of the wave):

$$u^{(0)}(x - \epsilon\theta(t), T) \approx u^{(0)}(x, T) - \epsilon u_x^{(0)}(x, T)\theta(t),$$

where  $\theta = \int_{-\infty}^t \alpha(\tau) d\tau$ . It is easy to show that the

mean-square [in the sense of (9)] phase fluctuation  $\langle \theta^2 \rangle$  is infinitely large. In fact,

$$\langle \theta^2 \rangle = \int_{-\infty}^0 \int_{-\infty}^0 \langle \alpha(\tau+t)\alpha(\tau'+t) \rangle d\tau d\tau' = \int_{-\infty}^0 \int_{-\infty}^0 W(\tau-\tau') d\tau d\tau'. \quad (11)$$

It is thus clear that the integral in (11) diverges

for any  $W(\tau)$ :  $\int_{-\infty}^{\infty} W(\tau) d\tau \neq 0$ . It is also clear that,

together with  $\langle \theta^2 \rangle$ , the quantity  $\langle (u^{(1)})^2 \rangle$  also diverges. Because of the nonlinearity of the original equation, this quantity appears in the boundedness condition of  $u^{(4)}$  in the limit  $t \rightarrow \infty$ . Hence secular terms appear in the higher approximations of perturbation theory when  $c^{(1)} = 0$  and, therefore, perturbation theory as a whole breaks down.<sup>2</sup> This was not noted in Ref. 5 or in later papers on this subject, causing the erroneous results. A natural condition removing the increase of  $u^{(4)}$  is (7): It can be assumed as a necessary condition for this reason. If this increase is ignored (as was done in Ref. 5), the resulting equation will describe the mean field  $\bar{u}$  (rather than the mean wave form  $u$ ) and will be in a reference frame  $(x, T)$  moving with the constant velocity  $c = 1$ :

$$\bar{u}_T + \frac{\sigma^2}{2} \bar{u}_{\bar{x}\bar{x}} + \bar{u} \bar{u}_{\bar{x}} + \int_0^{\infty} \bar{u}_{\bar{x}\bar{x}}(\bar{x} + 2\tau, T) W(\tau) d\tau + \bar{u}_{\bar{x}\bar{x}} \int_0^{\infty} W(\tau) d\tau = 0. \quad (12)$$

In order to compare our results with those obtained using the mean-field method, we calculate the mean field in a "field-averaged" reference frame in terms of the randomly oscillating reference frame (2):

$$\bar{u}(\bar{x}, T) = \int_{-\infty}^{\infty} u(\bar{x} - \epsilon\theta, T) \Gamma(\theta) d\theta, \quad (13)$$

where  $\Gamma(\theta)$  is the probability distribution of the phase, which is determined through the distribution of fluctuations in the speed of sound  $\Gamma(\alpha)$  and

the equation  $\theta(t) = \int_{-\infty}^t \alpha(\tau) d\tau$ . Similarly, it can be

shown<sup>6</sup> that if  $\Gamma(\alpha)$  is Gaussian, then (13) for  $\bar{u}(\bar{x}, T)$  satisfies an equation which is fundamentally different from the ordinary mean-field equation (12) (the difference lies in the different types of nonlinear terms). In the linear approximation, however, the results of the two approaches are identical, since there are no diverging (quadratic) terms of the type  $\langle (u^{(1)})^2 \rangle$  in perturbation theory for linear systems.

4. We consider now the case of purely spatial fluctuations of the speed of sound  $\alpha = \alpha(x)$ . This case can be included in the method described here if the substitution  $x \leftrightarrow t$  is carried out in all of the computations [for example, in place of (2) we

make the substitution  $x' = x, t' = t - \int \frac{dx}{c(x)}$ , and

so on]. We thus obtain an equation of the type (10), but one that describes the evolution with respect to  $X$  ( $X = \epsilon^2 x$  is the slow spatial coordinate):

$$u_X - \frac{\sigma^2}{2} u_t + uu_t - \int_0^{\infty} u_{tt}(X, t + 2\tau) W(\tau) d\tau = 0. \quad (14)$$

Finally, we note that our approach can be extended trivially to a weakly dispersive medium.

<sup>1</sup>We emphasize that the average of the correlation function is to be understood in the sense of (9).

<sup>2</sup>We note that  $u^{(4)}$  depends on  $t$  quadratically, and the condition  $|u^{(0)}| \gg \epsilon^4 |u^{(4)}|$  is violated for  $t \sim 1/\epsilon^2$  or, equivalently, when  $X = \epsilon^2 t \sim 1$ , i.e., the case of interest to us.

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<sup>3</sup>E. N. Pelinovskii, Trans. VIII All-Union Acoust. Conf., Nauka, Moscow (1973); p. 24.

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